Pizza time!

Closed book exam; no calculators. Adequately explain the steps you take and answer as much as you can (partial credit awarded).

1. Professor Z has invented a new system to cook pizza slices from the circular edge. The temperature satisfies

$$0 = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}$$

with

$$0 \le \theta \le \alpha, \quad u(1,\theta) = \frac{\theta}{\alpha}\left(1 - \frac{\theta}{\alpha}\right), \quad u(r,0) = u(r,\alpha) = 0,$$

where α is the (constant) angle of the wedge-shaped slice. Solve this problem using separation of variables. Verify Professor Z's assertion that the temperature of the midsection ($\theta = \frac{1}{2}\alpha$) is approximately equal to $Cr^{\pi/\alpha}$ to within a relative error of about 4%, where C is a constant that you should calculate.

2. After being cooked, the pizza slice cools according to

$$u_t = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} - u$$

with

$$0 \le \theta \le \alpha, \quad u(1,\theta) = u_\theta(r,0) = u_\theta(r,\alpha) = 0, \quad u(r,\theta,0) = u_0(r) + u_2(r)\cos(2\pi\theta/\alpha).$$

Solve this problem using separation of variables (it is fine if coefficients are expressed as integrals). Over long times, how is temperature distributed over the slice and what is the final decay rate?

Helpful information:

Fourier Series:

For a periodic function $f(x)$ with period $2L$, the Fourier series is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

with

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

For any non-negative constant ν, **Bessel's equation**

$$z^2y'' + zy' + (z^2 - \nu^2)y = 0.$$

has a regular solution, $y(z) = J_\nu(z)$, with $J_\nu(z) \propto z^\nu$ for $z \to 0$, and an infinite number of zeros at $z = z_{\nu n} > 0$, $n = 1, 2, ...$ (*i.e.* $J_\nu(z_{\nu n}) = 0$). We have $z_{\nu 1} > z_{01} \approx 2.405$ for any $\nu > 0$.

The **Sturm-Liouville ODE** is

$$[p(x)y']' + \lambda\sigma(x)y + q(x)y = 0, \qquad a < x < b,$$

with $\sigma(x) > 0$ and $p(x) > 0$. The associated expansion formula using the eigensolutions $\{\lambda_n, y_n(x)\}$ is

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \qquad c_n = \frac{\int_a^b f(x)y_n(x)\sigma(x)dx}{\int_a^b [y_n(x)]^2\sigma(x)dx}.$$

Helpful trig identities:

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B \qquad \& \qquad \sin(A \pm B) = \sin A\cos B \pm \cos A\sin B.$$

Midterm exam - solution

1. We separate variables: $u(r,\theta) = R(r)\Theta(\theta)$, giving

$$\frac{\Theta''}{\Theta} = -\frac{r(rR')'}{R} = -\lambda.$$

Hence $\Theta \propto \sin M\theta$ and $R \propto r^M$, with $M = m\pi/\alpha$ and $m = 1, 2, ...$, given that we want a non-trivial solution that is regular for $r \to 0$ and $\Theta(0) = \Theta(\alpha) = 0$. Hence

$$u(r,\theta) = \sum_{m=1}^{\infty} b_m r^M \sin M\theta \quad (M = m\pi\alpha^{-1}).$$

This solution is justified by extending everything as odd periodic functions wth period 2α, or by using Sturm-Liouville theory. Moreover, the coefficients b_m are given by

$$b_m = \int_0^\alpha \frac{\theta}{\alpha}\left(1 - \frac{\theta}{\alpha}\right)\sin M\theta \, d\theta \;\times\; \left[\int_0^\alpha \sin^2 M\theta \, d\theta\right]^{-1}$$

$$= 2\int_0^1 x(1-x)\sin m\pi x \, dx = \frac{4[1-(-1)^m]}{m^3\pi^3}$$

Along the midsection $\theta = \pi$, $\sin M\theta = \sin(\pi m/2)$ and so

$$u = \frac{8}{\pi^3}\left[r^a - \frac{r^{3a}}{27} + \frac{r^{5a}}{125} + ...\right]$$

with $a = \pi\alpha^{-1}$. Thus $u \approx Cr^a$, as required, with $C = 8/\pi^3$. Also, the next terms in the series are less than $1/25 \equiv 4\%$ at their greatest (negative) size at $r = 1$.

2. We separate variables for the PDE, $u = R(r)\Theta(\theta)T(t)$, finding

$$T_t = -(1+k^2)T, \qquad \Theta_{\theta\theta} = -M^2\Theta, \qquad r^2 X_{rr} + rX_r + \left(k^2r^2 - M^2\right)X = 0.$$

The solutions are $T \propto e^{-(1+k^2)t}$, $\Theta \propto \sin M\theta$ or $\cos M\theta$ if $M > 0$ or a constant if $M = 0$, and the Bessel function $J_M(kr)$. The boundary conditions in angle demand $M = m\pi/\alpha$ and $\Theta \propto \cos(m\pi\theta/\alpha)$ $(m = 1, 2, ...)$ or $\Theta = \textit{constant}$ $(m = 0)$, whereas that at $r = 1$ demands that $k = z_{Mn}$, the n^{th} zero of $J_M(z)$. However, the initial condition has only two pieces, one independent of θ and the other proportional to $\cos(2\pi\theta/\alpha)$. Therefore,

$$u = \sum_{n=1}^{\infty} c_{0n} J_0(z_{0n}r)e^{-(1+z_{0n}^2)t} + \cos(2\pi\theta/\alpha)\sum_{n=1}^{\infty} c_{\nu n} J_\nu(z_{\nu n}r)e^{-(1+z_{\nu n}^2)t}, \qquad \nu = 2\pi\alpha^{-1}.$$

Using the Sturm-Liouville expansion formula, the coefficients must be given by

$$c_{0n} = \int_0^1 u_0(r)J_0(z_{0n}r)rdr \;\times\; \left[\int_0^1 [J_0(z_{0n}r)]^2 rdr\right]^{-1}$$

$$c_{\nu n} = \int_0^1 u_2(r)J_\nu(z_{\nu n}r)rdr \;\times\; \left[\int_0^1 [J_\nu(z_{\nu n}r)]^2 rdr\right]^{-1}.$$

For large times, the first, $\theta-$independent term dominates with

$$u \approx c_{01}J_0(z_{01}r)e^{-at} \quad \text{with} \quad a = 1 + z_{01}^2 \quad (z_{01} \approx 2.405).$$