Math 400 - midterm

Closed book exam; no calculators. Adequately explain the steps you take and answer as much as you can (partial credit awarded).

1. Seeds blown by the wind from the mainland arrive on a circular desert island; the resulting vegetation colonizes the land. A biologist assumes that the plant density $u(r, \theta, t)$ evolves according to

$$u_t = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} + \gamma u, \quad u(1, \theta, t) = 0, \quad u(r, \theta, 0) = f(r)\sin 2\theta.$$

where γ is a constant representing the replication rate of the plants and $f(r)$ denotes some initial distribution. Solve this problem using separation of variables, and provide a condition on γ that ensures that the plants will eventually die out.

2. An acoustician models the ringing of an air bubble by

$$u_{tt} = \frac{1}{r^2}(r^2 u_r)_r - \frac{1}{r^2}\ell(\ell+1)u, \quad u(1, t) = u_t(r, 0) = 0, \quad u(r, 0) = 1, \quad u \text{ regular for } r \to 0,$$

where $\ell > 0$ is a constant. Solve this problem using separation of variables. Confirm that

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\sin z,$$

and hence provide a solution for $\ell = 0$ without any integrals and verify that this solution has the form, $[F(r+t) + F(r-t)]/(2r)$, for some function $F(z)$. What is $F(z)$ and can you explain this d'Alembert-like form?

Helpful information:

Fourier Series:

For a periodic function $f(x)$ with period $2L$, the Fourier series is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right],$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

For any non-negative constant ν, **Bessel's equation**

$$z^2 y'' + zy' + (z^2 - \nu^2)y = 0.$$

has a regular solution, $y(z) = J_\nu(z)$, with $J_\nu(z) \propto z^\nu$ for $z \to 0$, and a singular solution, $y(z) = Y_\nu(z)$. For $\nu = 2$, $J_2(z)$ has an infinite number of zeros, $z = z_n$, with the first at $z = z_1 \approx 5.14$. The more general ODE,

$$x^2 y'' + (1 - 2\alpha)xy' + (\omega^2\beta^2 x^{2\beta} + \alpha^2 - \nu^2\beta^2)y = 0,$$

with parameters α, ω, β and $\nu > 0$, has the solutions $x^\alpha J_\nu(\omega x^\beta)$ and $x^\alpha Y_\nu(\omega x^\beta)$.

The **Sturm-Liouville ODE** is

$$[p(x)y']' + \lambda\sigma(x)y + q(x)y = 0, \qquad a < x < b,$$

with $\sigma(x) > 0$ and $p(x) > 0$. The associated expansion formula using the eigensolutions $\{\lambda_n, y_n(x)\}$ is

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \qquad c_n = \frac{\int_a^b f(x)y_n(x)\sigma(x)dx}{\int_a^b [y_n(x)]^2\sigma(x)dx}.$$

Helpful trig identities:

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B \qquad \& \qquad \sin(A \pm B) = \sin A\cos B \pm \cos A\sin B.$$

Midterm exam - solution

1. (10 points) We separate variables for the PDE, $u = R(r)\Theta(\theta)T(t)$, finding

$$T_t = (\gamma - k^2)T, \qquad \Theta_{\theta\theta} = -m^2\Theta, \qquad r^2X_{rr} + rX_r + \left(k^2r^2 - m^2\right)X = 0.$$

Given $u(r,\theta,t)$ must be $2\pi-$periodic in angle and regular for $r \to 0$, the solutions are $T \propto e^{(\gamma-k^2)t}$, $\Theta \propto \sin m\theta$ or $\cos m\theta$ if $m = 1, 2, 0..$ or a constant if $m = 0$, and the Bessel function $J_m(kr)$ *(3 points)*. The boundary condition $R(1) = 0$ then implies that k must be a zero of $J_m(z)$. However, the initial condition is proportional to $\sin 2\theta$, so we may take $u(r,\theta,t) \propto \sin 2\theta$ and set $m = 2$. Moreover, $k = z_n$, the n^{th} (positive) zero of $J_2(z)$ (*i.e.* $J_2(z_n) = 0$) *(2 points)*. Hence, we write a solution suitable for the initial-value problem,

$$u = \sin 2\theta \sum_{n=1}^{\infty} c_n J_2(z_n r) e^{(\gamma - z_n^2)t}, \qquad c_n = \int_0^1 f(r) J_2(z_n r) r dr \ \times \ \left[\int_0^1 [J_2(z_n r)]^2 r dr\right]^{-1}$$

using the Sturm-Liouville expansion formula, and given that the radial problem is of SL form with weight function $\sigma = r$ and boundary conditions of type 2 (regularity) at $r = 0$ and type 1 (mixed, or Dirichlet) at $r = 1$ *(3 points)*. For large times, the first term of the series dominates with time-dependence $e^{(\gamma - z_1^2)t}$. Thus, plants die out unless $\gamma > z_1^2 \approx 5.14^2$ *(2 points)*.

2. (15 points) We separate variables: $u(r,t) = R(r)T(t)$, giving

$$\frac{T''}{T} = \frac{(r^2R')' - \ell(\ell+1)R}{r^2R} = -\lambda.$$

Hence $T(t)$ is proportional to either $\cos\omega t$ or $\sin\omega t$ if $\lambda = \omega^2 > 0$, or a linear function of t if $\lambda = 0$ *(1 point)*. The ODE for $R(r)$ can be written as

$$r^2R'' + 2rR' + [\omega^2r^2 - \ell(\ell+1)]R = 0,$$

which is a Sturm-Liouville ODE with weight function $\sigma(r) = r^2$ and boundary conditions of type 2 (regularity) at $r = 0$ and type 1 (mixed, or Dirichlet) at $r = 1$ *(2 points)*. It is also Bessel's equation in disguise – we have the more general ODE quoted in the helpful information, with

$$\alpha = -\frac{1}{2}, \quad \beta = 1, \quad \nu^2 = \ell(\ell+1) + \frac{1}{4} = \left(\ell + \frac{1}{2}\right)^2.$$

Hence,

$$\nu = \ell + \frac{1}{2}, \quad R(r) \propto r^{-\frac{1}{2}} J_\nu(\omega r)$$

(which does not diverge for $r \to 0$ in view of the behaviour of $J_\nu(z)$ for small argument). But the boundary condition $R(1) = 0$ implies that $J_\nu(\omega) = 0$. *i.e.* $\omega = \zeta_n$, the n^{th} zero of $J_\nu(z)$ (so $J_\nu(\zeta_n) = 0$). *(5 points)*

Bearing in mind the initial conditions, we now write the solution

$$u(r,t) = \frac{1}{\sqrt{r}} \sum_{n=0}^{\infty} c_n J_\nu(\zeta_n r)\cos(\zeta_n t), \quad c_n = \frac{\int_0^1 J_\nu(\zeta_n r) r^{\frac{3}{2}}\, dr}{\int_0^1 [J_\nu(\zeta_n r)]^2 r\, dr},$$

in view of the Sturm-Liouville expansion formula. *(2 points)*

The function $y = \sqrt{2/\pi z}\,\sin z$ satisfies

$$z(zy')' = z\left(\sqrt{\frac{2z}{\pi}}\cos z - \sqrt{\frac{1}{2\pi z}}\sin z\right)' = -z^2\sqrt{\frac{2}{\pi z}}\sin z + \frac{1}{4}\sqrt{\frac{2}{\pi z}}\sin z = -z^2y + \frac{1}{4}y,$$

which is Bessel's equation for $\nu = \frac{1}{2}$. For $\ell = 0$, we therefore have $\zeta_n = n\pi$ and

$$c_n = \frac{\int_0^1 \sin(n\pi r) r \, dr}{\int_0^1 [\sin(n\pi r)]^2 \, dr} = (-1)^{n+1} \sqrt{\frac{2}{n}}.$$

Therefore,

$$u(r,t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n\pi r} [\sin n\pi(r+t) + \sin n\pi(r-t)] \equiv \frac{1}{2r}[F(r+t) + F(r-t)], \quad F(z) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi} \sin n\pi z.$$

But in view of the initial condition, $F(z)$ must be the odd, period-2 extension of the function $F(r) = r$ (which can also verified by a direct computation). The form of this solution arises because $U(r,t) = ru$ satisfies $U_{tt} = U_{rr}$, which has d'Alembert's solution. *(5 points, but I will be impressed if anyone manages to wade through this last part)*.