Math 400 - midterm

Closed book exam; no calculators. Adequately explain the steps you take and answer as much as you can (partial credit awarded).

Consider diffusion on a disk of unit radius. The mission is to solve the PDE,

$$u_t = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}, \quad u(r,\theta,0) = 0, \quad u(1,\theta,t) = f(\theta),$$

where $f(\theta)$ is an odd, $2\pi-$periodic function of angle. A difficulty is that the boundary condition at $r = 1$ makes the problem inhomogeneous. Therefore to solve the full initial-value problem, we must first find a steady-state solution that we can employ to homogenize the boundary condition.

(a) Using separation of variables, find the steady-state solution, $u = U(r, \theta)$, writing your answer as a Fourier series in angle. Compute the coefficients for the special case with

$$f(\theta) = \begin{cases} -1, & -\pi < \theta < 0 \\ 1, & 0 < \theta < \pi \end{cases} \qquad (*)$$

(b) For general $f(\theta)$, exploit the identity $\sum_{n=1}^{\infty} z^n = z/(1-z)$ to sum the series and write your steady-state in terms of a single integral. What is $U(r, 0)$? Briefly comment on the limit of your integral for $r \to 1$.

(c) Using your solution for $U(r, \theta)$ from part (a), set $u(r, \theta, t) = U(r, \theta) + v(r, \theta, t)$, and write down the problem for $v(r, \theta, t)$. Solve this problem to find the solution to the original initial-value problem.

(d) For the special case with $f(\theta)$ given by equation $(*)$, use the helpful results provided below to evaluate all the integrals in the coefficients of your series solution.

Helpful information:

The **Fourier Series:** of a periodic function $f(x)$ with period $2L$ is given by is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right],$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

For $m = 0, 1, 2, ...$, **Bessel's equation**

$$z^2 y'' + zy' + (z^2 - m^2)y = 0.$$

has a regular solution, $y(z) = J_m(z)$, with $J_m(z) \propto z^m$ for $z \to 0$, and a singular solution, $y(z) = Y_m(z)$. The Bessel functions of the first kind, $J_m(z)$, satisfy the relations,

$$\frac{d}{dz}(z^{m+1}J_{m+1}) = z^{m+1}J_m$$

$$\int_0^z z[J_m(z)]^2 dz = \frac{1}{2}z^2[J_m'(z)]^2 + \frac{1}{2}\left(z^2 - m^2\right)[J_m(z)]^2.$$

The **Sturm-Liouville ODE** is

$$[p(x)y']' + \lambda\sigma(x)y + q(x)y = 0, \qquad a < x < b,$$

with $\sigma(x) > 0$ and $p(x) > 0$. The associated expansion formula using the eigensolutions $\{\lambda_n, y_n(x)\}$ is

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \qquad c_n = \frac{\int_a^b f(x)y_n(x)\sigma(x)dx}{\int_a^b [y_n(x)]^2\sigma(x)dx}.$$

Helpful trig identities:

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B \qquad \& \qquad \sin(A \pm B) = \sin A\cos B \pm \cos A\sin B.$$

Midterm exam - solution

(a). (8 points) For the steady-state solution, we set $U = R(r)\Theta(\theta)$, plug in to the PDE, and find

$$-\frac{\Theta''}{\Theta} = \frac{(rR')'}{R}, \quad \text{or} \quad \Theta = -m^2\Theta, \quad r^2R'' + rR' - m^2R = 0,$$

where m^2 is a separation constant *(2 points).*

The solution for $\Theta(\theta)$ must be $2\pi-$periodic, and so $m = 0, 1, 2, ...$ with either a constant solution (for $m = 0$), or a $\sin m\theta$ or a $\cos m\theta$ (for $m > 0$). But the boundary condition states that $U(1, \theta) = f(\theta)$ is an odd, $2\pi-$periodic function. Therefore, all we need are the $\sin m\theta$ solutions with $m > 1$. The $R-$ equation has Euler form, with power-type solutions, $R \propto r^{\pm m}$. Discarding the singular solutions for $r \to 0$, we now end up with the general solution (a Fourier sine series),

$$U = \sum_{m=1}^{\infty} b_m r^m \sin m\theta. \qquad (4\ \textit{points}).$$

Using Fourier series theory, we know that

$$b_m = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin m\theta\, d\theta. \qquad (1\ \textit{point}).$$

For the special case, we find

$$b_m = \frac{2}{m\pi}[1 - (-1)^m] \qquad (1\ \textit{point}).$$

(b). (7 points) We rewrite the general solution above as

$$U = \frac{2}{\pi}\int_0^{\pi} f(\hat\theta)\sum_{m=1}^{\infty} r^m \sin m\hat\theta \sin m\theta\, d\hat\theta$$

$$= \frac{1}{2\pi}\int_0^{\pi} f(\hat\theta)\sum_{m=1}^{\infty} r^m[e^{im(\theta-\hat\theta)} + e^{-im(\theta-\hat\theta)} - e^{im(\theta+\hat\theta)} - e^{-im(\theta+\hat\theta)}]\, d\hat\theta$$

$$= \frac{1}{2\pi}\int_0^{\pi} f(\hat\theta)\left[\frac{re^{i(\theta-\hat\theta)}}{1 - re^{i(\theta-\hat\theta)}} + \frac{re^{-i(\theta-\hat\theta)}}{1 - re^{-i(\theta-\hat\theta)}} - \frac{re^{i(\theta+\hat\theta)}}{1 - re^{i(\theta+\hat\theta)}} - \frac{re^{-i(\theta+\hat\theta)}}{1 - re^{-i(\theta+\hat\theta)}}\right] d\hat\theta$$

$$= \frac{1 - r^2}{2\pi}\int_0^{\pi} f(\hat\theta)\left[\frac{1}{1 + r^2 - 2r\cos(\theta-\hat\theta)} - \frac{1}{1 + r^2 - 2r\cos(\theta+\hat\theta)}\right] d\hat\theta$$

(4 points). The final simplications are not necessary.

Plugging in $\theta = 0$ gives $U(r, 0) = 0$ *(1 point).*

If we set $r = 1$, then the integrand appears to vanish everywhere, even though the integral should equal $f(\theta)$. However, the integrand also becomes undefined for $\theta \to \hat\theta$ and $r \to 1$. Thus, in the limit, the integrand must provide a representation of the delta function $\delta(\theta - \hat\theta)$ (periodically extended as an odd, $2\pi-$periodic function) *(2 points).*

(c). (12 points) We now set $u(r, \theta, t) = U(r, \theta) + v(r, \theta, t)$, to find the problem,

$$v_t = \nabla^2 v, \quad v(r, \theta, 0) = -U(r, \theta), \quad v(1, \theta, t) = 0$$

(2 points). Separating variables once more, we set $u = R(r)\Theta(\theta)T(t)$, to find

$$T_t = -k^2 T, \qquad \Theta_{\theta\theta} = -m^2\Theta, \qquad r^2X_{rr} + rX_r + \left(k^2r^2 - m^2\right)X = 0,$$

for two more separation constants m^2 and k^2 *(2 points)*. Again, we pick $m = 1, 2, ...$ and $Y \propto \sin m\theta$. The solution in time is now $T \propto e^{-k^2 t}$. Finally, that in radius is $X \propto J_m(kr)$, on recognizing that we now have Bessel's equation (for $z = kr$) and after eliminating the singular solution at $r = 0$. The boundary condition $v(1, \theta, t) = 0$, or $X(1) = 0$, now implies that k must be a zero of $J_m(z)$. Denoting the j^{th} such zero by z_{mj}, we now arrive at the general solution,

$$v = \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} c_{mj} J_m(z_{mj} r) e^{-z_{mj}^2 t} \sin m\theta$$

(4 points). Because the $X-$problem here is a Sturm-Liouville problem, and in view of the solution in (a), we may write

$$c_{mj} = \frac{b_m \int_0^1 r^m J_m(z_{mj} r) r dr}{\int_0^1 [J_m(z_{mj} r)]^2 r dr}$$

(4 points).

(d). (4 points) For the special case b_m is given analytically in (b). Moreover, from the helpful information, we observe that

$$\int_0^1 r^{m+1} J_m(z_{mj} r) dr = z_{mj}^{-m-2} \int_0^{z_{mj}} z^{m+1} J_m(z) dz = z_{mj}^{-1} J_{m+1}(z_{mj})$$

and

$$\int_0^1 [J_m(z_{mj} r)]^2 r dr = z_{mj}^{-2} \int_0^{z_{mj}} [J_m(z)]^2 z dz = \frac{1}{2} [J_m'(z_{mj})]^2.$$

Hence,

$$c_{mj} = \frac{4[1 - (-1)^m] J_{m+1}(z_{mj})}{m\pi z_{mj} [J_m'(z_{mj})]^2}.$$