Math 256. Midterm exam.

No formula sheet, books or calculators! Include this answer sheet with your answer booklet to receive credit for part I!

Name:

Part I

Circle what you think is the correct answer. +3 for a correct answer, −1 for a wrong answer, 0 for no answer.

1. The integrating factor for the ODE $y' = 5x^4 y - e^{x^5}$ is

$(a)\quad e^{5x^4} \qquad (b)\quad e^{-5x^4} \qquad (c)\quad e^{x^5} \qquad (d)\quad e^{-x^5} \qquad (e)$ *None of the above.*

2. The solution to the ODE $y' = 5x^4 y - e^{x^5}$ is

$(a)\quad C + xe^{-x^4} \qquad (b)\quad (C+x)e^{x^4} \qquad (c)\quad (C-x)e^{x^5} \qquad (d)\quad (x-C)e^{-x^5}$

(e) *None of the above,*

where C is an arbitrary constant.

3. The ODE $y' = y^2 \tan x$, with $y(0) = 1$, has the solution,

$(a)\quad \ln(\sin x + e) \qquad (b)\quad (1+\tan x)^{-1} \qquad (c)\quad e^{1-\cos x}$

$(d)\quad (1+\ln\cos x)^{-1} \qquad (e)$ *None of the above.*

4. The particular solution to $y'' + 4y = 9\sin x$, is

$(a)\quad 3\sin x \qquad (b)\quad 3\cos x - 2\sin x \qquad (c)\quad 3\sin x - 2\cos x$

$(d)\quad 3\sin 2x \qquad (e)$ *None of the above.*

5. The ODE $y'' + 4y = 9\sin x$, has the homogeneous solutions,

$(a)\quad Ae^{2x} + Be^{-2x} \qquad (b)\quad Ae^{2x} + Bxe^{2x} \qquad (c)\quad A\cos 2x + B\sin x$

$(d)\quad A\cos(2x - B) \qquad (e)$ *None of the above,*

where A and B are arbitrary constants.

Part II

Answer in full (i.e. give as many arguments, explanations and steps as you think is needed for a normal person to understand your logic). Answer as much as you can; partial credit awarded.

1. *(8 marks)* (a) By treating the problem as separable, solve the first-order ODE

$$y' = \frac{1+y}{x-2}, \quad y(0) = 1.$$

(b) Now solve the problem again using an integrating factor.

Verify that your solution is correct by showing that it solves the ODE and matches starting condition.

2. *(8 marks)* Solve the ODE,

$$y'' + 2y' + 17y = e^{-x}\sin x.$$

Without solving the problem, indicate what trial particular solution you would have chosen had the right-hand side actually been $e^{-x}\sin 4x$.

Helpful trig identities:

$$\sin 0 = \sin\pi = 0, \quad \sin(\pi/2) = 1 = -\sin(3\pi/2),$$

$$\cos 0 = -\cos\pi = 1, \quad \cos(\pi/2) = \cos(3\pi/2) = 0,$$

$$\sin(-A) = -\sin A, \quad \cos(-A) = \cos A, \quad \sin^2 A + \cos^2 A = 1,$$

$$\sin(2A) = 2\sin A\cos A, \quad \sin(A+B) = \sin A\cos B + \cos A\sin B,$$

$$\cos(2A) = \cos^2 A - \sin^2 A, \quad \cos(A+B) = \cos A\cos B - \sin A\sin B,$$

$$\frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cos x = -\sin x.$$

Solutions

Part I:

1. Answer (d).

$$p = -5x^4, \quad \int p dx = -x^5, \quad I = e^{-x^5}$$

2. Answer (c).

$$y = \frac{C}{I} - \frac{1}{I} \int e^{x^5} I dx = (C - x)e^{x^5} \quad (-C \ arbitrary)$$

3. Answer (d).

$$\int \frac{dy}{y^2} = \int \frac{\sin x}{\cos x} \, dx \quad \rightarrow \quad \frac{1}{y} = C + \ln \cos x$$

But $y(0) = 1$ and so $C = 1$ and $y = (1 + \ln \cos x)^{-1}$.

4. Answer (a).

$$Try \ y_p = d \sin x \quad \rightarrow \quad 3d \sin x = 9 \sin x \quad \rightarrow \quad d = 3$$

5. Answer (d).

$$Aux. \ Eq. : \quad m^2 = -4 \quad \rightarrow \quad y_h = A \cos(2x - B) \quad (A, B \ arbitrary).$$

Part II:

1. Treating the ODE as separable:

$$\int \frac{dy}{1+y} = \int \frac{dx}{x-2} \quad \rightarrow \quad \ln|1+y| = A + \ln|x-2| \quad \rightarrow \quad y = C(x-2) - 1,$$

with $C = \pm e^A$.

Treating the ODE as linear: the integrating factor is

$$I = \exp - \int \frac{dx}{x-2} = (x-2)^{-1}.$$

Hence

$$y = \frac{C}{I} + \frac{1}{I}\int \frac{I dx}{x-2} = C(x-2) + (x-2)\int \frac{dx}{(x-2)^2} = C(x-2) - 1.$$

Last we apply the initial condition, $y(0) = 1$, which implies that $C = -1$ and $y = 1 - x$. Substituting this back into the ODE and IC gives $-1 = -1$ and $1 = 1$, as required.

2. For the homogeneous solutions, we solve the auxiliary equation,

$$m^2 + 2m + 17 = (m+1)^2 + 4^2 = 0 \quad \rightarrow \quad m = -1 \pm 2i$$

Hence the homogeneous solutions are

$$y_h = (A\cos 4x + B\sin 4x)e^{-x}.$$

As a trial particular solution we pose $y_p = (d_1 \sin x + d_2 \cos x)e^{-x}$. Plugging this into the ODE gives

$$-2(d_1\cos x - d_2 \sin x)e^{-x} + 2(d_1\cos x - d_2\sin x - d_1 \sin x - d_2\cos x)e^{-x} + 17(d_1\sin x + d_2\cos x)e^{-x} = e^{-x}\sin x,$$

which imply that $15d_1 = 1$ and $15d_2 = 0$. We then arrive at the general solution

$$y = (A\cos 4x + B\sin 4x)e^{-x} + \frac{1}{15}xe^{-x}.$$

Had the RHS been $e^{-x}\sin 4x$ (which is one of the homogeneous solutions), we should try

$$y_p = xe^{-x}(d_1\sin 4x + d_2\cos 4x).$$

(Although this is not needed, one can plug this into the ODE and match up terms, to find that $d_1 = 0$ and $d_2 = -\frac{1}{8}$.)