Math 256. Midterm exam.

No formula sheet, books or calculators! Include this answer sheet with your answer booklet to receive credit for part I!

Name:

Part I

Circle what you think is the correct answer. +3 for a correct answer, −1 for a wrong answer, 0 for no answer.

1. The integrating factor for the ODE $y' = 3x^2y + 2e^{x^3}$ is

$(a)\quad e^{3x^2} \qquad (b)\quad e^{-3x^2} \qquad (c)\quad e^{x^3} \qquad (d)\quad e^{-x^3} \qquad (e)\quad \textit{None of the above.}$

2. The solution to the ODE $y' = 3x^2y + 2e^{x^3}$ is

$(a)\quad C + 2xe^{-x^2} \qquad (b)\quad (C + 2x)e^{x^2} \qquad (c)\quad (C + 2x)e^{-x^3} \qquad (d)\quad (2x - C)e^{x^3}$

$(e)\quad \textit{None of the above,}$

where C is an arbitrary constant.

3. The ODE $y' = -y^2 \cos x$, with $y(0) = 1$, has the solution,

$(a)\quad \ln(\sin x + e) \qquad (b)\quad (1 + \sin x)^{-1} \qquad (c)\quad e^{\sin x}$

$(d)\quad \ln(\cos x) \qquad (e)\quad \textit{None of the above.}$

4. The particular solution to $y'' + 49y = 96 \sin x$, is

$(a)\quad 2 \sin x \qquad (b)\quad \cos x - 2 \sin x \qquad (c)\quad \sin x - 2 \cos x$

$(d)\quad \sin 2x \qquad (e)\quad \textit{None of the above.}$

5. The ODE $y'' + 49y = 96 \sin x$, has the homogeneous solutions,

$(a)\quad Ae^{7x} + Be^{-7x} \qquad (b)\quad Ae^{7x} + Bxe^{7x} \qquad (c)\quad A \cos 7x + B \sin x$

$(d)\quad A \cos(7x - B) \qquad (e)\quad \textit{None of the above,}$

where A and B are arbitrary constants.

Part II

Answer in full (i.e. give as many arguments, explanations and steps as you think is needed for a normal person to understand your logic). Answer as much as you can; partial credit awarded.

1. *(8 marks)* (a) By treating the problem as separable, solve the first-order ODE

$$y' = (x^2 - 1)(1 + y).$$

(b) Now solve the problem again using an integrating factor.

2. *(8 marks)* Solve the ODE,

$$y'' + 8y' + 17y = xe^{-4x}.$$

Without solving the problem, indicate what trial particular solution you would have chosen had the right-hand side actually been $e^{-4x}\sin x$.

Helpful trig identities:

$$\sin 0 = \sin\pi = 0, \quad \sin(\pi/2) = 1 = -\sin(3\pi/2),$$
$$\cos 0 = -\cos\pi = 1, \quad \cos(\pi/2) = \cos(3\pi/2) = 0,$$
$$\sin(-A) = -\sin A, \quad \cos(-A) = \cos A, \quad \sin^2 A + \cos^2 A = 1,$$
$$\sin(2A) = 2\sin A\cos A, \quad \sin(A+B) = \sin A\cos B + \cos A\sin B,$$
$$\cos(2A) = \cos^2 A - \sin^2 A, \quad \cos(A+B) = \cos A\cos B - \sin A\sin B,$$
$$\frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cos x = -\sin x.$$

Solutions

Part I:

1. Answer (d).

$$p = -3x^2, \quad \int p dx = -x^3, \quad I = e^{-x^3}$$

2. Answer (d).

$$y = -\frac{C}{I} + \frac{1}{I}\int 2e^{x^3} I dx = (2x - C)e^{x^3} \quad (-C\ arbitrary)$$

3. Answer (b).

$$-\int \frac{dy}{y^2} = \int \cos x\, dx \quad \to \quad \frac{1}{y} = C + \sin x$$

But $y(0) = 1$ and so $y = (1 + \sin x)^{-1}$.

4. Answer (a).

$$Try\ y_p = d\sin x \quad \to \quad 48d\sin x = 96\sin x \quad \to \quad d = 2$$

5. Answer (d).

$$Aux.\ Eq.: \quad m^2 = -49 \quad \to \quad y_h = A\cos(7x - B) \quad (-B\ arbitrary).$$

Part II:

1. Treating the ODE as separable:

$$\int \frac{dy}{1+y} = \int (x^2-1)dx \quad \to \quad \ln|1+y| = A + \frac{x^3}{3} - x \quad \to \quad y = C\exp\left(\frac{x^3}{3} - x\right) - 1,$$

with $C = \pm e^A$.

Treating the ODE as linear: the integrating factor is

$$I = \exp \int (1-x^2)dx = e^{x-x^3/3}.$$

Hence

$$y = \frac{C}{I} + \frac{1}{I}\int I(x^2-1)dx = Ce^{x-x^3/3} - e^{x-x^3/3}\int (1-x^2)e^{x-x^3/3}dx = Ce^{x-x^3/3} - e^{x-x^3/3}\int e^u du$$

with $u = x - x^3/3$. Hence

$$y = Ce^{x-x^3/3} - 1.$$

2. For the homogeneous solutions, we solve the auxiliary equation,

$$m^2 + 8m + 17 = (m+4)^2 + 1 = 0 \quad \to \quad m = -4 \pm i$$

Hence the homogeneous solutions are

$$y_h = (A\cos x + B\sin x)e^{-4x}.$$

As a trial particular solution we pose $y_p = (d_1 x + d_2)e^{-4x}$. Plugging this into the ODE gives

$$(16d_1x + 16d_2 - 8d_1)e^{-4x} + 8(d_1 - 4d_1x - 4d_2)e^{-4x} + 17(d_1x + d_2)e^{-4x} = xe^{-4x},$$

which imply that $d_1 = 1$ and $d_2 = 0$. We then arrive at the general solution

$$y = (A\cos x + B\sin x)e^{-4x} + xe^{-4x}.$$

Had the RHS been $e^{-4x}\sin x$ (which is one of the homogeneous solutions), we should try

$$y_p = xe^{-4x}(d_1\sin x + d_2\cos x).$$

(Although this is not needed, one can plug this into the ODE and match up terms, to find that $d_1 = 0$ and $d_2 = \frac{1}{2}$.)