Math 256. Midterm 2.

No formula sheet, books or calculators! Include this exam sheet with your answer booklet!

Name:

Part I

Circle what you think is the correct answer. +3 for a correct answer, -1 for a wrong answer, 0 for no answer.

1. The system

$$\mathbf{y}' = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \mathbf{y}$$

has the general solution,

(a) $\mathbf{u}_1 e^{t} + \mathbf{u}_2 e^{3t}$ (b) $\mathbf{u}_1 + \mathbf{u}_2 e^{4t}$ (c) $\mathbf{u}_1 + \mathbf{u}_2 e^{-4t}$ (d) $\mathbf{u}_1 e^{4t} + \mathbf{u}_2 e^{2t}$ (e) $\mathbf{u}_1 e^{3t} + \mathbf{u}_2 e^{-t}$

for two constant vectors $\mathbf{u}_1$ and $\mathbf{u}_2$.

2. If λ_1 and λ_2 are the eigenvalues of the constant matrix A, the particular solution to

$$\mathbf{x}'' = A\mathbf{x} + \mathbf{f}_0 \sin \omega t,$$

where ω is a constant and $\mathbf{f}_0$ a constant vector, is given by

(a) $-(\omega^2 I + A)^{-1} \mathbf{f}_0 \sin \omega t$ *for any* ω, (b) $-(\omega^2 I + A)^{-1} \mathbf{f}_0 \sin \omega t$ *for* $-\omega^2 \neq \lambda_1$ *or* λ_2,

(c) $-(\omega^2 I + A)^{-1} \mathbf{f}_0 \sin \omega t$ *for* $\omega \neq \lambda_1$ *or* λ_2, (d) $(\omega^2 I + A)^{-1} \mathbf{f}_0 \sin \omega t$ *for not all* ω,

(e) *None of the above.*

3. The Laplace transform of $2e^{-3t} \sin 3t$ is

(a) $\dfrac{2(s-3)}{(s-3)^2+9}$ (b) $\dfrac{2(s+3)}{(s+3)^2+9}$ (c) $\dfrac{6}{(s-3)^2+9}$ (d) $\dfrac{6}{s^2+6s+18}$

(e) *None of the above.*

4. The inverse Laplace transform of $(s-4)/[(s+1)(s^2+4)]$ is

(a) $e^{2t} - \cos t$ (b) $4e^{-t} - \cos t$ (c) $\cos 2t - e^{-t}$ (d) $e^{t} + \sin 2t$ (e) *None of the above.*

Part II

Answer in full (i.e. give as many arguments, explanations and steps as you think is needed for a normal person to understand your logic). Answer as much as you can; partial credit awarded.

1. *(10 points)* The positions of two atoms in a molecule, $x(t)$ and $y(t)$, satisfy the ODEs,

$$x'' = 2y - 5x, \quad y'' = 2x - 8y + \sin t.$$

Write these equations as a system and then find the general solution using the eigenvalues and eigenvectors of the constant matrix that appears in your system. Does the irradiance, modelled by the $\sin t$ forcing term, resonantly drive the molecular motions?

2. *(8 points)* Using Laplace transforms, solve the ODE

$$y'' + 6y' + 10y = e^{-3t}\cos t, \qquad y(0) = y'(0) = 1.$$

Useful Laplace Transforms

$$f(t) \quad \to \quad \bar{f}(s)$$

$$1 \quad \to \quad 1/s$$

$$t^n, \quad n = 0, 1, 2, ... \quad \to \quad n!/s^{n+1}$$

$$e^{at} \quad \to \quad 1/(s-a)$$

$$\sin at \quad \to \quad a/(s^2 + a^2)$$

$$\cos at \quad \to \quad s/(s^2 + a^2)$$

$$t \sin at \quad \to \quad 2as/(s^2 + a^2)^2$$

$$t \cos at \quad \to \quad (s^2 - a^2)/(s^2 + a^2)^2$$

$$y'(t) \quad \to \quad s\bar{y}(s) - y(0)$$

$$y''(t) \quad \to \quad s^2\bar{y}(s) - y'(0) - sy(0)$$

$$e^{at} f(t) \quad \to \quad \bar{f}(s-a)$$

$$f(t-a)H(t-a) \quad \to \quad e^{-as}\bar{f}(s)$$

Helpful trig identities:

$$\sin 0 = \sin \pi = 0, \quad \sin(\pi/2) = 1 = -\sin(3\pi/2),$$
$$\cos 0 = -\cos \pi = 1, \quad \cos(\pi/2) = \cos(3\pi/2) = 0,$$
$$\sin(-A) = -\sin A, \quad \cos(-A) = \cos A, \quad \sin^2 A + \cos^2 A = 1,$$
$$\sin(2A) = 2\sin A \cos A, \quad \sin(A+B) = \sin A \cos B + \cos A \sin B,$$
$$\cos(2A) = \cos^2 A - \sin^2 A, \quad \cos(A+B) = \cos A \cos B - \sin A \sin B,$$