Math 256. Midterm 2.

No formula sheet, books or calculators! Include this exam sheet with your answer booklet!

Name:

Part I

Circle what you think is the correct answer. +3 for a correct answer, −1 for a wrong answer, 0 for no answer.

1. The system

$$\mathbf{y}' = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \mathbf{y}$$

has the general solution,

(a) $\mathbf{u}_1 e^t + \mathbf{u}_2 e^{-t}$ (b) $\mathbf{u}_1 + \mathbf{u}_2 e^{4t}$ (c) $\mathbf{u}_1 + \mathbf{u}_2 e^{-4t}$ (d) $\mathbf{u}_1 e^{2t} + \mathbf{u}_2 e^{-2t}$ (e) $\mathbf{u}_1 e^t + \mathbf{u}_2 e^{-4t}$

for two constant vectors $\mathbf{u}_1$ and $\mathbf{u}_2$.

2. The constant matrix A has eigenvalues λ_1 and λ_2, and $\mathbf{f}_0$ is a constant vector. The particular solution to

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}_0 \cos t,$$

is given by

(a) $-(I + A)^{-1}\mathbf{f}_0 \cos t$, *provided* $\lambda_1 \neq -1$ & $\lambda_2 \neq -1$,

(b) $\mathbf{d}_1 \cos t + \mathbf{d}_2 \sin t$, *for two constant vectors* $\mathbf{d}_1$ & $\mathbf{d}_2$, *and all matrices* A,

(c) $\mathbf{d}_1 \cos t + \mathbf{d}_2 \sin t$, *for two constant vectors* $\mathbf{d}_1$ & $\mathbf{d}_2$, *and provided* $\det(A^2 + I) \neq 0$,

(d) $\mathbf{d}_1 \cos t + \mathbf{d}_2 \sin t$, *for two constant vectors* $\mathbf{d}_1$ & $\mathbf{d}_2$, *and provided* $\det(A^2 - I) \neq 0$,

(e) $\mathbf{d}_1 \cos t$, *for a constant vector* $\mathbf{d}_1$ *and provided* $\lambda_1 \neq 0$,

(f) *a pink elephant.*

3. The Laplace transform of the ODE

$$y'' - 6y' + 9y = e^{-t}, \quad y(0) = y'(0) = 0,$$

yields the $\bar{y}(s)$ given by

(a) $\dfrac{1}{(s^2 + 6s + 9)}$ (b) $\dfrac{1}{s(s^2 + 6s + 9)}$ (b) $\dfrac{1}{(s-3)(s+1)}$ (c) $\dfrac{1}{(s+3)^2(s-1)}$

(d) $\dfrac{1}{(s-3)^2(s+1)}$ (e) *None of the above.*

4. The inverse Laplace transform of $6/[(s^2 + 4)(s^2 + 1)]$ is

(a) $\cos 2t + \cos t$ (b) $2\cos 2t - \cos t$ (c) $2\sin t - \sin 2t$ (d) $\cos 2t + 2\sin t$

(e) *None of the above.*

Part II

Answer in full (i.e. give as many arguments, explanations and steps as you think is needed for a normal person to understand your logic). Answer as much as you can; partial credit awarded.

1. *(12 points)* The positions of two atoms in a molecule, $x(t)$ and $y(t)$, satisfy the ODEs,

$$2x'' = 3y - 5x + 2\sin 3t, \quad 2y'' = 3x - 5y + 2\sin 3t.$$

Write these equations as a system and then find the general solution using the eigenvalues and eigenvectors of the constant matrix that appears in your system. To avoid some algebra, you may write the particular solution formally using a matrix inverse. Do the $\sin 3t$ forcing terms (modelling irradiance) resonantly drive molecular motions?

2. *(10 points)* Using the definition of the Laplace transform, establish the first shifting theorem and show that $\mathcal{L}\{t^n\} = n!/s^{n+1}$, where n is an integer. Hence compute the Laplace transform of $t^n e^{at}$ where a is a constant. Using Laplace transforms, solve the ODE

$$y'' + 2y' + y = t^n e^{-t}, \qquad y(0) = y'(0) = 1.$$

Useful Laplace Transforms

$f(t)$	$\rightarrow$	$\bar{f}(s)$
1	$\rightarrow$	$1/s$
$t^n, \quad n = 0, 1, 2, ...$	$\rightarrow$	$n!/s^{n+1}$
e^{at}	$\rightarrow$	$1/(s-a)$
$\sin at$	$\rightarrow$	$a/(s^2+a^2)$
$\cos at$	$\rightarrow$	$s/(s^2+a^2)$
$t\sin at$	$\rightarrow$	$2as/(s^2+a^2)^2$
$t\cos at$	$\rightarrow$	$(s^2-a^2)/(s^2+a^2)^2$
$y'(t)$	$\rightarrow$	$s\bar{y}(s) - y(0)$
$y''(t)$	$\rightarrow$	$s^2\bar{y}(s) - y'(0) - sy(0)$
$e^{at}f(t)$	$\rightarrow$	$\bar{f}(s-a)$
$f(t-a)H(t-a)$	$\rightarrow$	$e^{-as}\bar{f}(s)$

Helpful trig identities:

$$\sin 0 = \sin\pi = 0, \quad \sin(\pi/2) = 1 = -\sin(3\pi/2), \quad \cos 0 = -\cos\pi = 1, \quad \cos(\pi/2) = \cos(3\pi/2) = 0,$$

$$\sin(-A) = -\sin A, \quad \cos(-A) = \cos A, \quad \sin^2 A + \cos^2 A = 1,$$

$$\sin(2A) = 2\sin A\cos A, \quad \sin(A+B) = \sin A\cos B + \cos A\sin B,$$

$$\cos(2A) = \cos^2 A - \sin^2 A, \quad \cos(A+B) = \cos A\cos B - \sin A\sin B,$$

Solutions:

Part I:

(1) The eigenvalues of the matrix are 0 and 4, so (b) solves the system.

(2) We need a trial particular solution of the form $\mathbf{d}_1 \cos t + \mathbf{d_2} \sin t$. Plugging in, we find $A\mathbf{d}_2 = -\mathbf{d}_1$ and $A\mathbf{d}_1 = \mathbf{d}_2 - \mathbf{f}_0$. Hence $\mathbf{d}_1 = -(A^2+I)^{-1}A\mathbf{f}_0$, $\mathbf{d}_2 = (A^2+I)^{-1}\mathbf{f}_0$, both of which can be found if $\det(A^2+I) \neq 0$. This is answer (c).

(3) We zap the ODE with the Laplace transform: $(s^2 - 6s + 9)\bar{y}(s) = (s+1)^{-1}$, implying answer (d).

(4) A quick partial fraction gives

$$\frac{6}{(s^2+4)(s^2+1)} = \frac{2}{s^2+1} - \frac{2}{s^2+4}$$

which imply answer (c).

Part II:

(1) The system is

$$\frac{d^2}{dt^2}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{5}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{5}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 3t.$$

(1 point)

To find the homogeneous solutions, we put $\mathbf{x} = \mathbf{v}e^{mt}$, to obtain $m^2\mathbf{v} = A\mathbf{v}$. Hence $m^2 = \lambda$, with λ being one of the eigenvalue of A and $\mathbf{v}$ being the corresponding eigenvector. (1 point)

The eigenvalues of the matrix A are given by $\det(A - \lambda I) = 0$, or

$$\left(\frac{5}{2} + \lambda\right)^2 - \frac{9}{4} = 0, \quad or \quad \lambda = \pm\frac{3}{2} - \frac{5}{2} = -1 \ or \ -4.$$

(2 points)

The corresponding eigenvectors follow from $(A - \lambda I)\mathbf{v} = \mathbf{0}$:

$$\lambda = -1, \quad \begin{pmatrix} -\frac{5}{2} - \lambda & \frac{3}{2} \\ \frac{3}{2} & -\frac{5}{2} - \lambda \end{pmatrix}\mathbf{v} = \frac{3}{2}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\mathbf{v} = 0, \quad or \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

and

$$\lambda = -4, \quad \begin{pmatrix} -\frac{5}{2} - \lambda & \frac{3}{2} \\ \frac{3}{2} & -\frac{5}{2} - \lambda \end{pmatrix}\mathbf{v} = \frac{3}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\mathbf{v} = 0, \quad or \quad \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(2 points)

Since $m = \pm i$ or $\pm 2i$, we therefore have

$$\mathbf{x}_h = (A_1 \cos t + B_1 \sin t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (A_1 \cos 2t + B_1 \sin 2t)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(2 points)

None of the homogeneous solutions are equal $\sin 3t$, so there is no resonance, and we can pose a trial particular solution with the form $\mathbf{d} \sin 3t$. Plugging in, we obtain

$$\mathbf{d} = -(A + 9I)^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(2 points)

The general solution is therefore

$$\mathbf{x} = (A_1 \cos t + B_1 \sin t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (A_1 \cos 2t + B_1 \sin 2t)\begin{pmatrix} 1 \\ -1 \end{pmatrix} - (A + 9I)^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 3t,$$

(1 point)

And as said already, there is no resonance. (1 point)

(2) For the identities, using the definition of the transform, we have

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n dt = -\frac{1}{s}[t^n e^{-st}]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1} dt$$

$$= -\frac{n}{s^2}[t^n e^{-st}]_0^\infty + \frac{n(n-1)}{s^2}\int_0^\infty e^{-st} t^{n-2} dt = \ldots = -\frac{n!}{s^n}[te^{-st}]_0^\infty + \frac{n!}{s^n}\int_0^\infty e^{-st} dt = \frac{n!}{s^{n+1}}$$

and

$$\mathcal{L}\{e^{at} y(t)\} = \int_0^\infty e^{-(s-a)t} y(t) dt \equiv \bar{f}(s-a).$$

(3 points)

For the function, from the table $\mathcal{L}\{t^n\} = n! s^{-n-1}$, and so

$$\mathcal{L}\{e^{at} t^n\} = \frac{n!}{(s-a)^{n+1}}.$$

(1 point)

We now zap the ODE:

$$(s^2 + 2s + 1)\bar{y} - s - 3 = \frac{n!}{(s+1)^{n+1}}, \quad or \quad \bar{y} = \frac{1}{(s+1)} + \frac{2}{(s+1)^2} + \frac{(n+2)!}{(n+1)(n+2)(s+1)^{n+3}}.$$

(3 points)

Undoing the transform:

$$y(t) = e^{-t} + 2te^{-t} + \frac{t^{n+2} e^{-t}}{(n+1)(n+2)}.$$

(3 points)