Math 256. Midterm 2.

No formula sheet, books or calculators! Include this exam sheet with your answer booklet!

Name:

Part I

Circle what you think is the correct answer. +3 for a correct answer, −1 for a wrong answer, 0 for no answer.

1. The system

$$\mathbf{y}' = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \mathbf{y}$$

has the general solution,

$(a)\quad \mathbf{u}_1 e^t + \mathbf{u}_2 e^{3t} \qquad (b)\quad \mathbf{u}_1 + \mathbf{u}_2 e^{7t} \qquad (c)\quad \mathbf{u}_1 t + \mathbf{u}_2 e^{6t} \qquad (d)\quad \mathbf{u}_1 e^{4t} + \mathbf{u}_2 e^{2t} \qquad (e)\quad \mathbf{u}_1 e^{3t} + \mathbf{u}_2 e^{-t}$

for two constant vectors $\mathbf{u}_1$ and $\mathbf{u}_2$.

2. For a constant matrix A and vector $\mathbf{f}_0$, the particular solution to

$$\mathbf{x}' = A\mathbf{x} - \mathbf{f}_0 e^t,$$

is given by

$(a)\quad (A-I)^{-1}\mathbf{f}_0 e^t \ \ always, \qquad (b)\quad (A-I)^{-1}\mathbf{f}_0 e^t \ \ sometimes,$

$(c)\quad (A+I)^{-1}\mathbf{f}_0 e^t \ \ if\ no\ eigenvalue\ of\ A\ is\ equal\ -1, \qquad (d)\quad (I+A)^{-1}\mathbf{f}_0 \sin t,$

$(e)\quad None\ of\ the\ above.$

3. The Laplace transform of $e^{-t}\sin 8t$ is

$(a)\quad \dfrac{8}{(s-1)^2+64} \qquad (b)\quad \dfrac{s-1}{(s-1)^2+64} \qquad (c)\quad \dfrac{8}{s^2+2s+65} \qquad (d)\quad \dfrac{s+1}{s^2+2s+65}$

$(e)\quad None\ of\ the\ above.$

4. The inverse Laplace transform of

$$\frac{4}{s^4-1}$$

is

$(a)\quad e^t - 2\sin t \qquad (b)\quad e^{-t} - 2\cos t \qquad (c)\quad \cos 2t - e^{-t} - e^t,$

$(d)\quad e^t - e^{-t} - 2\sin t \qquad (e)\quad None\ of\ the\ above.$

Part II

Answer in full (i.e. give as many arguments, explanations and steps as you think is needed for a normal person to understand your logic). Answer as much as you can; partial credit awarded.

1. *(10 points)* The positions of two masses, $x(t)$ and $y(t)$, satisfy the ODEs,

$$x'' + 2y + x = 0, \quad y'' + 3x + 6y = 0.$$

Write these equations as a system and then find the general solution using the eigenvalues and eigenvectors of the constant matrix that appears in your system. If the mass at position $x(t)$ is further driven by an additional force $f(t) = \sin t$, what is the particular solution to the system, and does resonance occur? For the particular solution, you may avoid any detailed algebra and quote the answer in terms of a matrix inverse.

2. *(8 points)* Using the definition of the Laplace transform, show that $\mathcal{L}\{e^{at}y(t)\} = \overline{y}(s-a)$, where a is a constant. Hence compute $\mathcal{L}\{t^n e^{-t}\}$, where n is an integer. Using Laplace transforms, solve the ODE

$$y'' + 2y' + y = t^4 e^{-t}, \qquad y(0) = 0, \qquad y'(0) = 1.$$

Useful Laplace Transforms

$f(t)$		$\bar{f}(s)$
1	$\rightarrow$	$1/s$
$t^n, \quad n = 0, 1, 2, ...$	$\rightarrow$	$n!/s^{n+1}$
e^{at}	$\rightarrow$	$1/(s-a)$
$\sin at$	$\rightarrow$	$a/(s^2+a^2)$
$\cos at$	$\rightarrow$	$s/(s^2+a^2)$
$t \sin at$	$\rightarrow$	$2as/(s^2+a^2)^2$
$t \cos at$	$\rightarrow$	$(s^2-a^2)/(s^2+a^2)^2$
$y'(t)$	$\rightarrow$	$s\overline{y}(s) - y(0)$
$y''(t)$	$\rightarrow$	$s^2\overline{y}(s) - y'(0) - sy(0)$
$e^{at}f(t)$	$\rightarrow$	$\bar{f}(s-a)$
$f(t-a)H(t-a)$	$\rightarrow$	$e^{-as}\bar{f}(s)$

Helpful trig identities:

$$\sin 0 = \sin \pi = 0, \quad \sin(\pi/2) = 1 = -\sin(3\pi/2),$$
$$\cos 0 = -\cos \pi = 1, \quad \cos(\pi/2) = \cos(3\pi/2) = 0,$$
$$\sin(-A) = -\sin A, \quad \cos(-A) = \cos A, \quad \sin^2 A + \cos^2 A = 1,$$
$$\sin(2A) = 2\sin A \cos A, \quad \sin(A+B) = \sin A \cos B + \cos A \sin B,$$
$$\cos(2A) = \cos^2 A - \sin^2 A, \quad \cos(A+B) = \cos A \cos B - \sin A \sin B,$$

Answers

Part I:

1. Eigenvalues:

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 6-\lambda \end{vmatrix} = \lambda(\lambda - 7)$$

so (b) works.

2. For the particular solution, we would try $\mathbf{d}e^t$ to obtain $\mathbf{d} = (a - I)^{-1}\mathbf{f}_0$, which is fine unless A has an eigenvalue of unity. So (b) works.

3. We know that $\mathcal{L}\{\sin 8t\} = 8/(s^2 + 64)$. Shifting: $\mathcal{L}\{e^{-t} \sin 8t\} = 8/[(s + 1)^2 + 64]$. Expanding the square gives (c).

4. Partial fractioning:

$$\frac{4}{(s^2+1)(s^2-1)} = \frac{2}{s^2-1} - \frac{2}{s^2+1} = \frac{1}{s-1} - \frac{1}{s+1} - \frac{2}{s^2+1}$$

Undoing the transform gives (d).

Part II:

1. Rewriting the ODEs in matrix/vector form:

$$\frac{d^2}{dt^2}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -3 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix has eigenvalues:

$$\begin{vmatrix} -1-\lambda & -2 \\ -3 & -6-\lambda \end{vmatrix} = \lambda(\lambda + 7)$$

For $\lambda = 0$:

$$\begin{pmatrix} -1 & -2 \\ -3 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

For $\lambda = -7$:

$$\begin{pmatrix} 6 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

The solutions of the systems are given by $\mathbf{x} = \mathbf{v}e^{mt}$ with $\mathbf{v}$ and m satisfying $m^2\mathbf{v} = A\mathbf{v}$ (so $m^2 = \lambda$ and $\mathbf{v}$ is the corresponding eigenvector). The general solution is therefore

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}(C\cos\sqrt{7}t + D\sin\sqrt{7}t) + \begin{pmatrix} 2 \\ -1 \end{pmatrix}(A + Bt)$$

(the zero eigenvalue requiring the additonal factor of t to arrive at the second independent solution). When the system is modified to

$$\ddot{\mathbf{x}} = A\mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin t,$$

the inhomogeneous term does not correspond to a homogeneous solution, and so we can get away with a trial particular solution of $\mathbf{x}_p = \mathbf{d}\sin t$ (there being no first derivatives). This gives

$$\mathbf{d} = (A + I)^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Because $\sin t$ does not correspond to a homogeneous solution, there is no resonance.

2. From the definition of the Laplace transform,

$$\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{-(s-a)t}f(t)dt = \overline{f}(s - a)$$

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n dt = [(-s)^{-1} e^{-st} t^n]_0^\infty + \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} dt = \frac{n}{s} \mathcal{L}\{t^{n-1}\}$$

Repeating the integration by parts n times therefore gives the result $\mathcal{L}\{t^n\} = n! s^{-n-1}$. If we now use the shifting theorem (with $a = -1$), we arrive at

$$\mathcal{L}\{e^{-t} t^n\} = \frac{n!}{(s+1)^{n+1}}.$$

Laplace transforming the ODE gives

$$(s^2 + 2s + 1)\overline{y} - 1 = \frac{4!}{(s+1)^5} \quad \rightarrow \quad \overline{y} = \frac{1}{(s+1)^2} + \frac{6!}{30(s+1)^7}$$

Hence

$$y(t) = te^{-t} + \frac{t^6 e^{-t}}{30}.$$