Math 400 - N. Balmforth (njbmath.ubc.ca, Math 229C)

Pre-amble

The course webpage: www.math.ubc.ca/˜njb/Math400.htm

Suggested text: R. Haberman, *Applied PDEs*

Assessment: coursework, midterm, final (breakdown to be determined)

Office hours: to be determined

Main topics:
- Separation of variables and eigenfunction expansions (§2, 4, 7)
- Fourier and Laplace Transforms (§10, 13)
- Method of characteristics (§12)

(chapters numbers refer to the 4^{th} edition of Haberman's book)

Definitions and notation

The essential problem:
- To find a function [$u(x, y)$ or $u(x, t)$ or $u(x, y, t)$ *etc.*] of multiple independent variables [(x, y) or (x, t) or (x, y, t) *etc.*]
- The differential equation: an algebraic relation between the independent variables, u and its derivatives:

$$\begin{cases} F(x, t, u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, ...) = 0, \quad \text{or} \\ F(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}, ...) = 0, \quad \text{or} \\ F(x, y, t, u, u_x, u_y, u_t, ...) = 0. \end{cases}$$

- Shorthand notation for derivatives: for $u(x, y)$,

$$u_x = \left.\frac{\partial u}{\partial x}\right|_y \equiv \frac{\partial u}{\partial x}, \quad u_y = \left.\frac{\partial u}{\partial y}\right|_x \equiv \frac{\partial u}{\partial y}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{yy} = \frac{\partial^2 u}{\partial y^2}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y},$$

- Boundary and/or initial conditions on the edges of the domain of the independent variables over which the PDE is to be solved

The independent variable t is usually a time-like quantity, requiring suitable initial conditions. The variables x, y *etc.* are typically space-like variables, demanding boundary data.

Organization

* Order The number of derivatives on the most differentiated term, for each independent variable

* Dimension: The number of independent variables

* Domain: The window of space and time for the solution

* Linear versus nonlinear: The PDE is linear if F is a linear function of u and its derivatives

e.g. The wave equation

$$\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{or} \qquad u_{tt} = C^2 u_{xx}$$

where the wavespeed C is a prescribed coefficient, is linear, second-order in space (x) and time (t), and has two dimensions (x and t). It is normally solved on the domain $t \geq 0$ and $a \leq x \leq b$, with initial conditions on $u(x,0)$ and $u_t(x,0)$, and boundary conditions at $x = a$ and $x = b$. Wave-like solutions include the travelling or standing waves,

$$u(x,t) = \sin(x - Ct) \qquad \text{or} \qquad u(x,t) = \sin x \sin Ct.$$

A more general solution is d'Alembert's:

$$u(x,t) = f(x - Ct) + g(x + Ct),$$

where $f(z)$ and $g(z)$ are arbitrary functions. All these solutions can be verified by plugging and chugging with the PDE. At $t = 0$, $u(x,0) = f(x) + g(x)$ and $u_t(x,0) = C[g'(x) - f'(x)]$, indicating that initial conditions on $u(x,0)$ and $u_t(x,0)$ are sufficient to pin down a unique solution (*i.e.* find $f(z)$ and $g(z)$). D'Alembert's solution is suitable for an infinite spatial domain $-\infty < x < \infty$, in which case the boundary conditions in x are not relevant at finite times.

The Korteweg de Vries equation, $u_t + uu_x = u_{xxx}$, is two-dimensional, first-order in time, third-order in space; it is nonlinear because of the uu_x term. At the initial moment $u(x,0)$ is normally provided, and boundary conditions are needed in x.

Linear PDEs have the property that knowns solutions can be linearly superposed to generate new solutions. *i.e.* if $u_1(x,y)$ and $u_2(x,y)$ satisfy the PDE, then so does $a_1u_1 + a_2u_2$ if a_1 and a_2 are arbitrary constants. Much of Math400 will be concerned with these linear PDEs; only when we consider the method of characteristics will we advance into the realm of nonlinear PDEs.

Classification

The general form of a linear, second-order, two-dimensional PDE for $u(x,y)$ is

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g.$$

The coefficients $(a, b, ..., g)$ may be functions of the independent variables x and y (but not u and its derivatives). The principal part of this PDE consists of the three second-order derivative terms:

$$au_{xx} + bu_{xy} + cu_{yy}$$

Introduce the replacements, $\partial/\partial x \to \xi_x$ and $\partial/\partial y \to \xi_y$:

$$(a\lambda^2 + b\lambda + c)\xi_y^2 u, \quad \lambda = \frac{\xi_x}{\xi_y}.$$

The PDE is classified according to the roots of the first factor: if the roots are real, $b^2 > 4ac$, the PDE is said to be *hyperbolic*; if the roots are complex and $b^2 < 4ac$, the PDE is *elliptic*. Finally, if $b^2 = 4ac$, and the roots are real and equal, the PDE is *parabolic*.

Laplace's equation or the Helmholtz equation,

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \qquad \text{or} \qquad \nabla^2 u = u_{xx} + u_{yy} = \rho(x,y)$$

where $\rho(x,y)$ is a prescribed function, bave $a = c = 1$ and $b = 1$, rendering them elliptic; the wave equation (with unit wavespeed)

$$u_{xx} = u_{tt}$$

has $a = 1 = -c$ and $b = 0$, indicating that it is hyperbolic (identifying t with y). The diffusion equation

$$u_t = \kappa u_{xx}$$

(the coefficient κ, or "diffusivity", is a physical constant), has $b = c = 0$, and so it is parabolic.

Classification has something to do with the geometry of the solution $u(x, y)$, which can be thought of as a surface above the $(x, y)-$plane. The PDEs of each of these classes share many common properties. Consequently, we can understand much about any linear, second-order, two-dimensional PDEs simply by looking at the canonical three examples given above: Laplace's equation, the wave equation and the diffusion equation.

Heat or diffusion equation

Consider a long thin rod along which heat diffuses. Let $T(x,t)$ denote the temperature at position x and time t. The heat content of the element $a \le x \le b$ is given by

$$\int_a^b \rho c_p T(x,t)\,\mathrm{d}x$$

where ρ and c_p are density and specific heat. This heat content changes according to the flux of heat in and out, and due to any sources or sinks. If $S(x,t)$ denotes the (line) density of any sources and sinks, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \rho c_p T(x,t)\,\mathrm{d}x = \mathit{flux\ in} - \mathit{flux\ out} + \int_a^b S(x,t)\,\mathrm{d}x.$$

According to Fourier's law, the flux is proportional to temperature gradient and heat flows from hot to cold: $\mathit{flux} = -k\frac{\partial T}{\partial x}$, where k is a physical constant. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b \rho c_p\,\mathrm{d}x = -\left[k\frac{\partial T}{\partial x}\right]_{x=a} + \left[k\frac{\partial T}{\partial x}\right]_{x=b} + \int_a^b S\,\mathrm{d}x \equiv \int_a^b \left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + S\right]\,\mathrm{d}x.$$

We may bring the time derivative inside the integral, in which case it should be interpreted as a partial derivative, holding x fixed. Then,

$$\int_a^b (\rho c_p T_t - k T_{xx} - S)\,\mathrm{d}x = 0.$$

But, because the endpoints a and b are arbitrary, this can only hold if the integrand itself vanishes eveywhere. Thus,

$$T_t = \kappa T_{xx} + s, \qquad \kappa = \frac{k}{\rho c_p}, \quad s = \frac{S}{\rho c_p}.$$

This must be solved subject to an initial temperature profile $T(x, 0)$, and boundary conditions, such as fixed temperature $T = T_0$ at (say) $x = 0$ and $x = L$.

As a last step, one can rescale x to resize the domain, rescale t to eliminate the coefficient κ (provided it is constant), and adjust the dependent variable $T(x,t)$ to remove the reference temperature and its physical units: $T = T_0 + T_0 u(x,t)$. The *dimensionless* mathematical problem that we end up with is then more compact and something like

$$u_t = u_{xx} + q(x,t), \quad u(x,0) = f(x), \quad 0 \le x \le \pi, \quad u(0,t) = u(\pi, 0) = 0,$$

where the (rescaled, dimensionless) initial profile $f(x)$ and source density $q(x,t)$ are prescribed, and we choose a scaled domain length of π because of the Fourier series theory to appear shortly. This is not to say that the coefficients of the original PDE are not important: they contain essential physical information regarding dimensional units, which is critical in any application of the mathematical analysis. But for the purposes of Math400, we will not be concerned about that, and focus on dimensionless mathematical problems.