## Laplace's equation for a disk

In circular polar coordinates, Laplace's equation $\nabla^2 u = 0$ takes the specific form,

$$\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} = 0, .$$

The solution $u(r,\theta)$ should be $2\pi-$periodic in angle $\theta$ if we deal with an entire disk, and be regular at the origin $r=0$. (We translate back to Cartesian coordinates using $x = r\cos\theta$ and $y = r\sin\theta$.) A typical Math400-style problem would be to supplement the PDE and these conditions with the additional boundary condition

$$u(1,\theta) = f(\theta)$$

for some prescribed boundary data $f(\theta)$, in the case that we can scale the disk's radius to unity. Because there is no further constraint on the $2\pi-$ periodic function $f(\theta)$, it has the full Fourier series,

$$f(\theta) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta),$$

with the usual integrals prescribing the coefficients:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\,\mathrm{d}\theta, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,\mathrm{d}\theta, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,\mathrm{d}\theta. \tag{1}$$

Let's solve the PDE using an eigenfunction expansion:

$$u(r,\theta) = \tfrac{1}{2}A_0(r) + \sum_{n=1}^{\infty}[A_n(r)\cos n\theta + B_n(r)\sin n\theta].$$

In other words, we represent the angle-dependence of the solution in terms of a Fourier series (no extensions are needed as the problem is already periodic!). Although we can avoid differentiating term-by-term by using projection, let's just plug and chug (after multiplying by $r^2$):

$$\tfrac{1}{2}r(rA_0')' + \sum_{n=1}^{\infty}[r(rA_n')'\cos n\theta + r(rB_n')'\sin n\theta] = \sum_{n=1}^{\infty} n^2(A_n\cos n\theta + B_n \sin n\theta),$$

prime again referring to derivatives with respect to argument (*i.e.* radial derivatives for coefficients of the Fourier series). Hence,

$$r(rA_0')' = 0, \qquad r(rA_n')' = n^2 A_n \qquad \& \qquad r(rB_n')' = n^2 B_n. \tag{2}$$

The ODEs for $A_n(r)$ and $B_n(r)$ are the same. In particular,

$$r^2 A_n'' + rA_n' - n^2 A_n = 0.$$

This is an *Euler equation*, in which the coefficients are not constant (ouch), but for each derivative, there is a corresponding factor of $r$. This means that we may find a general solution by posing $A_n = Cr^\alpha$, where $C$ and $\alpha$ are constants, which implies

$$[\alpha(\alpha-1) + \alpha - n^2]Cr^\alpha = (\alpha^2 - n^2)Cr^\alpha = 0$$

(each derivative lowers the power in $R(r)$ by one, but then each factor of $r$ in the adjacent coefficient restores the power back to what it was, ensuring that $Cr^\alpha$ works as a trial solution). Thus, for something non-trivial, $\alpha = \pm n$ but $C$ remains arbitrary, and so $A_n \propto r^{\pm n}$ and $B_n \propto r^{\pm n}$. However, we also need a regular solution for $r \to 0$, so we must jetison the negative powers. Hence, both $A_n$ and $B_n$ are proportional to $r^n$. The other coefficient $A_0$ can be found more directly: two integrals of the first equation in (2) furnishes

$$A_0(r) = \textit{constant} + \textit{constant} \times \ln r.$$

Again, though, we should delete one of these solutions, the logarithm, as it is bad for $r \to 0$. Thus, $A_0 = \textit{constant}$.

Finally, since $u(r, \theta)$ should match up with $f(\theta)$ for $r = 1$, we must have that $A_0(1) = a_0$, $A_n(1) = a_n$ and $B_n(1) = b_n$, and so

$$u(r, \theta) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta). \tag{3}$$

### Summing the series

Given the definitions of the coefficients of the Fourier series for $f(\theta)$ in (1), we have

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\hat{\theta})\, \mathrm{d}\hat{\theta} + \sum_{n=1}^{\infty} \frac{r^n}{\pi} \int_{-\pi}^{\pi} f(\hat{\theta})(\cos n\theta \cos n\hat{\theta} + \sin n\theta \sin n\hat{\theta}) \mathrm{d}\hat{\theta}. \tag{4}$$

Note that it is essential here to distinguish between the $\theta$ of the Fourier series in the solution (3) and the integration variable that appears in the coefficients in (1); we have accomplished that effortlessly in (4) by decorating the integration variable by a hat.

The formula for the solution in (4) establishes two things:

- first, it clearly exposes an issue with our solution strategies: both separation of variables (which we did not use, but follows a parallel path to that used above)[1] and the eigenfunction expansion develop the solution as an infinite series. Worse, all the coefficients must be computed from integrals. Even if we can compute those integrals explicitly for a given $f(\theta)$, the infinite series is hard to work with.
- second, it indicates that a clever use of trig relations might simplify things.

In particular, $\cos(A-B) = \cos A \cos B + \sin A \sin B$ and so the integral within the sum is equivalent to

$$\int_{-\pi}^{\pi} f(\hat{\theta}) \cos n(\theta - \hat{\theta}) \mathrm{d}\hat{\theta}. \tag{5}$$

Now we recall that $\cos A = \frac{1}{2}(e^{iA} + e^{-iA})$. Interchanging the sum and integral in (4), and using these helpful relations now furnishes

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\hat{\theta}) \left\{ 1 + \sum_{n=1}^{\infty} \left[ \left(re^{iA}\right)^n + \left(re^{-iA}\right)^n \right] \right\} \mathrm{d}\hat{\theta}, \qquad A = \theta - \hat{\theta}.$$

[1] We pose $u(r, \theta) = R(r)\Theta(\theta)$; the PDE can then be manipulated into

$$\frac{r(rR')'}{R} = -\frac{\Theta''}{\Theta}.$$

Since this is a function of $r$ on the left, but a function of $\theta$ on the right, both sides must equal a separation constant. At this stage, since we know that we will need $2\pi$−periodic functions as solutions for $\Theta(\theta)$, we set that separation constant to be 0 or $-n^2$, with $n = 1, 2, \ldots$. The first of these choices gives $\Theta = \textit{constant}$; the second gives $\Theta \propto \sin n\theta$ or $\cos n\theta$. The ODEs that we need to solve for $R(r)$ just correspond to those in (2), for the two choices of separation constant. Hence $R = \textit{constant}$ or $R \propto r^n$, using the same arguments. Assembling the general solution from all of this gives (3).

But

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{or} \qquad \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n.$$

Hence we can sum the series to write

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\hat\theta)\left[1 + \frac{re^{iA}}{1-re^{iA}} + \frac{re^{-iA}}{1-re^{-iA}}\right] \mathrm{d}\hat\theta = \frac{1-r^2}{2\pi}\int_{-\pi}^{\pi} \frac{f(\hat\theta)\,\mathrm{d}\hat\theta}{1+r^2-2r\cos(\theta-\hat\theta)}.$$

This is Poisson's solution to the Laplace equation for a unit disk. We have successfully reduced the infinite series with all its integrals for the coefficients to a single integral. "Wow" as Jurgen Klopp would say.

Poisson's solution takes the form of an integral of the form

$$u(r,\theta) = \int_{-\pi}^{\pi} G(r,\theta-\hat\theta) f(\hat\theta)\,\mathrm{d}\hat\theta.$$

The factor $G(r,\theta-\hat\theta)$ comes from monkeying around with the Fourier series in the manipulations we conducted above. That is equivalent to saying that it is a property of the detailed form of the PDE that we started with (*i.e.* Laplace's equation in circular polar coordinates). Once we have figured out this function, however, to get the solution of the PDE for any boundary function $f(\hat\theta)$, we just do the integral in $\hat\theta$. This idea goes well beyond Laplace's equation. In this sense, $G(r,\theta-\hat\theta)$ corresponds to a type of Green function for the PDE, and the solution above corresponds to expressing it as a "boundary integral." Such notions play a prominent role in Math 401. For Math400, however, the fact that we managed to sum the series and write the solution as a single integral is mainly "cool", but one wonders whether it can be done for other problems. In fact, the series solution can also be summed for the wave equation.

### Summing the series for the wave equation; d'Alembert again

Consider the problem:

$$u_{tt} = c^2 u_{xx}, \qquad 0 \le x \le \pi, \qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = f(x), \qquad u_t(x,0) = g(x).$$

We have left a wavespeed constant $c$ in the PDE here, for no particular reason other than one often sees it there before scaling it away (an exercise that we will refrain from here, for once).

We can solve the PDE either by separation of variables or an eigenfunction expansion. Separation of variables with $u(x,t) = X(t)T(t)$ gives

$$\frac{X''}{X} = \frac{T''}{c^2 T} = -n^2, \quad n = 1, 2, ...,$$

with a sensible, and by now very obvious, choice for the separation constant. Hence the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} (b_n \cos nct + B_n \sin nct) \sin nx. \tag{6}$$

The arbitrary constants, $b_n$ and $B_n$, need to be picked to satisfy the initial conditions. By taking odd, $2\pi-$periodic extensions, we may express those initial conditions in the form

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} f_n \sin nx \qquad \& \qquad u_t(x,0) = g(x) = \sum_{n=1}^{\infty} g_n \sin nx.$$

Hence, $b_n \equiv f_n$ and $B_n = g_n/(nc)$.

All this indicates that

$$u(x,t) = \sum_{n=1}^{\infty} \left( f_n \cos nct + \frac{g_n}{nc} \sin nct \right) \sin nx.$$

Again, this is an infinite series with coefficients given by integrals. For the first term in the sum, we use the trig relation, $\sin A \cos B = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$, to write

$$\frac{1}{2}\sum_{n=1}^{\infty} f_n \sin n(x+ct) + \frac{1}{2}\sum_{n=1}^{\infty} f_n \sin n(x-ct).$$

But the Fourier series itself indicates that

$$f(z) = \sum_{n=1}^{\infty} f_n \sin nz,$$

irrespectively of what $z$ actually is. Hence the first part of the solution is simply

$$\tfrac{1}{2}f(x+ct) + \tfrac{1}{2}f(x-ct).$$

The second part,

$$\sum_{n=1}^{\infty} \frac{g_n}{nc} \sin nct \sin nx \equiv \sum_{n=1}^{\infty} \frac{g_n}{2nc}[\cos n(x-ct) - \cos n(x+ct)]$$

(using $\sin A \sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$), is a bit more complicated because of the factor of $n$ in the denomintor. We deal with this term by first noting that

$$\int_{z_1}^{z_2} g(z)\,\mathrm{d}z = \sum_{n=1} g_n \int_{z_1}^{z_2} \sin nz\,\mathrm{d}z = \sum_{n=1} \frac{g_n}{n}(\cos nz_1 - \cos nz_2).$$

Thus,

$$u(x,t) = \tfrac{1}{2}f(x+ct) + \tfrac{1}{2}f(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,\mathrm{d}z. \tag{7}$$

But, because the right-hand side here can be simply thought of as the sum of a function of $x - ct$ and another function of $x + ct$, *i.e.* $u(x,t) = F(x-ct) + G(x+ct)$, we have arrived at d'Alembert's solution.

### More directly

To obtain d'Alembert's solution more directly, we note that we can change variables to solve $u_{tt} = c^2 u_{xx}$ much more easily: let $\xi = x - ct$ and $\eta = x + ct$. Changing variables, we then set $u(x,t) = U(\xi,\eta)$. The chain rule now implies that

$$\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi} \qquad \& \qquad \frac{\partial}{\partial t} \equiv c\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right).$$

Under the change of variable, the wave equation therefore becomes

$$c^2\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right)^2 U = c^2\left(\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi}\right)^2 U \qquad \longrightarrow \qquad 4c^2 U_{\xi\eta} = 0.$$

Thus, $U = F(\xi) + G(\eta)$. The initial conditions now demand that

$$F(x) + G(x) = f(x) \qquad \& \qquad -cF'(x) + cG'(x) = g(x), \quad \text{or} \quad F(x) - G(x) = -\frac{1}{c}\int g \, \mathrm{d}x + C,$$

for some arbitrary constant of integration $C$. Hence,

$$F(x) = \tfrac{1}{2}f(x) - \frac{1}{2c}\int g \, \mathrm{d}x + \tfrac{1}{2}C \qquad \& \qquad G(x) = \tfrac{1}{2}f(x) + \frac{1}{2c}\int g \, \mathrm{d}x - \tfrac{1}{2}C. \tag{8}$$

The final constants $\pm\frac{1}{2}C$ are irrelevant as we require the combo $F(x-ct) + G(x+ct)$, and we now arrive at (7).

Note that our derivation of d'Alembert's solution here pays no attention to the boundary conditions. In fact, it looks like we have a solution for an infinite line; the two pieces to the solution ($F(x-ct)$ and $G(x+ct)$) correspond to disturbances that propagate at fixed speed $c$ either to the right or left. The two pieces look free to travel off to $x \to \pm\infty$. Equation (8) merely indicates how the specific initial conditions split up into the two disturbances. This is illustrated in figure 1.

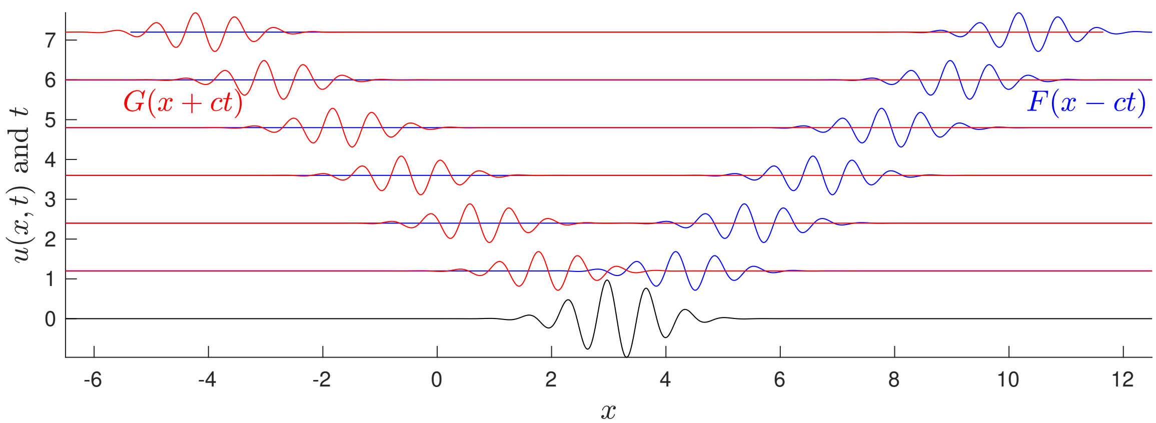


Figure 1: Graphical illustration of d'Alembert's solution without any boundaries.

However, our solution of the wave equation by separation of variables for the interval $0 \le x \le \pi$ also ended up giving d'Alembert's solution, once we suitably extended all the functions outside the original interval in such a way to satisfy the boundary conditions. All this leads to a physical interpretation of the odd, $2\pi-$periodic extension of the various functions of the original wave-equation problem. Key is an insight into how the boundary conditions can be taken care of by using the concept of images; *cf.* figure 2. But altogether this involves a lot of drawing that I will do on the blackboard. Figure 3 presents the general idea.

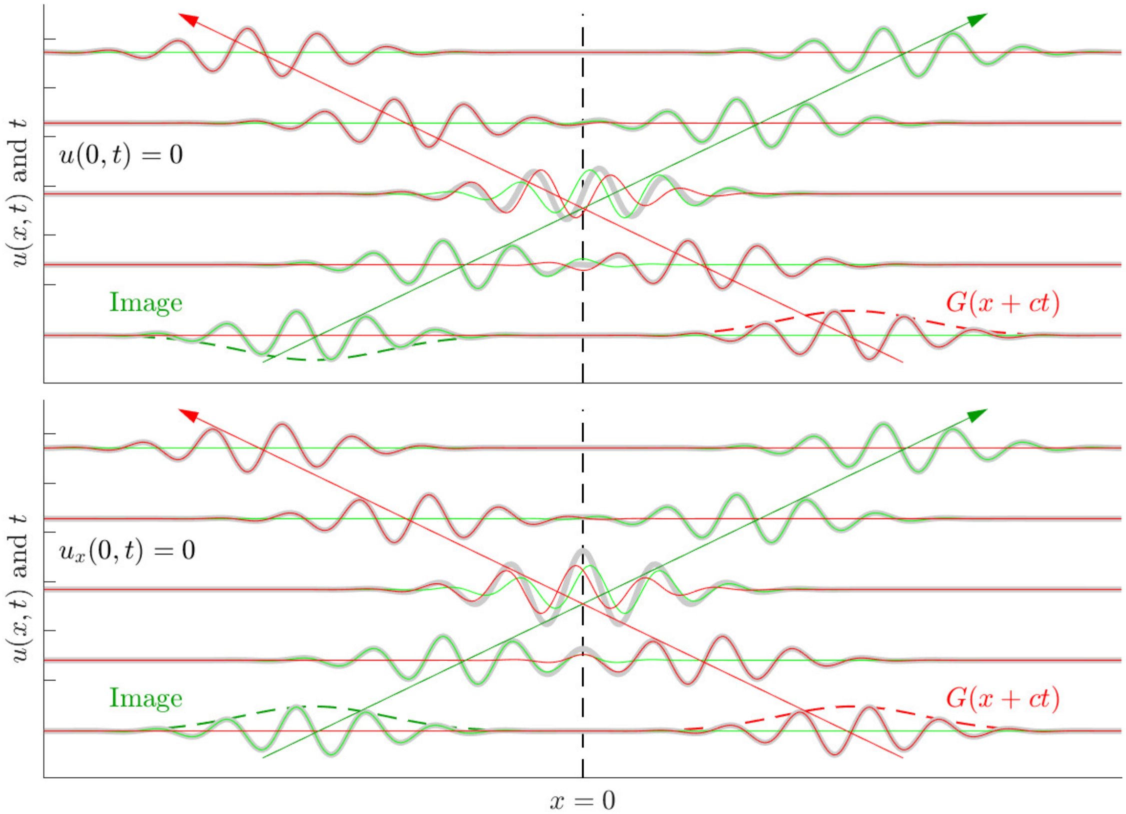


Figure 2: Image systems for Dirichlet and Neumann boundary conditions at $x = 0$. Red shows the solution $G(x+ct)$ incident from the left; green is the image which travels to the right; the combination of both, the actual solution, is the thick grey line (dashed lines indicate the original envelopes of the wave packets, with less wiggles). For $u(0,t) = 0$, there is an odd mirror image, leading to perfect destructive interference on the boundary. For $u_x(0,t) = 0$, there is an even mirror image, leading to perfect constructive interference on the boundary.

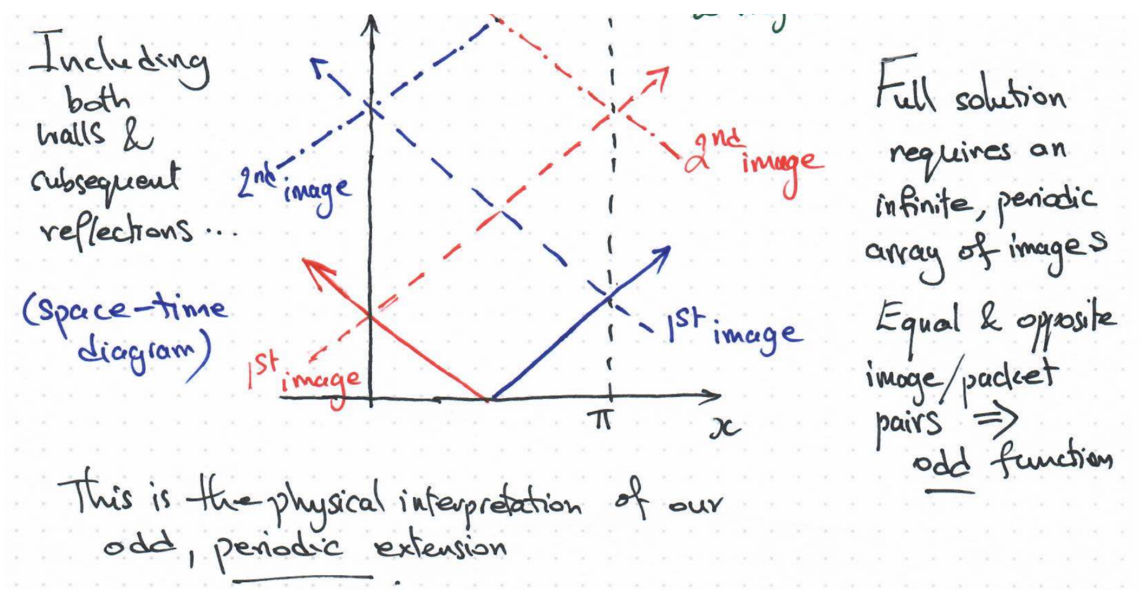


Figure 3: Image systems for Dirichlet and Neumann boundary conditions at $x = 0$ and $x = \pi$.