Systems :

$$\frac{d}{dt}\underset{\sim}{x} = A\underset{\sim}{x} \quad \text{Homog.} \qquad \left(\text{OR} \quad \frac{d^2}{dt^2}\underset{\sim}{x} = A\underset{\sim}{x}\right)$$

$$\frac{d}{dt}\underset{\sim}{x} = A\underset{\sim}{x} + \underset{\sim}{f}(t) \qquad \text{Inhomog.}$$

Pose $\underset{\sim}{x} = \underset{\sim}{c}e^{mt} \rightarrow$ $\underset{\sim}{c}$ is an eigenvector, m given by eigenvalue } of A

$(A\underset{\sim}{e} = \lambda\underset{\sim}{e}) \rightarrow$ gives the homog. sols.

For a particular solution, use a trial solution based on the form of $\underset{\sim}{f}(t)$.

eg. $\frac{d}{dt}\underset{\sim}{x} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}\underset{\sim}{x}$, $\underset{\sim}{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 2$, $\underset{\sim}{e} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\lambda = 3$, $\underset{\sim}{e} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Gen. Sol. $\underset{\sim}{x} = A_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{2t} + A_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{3t}$

Then $A_1 + A_2 = 1$, $-A_1 - 2A_2 = 1$ $\Rightarrow A_1 = 3, A_2 = -2$, $\underset{\sim}{x} = \begin{pmatrix} 3e^{2t} - 2e^{3t} \\ -3e^{2t} + 4e^{3t} \end{pmatrix}$

eg. $\frac{d}{dt}\underset{\sim}{x} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\underset{\sim}{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}t$

Homog. sols. $A_1\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-t} + A_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{3t}$

Part. sol. try $\underset{\sim}{x}_p = t\underset{\sim}{a} + \underset{\sim}{b}$ $\Rightarrow$ $\underset{\sim}{a} = A(t\underset{\sim}{a} + \underset{\sim}{b}) + \begin{pmatrix} 0 \\ t \end{pmatrix}$

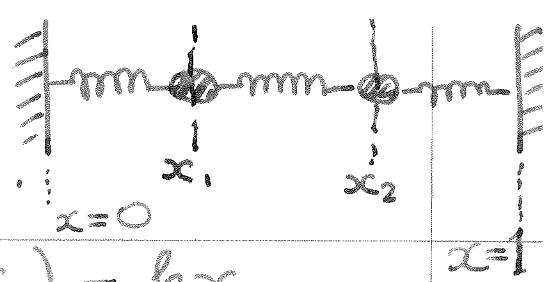
$\therefore \underset{\sim}{a} = A\underset{\sim}{b}$ and $A\underset{\sim}{a} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Solve for $\underset{\sim}{a}$, then $\underset{\sim}{b}$: $\underset{\sim}{x}_p = \frac{t}{3}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{1}{9}\begin{pmatrix} 2 \\ -5 \end{pmatrix}$

* apply any ICs after finding $\underset{\sim}{x}_p$!

Coupled Mass-spring system

Spring const. k, masses m.



Newton:
$$m\ddot{x}_1 = k(x_2 - x_1) - kx_1$$
$$m\ddot{x}_2 = k(1 - x_2) - k(x_2 - x_1)$$

$\omega^2 = k/m$, $\quad x_1 = \frac{1}{3} + y_1$, $\quad x_2 = \frac{2}{3} + y_2$

$$\rightarrow \frac{d^2}{dt^2}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -2\omega^2 & \omega^2 \\ \omega^2 & -2\omega^2 \end{pmatrix}}_{A}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Eigenvalues/vectors: $\lambda = -\omega^2$, $\underline{e} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = -3\omega^2$, $\underline{e} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\underset{\sim}{y} = \underset{\sim}{c}\, e^{mt} \implies m^2 = \lambda \implies m = \pm i\omega$ or $\pm\sqrt{3}i\omega$

Gen. Sol. $\underset{\sim}{y} = \left(A_1 e^{i\omega t} + A_2 e^{-i\omega t}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left(B_1 e^{\sqrt{3}i\omega t} + B_2 e^{-\sqrt{3}i\omega t}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

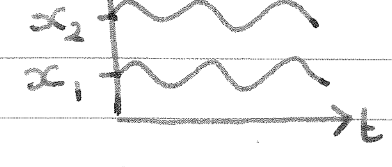
→ complex form

$= R_1 \cos(\omega t + \theta_1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + R_2 \cos(\sqrt{3}\omega t + \theta_2)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

→ real form.

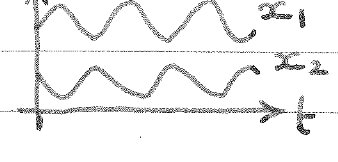
The two sols. are the "normal modes"

$\underline{\lambda = -\omega^2}$



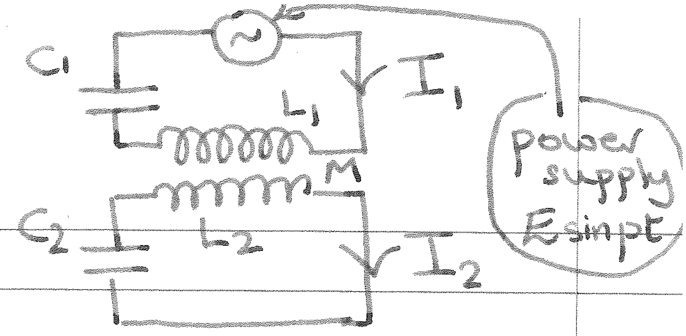
$y_1 = y_2$; <u>in phase</u>.

$\underline{\lambda = -3\omega^2}$



$y_1 = -y_2$; <u>out of phase</u>.

Coupled electrical circuits



$$\boxed{\begin{aligned} L_1\frac{d^2I_1}{dt^2} + M\frac{d^2I_2}{dt^2} + \frac{I_1}{C_1} &= Ep\cos pt \\ L_2\frac{d^2I_2}{dt^2} + M\frac{d^2I_1}{dt^2} + \frac{I_2}{C_2} &= 0 \end{aligned}}$$

$y_1 = L_1I_1 + MI_2$

$y_2 = L_2I_2 + MI_1$

$\omega_1^2 = \left[C_1L_1\left(1 - M^2/L_1L_2\right)\right]^{-1}$

$\omega_2^2 = \left[C_2L_2\left(1 - M^2/L_1L_2\right)\right]^{-1}$

$\alpha_1 = \frac{M}{L_1}, \quad \alpha_2 = \frac{M}{L_2}$

$$\frac{d^2}{dt^2}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} Ep\cos pt \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} -\omega_1^2 & \alpha_1\omega_1^2 \\ \alpha_2\omega_2^2 & -\omega_2^2 \end{pmatrix}$$

eg. $p=1$, $E=1$, $\omega_1^2=\omega_2^2=3$, $\alpha_1=\alpha_2=\frac{1}{3}$

$$\ddot{\underset{\sim}{y}} = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\underset{\sim}{y} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos t$$

$-\lambda = 2$, $\underline{e} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ & $-\lambda = 4$, $\underline{e} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Homog. Sols. $\underset{\sim}{y} = R_1\cos(\sqrt{2}\,t + \theta_1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + R_2\cos(2t + \theta_2)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Partic. Sol. : try $y_p = \underset{\sim}{a}\cos t$

$$\therefore \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}\underset{\sim}{a} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\underset{\sim}{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\rightarrow \underset{\sim}{a} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$

$$\therefore \underset{\sim}{y} = R_1\cos(\sqrt{2}\,t + \theta_1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + R_2\cos(2t + \theta_2)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\cos t$$