Systems (Review) $\quad \frac{d}{dt}\mathbf{x} = A\mathbf{x} + \mathbf{f}(t)$

- n^{th}-order ODE converted to a system with $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x \\ x' \\ \vdots \\ x^{(n-1)} \end{pmatrix}$
- Homog. sols given by eigenvalues/vectors of A.
- Part sols. can be found by posing trial sol.

eg. $\mathbf{f} = \mathbf{F}e^{rt}$

Use $\mathbf{x} = \mathbf{a}e^{rt} \longrightarrow \quad r\mathbf{a} = A\mathbf{a} + \mathbf{F}$

eg. $\mathbf{f} = \mathbf{F}_0 + t\mathbf{F}_1 + \ldots + t^N\mathbf{F}_N$

Use $\mathbf{x} = \mathbf{a}_0 + t\mathbf{a}_1 + \ldots + t^N\mathbf{a}_N \longrightarrow$

$$\mathbf{a}_1 + 2t\mathbf{a}_2 + \ldots + Nt^{N-1}\mathbf{a}_N = A(\mathbf{a}_0 + t\mathbf{a}_1 + \ldots t^N\mathbf{a}_N) + \mathbf{F}_0 + t\mathbf{F}_1 + \ldots t^N\mathbf{F}_N$$

match up powers:

$$\left.\begin{aligned} \mathbf{0} &= A\mathbf{a}_N + \mathbf{F}_N \\ N\mathbf{a}_N &= A\mathbf{a}_{N-1} + \mathbf{F}_{N-1} \\ &\vdots \\ 2\mathbf{a}_2 &= A\mathbf{a}_1 + \mathbf{F}_1 \end{aligned}\right\} \downarrow \quad \text{solve for } \mathbf{a}_N \text{ then } \mathbf{a}_{N-1} \ldots \ldots \mathbf{a}_0$$

$$\mathbf{a}_1 = A\mathbf{a}_0 + \mathbf{F}_0$$

eg. $\mathbf{f} = \mathbf{F}\cos\omega t \quad$ (or $\sin\omega t$)

Use $\mathbf{x} = \mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t \rightarrow$

$$-\omega\mathbf{a}\sin\omega t + \omega\mathbf{b}\cos\omega t = A\mathbf{a}\cos\omega t + A\mathbf{b}\sin\omega t + \mathbf{F}\cos\omega t$$

match up sin's & cos's ...

$$-\omega\mathbf{a} = A\mathbf{b}$$
$$\omega\mathbf{b} = A\mathbf{a} + \mathbf{F}$$

If $d^2\mathbf{x}/dt^2 = A\mathbf{x} + \mathbf{F}\cos\omega t$, then $\mathbf{x} = \mathbf{a}\cos\omega t$ suffices.

The linear systems remain solvable, but invertibility problems arise when $\mathbf{f}(t)$ is a homogeneous solution. Extra factors of t may then be needed.

Laplace Transforms

Converts a differential eq. to an algebraic problem for the transform of the unknown funk.

- No need to pose a trial sol.
- No need to split up sols. into homog. & particular
- Initial conditions imposed immediately.

* One must invert the transform.

$$\mathcal{L}\{y(t)\} = \bar{y}(s) = \int_0^\infty e^{-st} y(t)\, dt$$

eg. $\mathcal{L}\{1\} = \frac{1}{s}$

$$\mathcal{L}\{e^{mt}\} = \frac{1}{s-m} \quad \text{provided Real}(s) > m.$$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = s\bar{y}(s) - y(0)$$

$$\mathcal{L}\left\{d^2y/dt^2\right\} = s^2\bar{y}(s) - sy(0) - y'(0)$$

Transform of an ODE: $ay'' + by' + cy = f(t)$

$$\longrightarrow \quad \bar{y}(s) = \frac{\bar{f}(s) + (as+b)y(0) + ay'(0)}{as^2 + bs + c}$$

Strategy:
- compute $\bar{f}(s)$ from $f(t)$
- use initial conditions to obtain $\bar{y}(s)$
- invert $\bar{y}(s)$ to find $y(t)$.

Formally,

$$y(t) = \mathcal{L}^{-1}\{\bar{y}(s)\} = \frac{1}{2\pi i}\int_{\mathcal{C}} e^{st}\, \bar{y}(s)\, ds$$

$\mathcal{C}$ = Bromwich contour.

This is too difficult to use; exploit a table of inverses instead.

Table

$y(t)$	$\bar{y}(s)$
1	$1/s$
e^{mt}	$1/s-m$
t^n	$n!/s^{n+1}$

and more ...

eg. $y'' - 2y' - 3y = 0 \,, \quad y(0)=0, y'(0)=1$

$$\bar{y} = \frac{1}{(s+1)(s-3)} = \left(\frac{1}{s-3} - \frac{1}{s+1}\right)\frac{1}{4}$$

$$\therefore \; y = \frac{1}{4}\left(e^{3t} - e^{-t}\right)$$

In view of the definition of $\mathcal{L}\{y(t)\}$,

$$\mathcal{L}\{Af(t) + Bg(t)\} = A\,\mathcal{L}\{f(t)\} + B\,\mathcal{L}\{g(t)\}$$

(LINEARITY)

$$= A\bar{f}(s) + B\bar{g}(s)$$

More transforms:

$$y(t) = \sin\omega t \,, \quad \bar{y}(s) = \frac{\omega}{\omega^2+s^2}$$

$$y(t) = \cos\omega t \,, \quad \bar{y}(s) = \frac{s}{\omega^2+s^2}$$

eg. $y'' + y = 0 \,, \quad y(0) = 0 \,, \; y'(0) = 1$

$$\bar{y}(s) = \frac{1}{s^2+1} \Rightarrow y(t) = \sin t$$

$-5t$

eg. $y' + 5y = 2 \,, \; y(0) = 1 \longrightarrow \bar{y} = \frac{2+s}{s(5+s)} \longrightarrow y = \frac{2}{5} + \frac{3}{5}e^{-5t}$