Helpful tools for inverting Laplace transforms:

- partial fractions
- completing a square; 1st shifting thm.

eg. $$\bar{y}(s) = \frac{6+5s}{s(s^2+4)} = \frac{3}{2s} - \frac{3s}{2(s^2+4)} + \frac{5}{s^2+4}$$

$\therefore$ $$y'' + 4y = 6, \quad y(0)=0, \; y'(0)=5$$

has solution $$y(t) = \frac{3}{2} - \frac{3}{2}\cos 2t + \frac{5}{2}\sin 2t$$

1st shifting theorem: $$\mathcal{L}\{y\} = \int_0^\infty e^{-st} y(t)\,dt = \bar{y}(s)$$

$$\mathcal{L}\{e^{at}y\} = \int_0^\infty e^{-(s-a)t} y(t)\,dt = \bar{y}(s-a)$$

eg. $$\frac{3}{s^2-4s+13} \equiv \frac{3}{(s-2)^2+3^2}.$$ But $$\mathcal{L}\{\sin\omega t\} = \frac{\omega}{\omega^2+s^2}$$

so

$$= \mathcal{L}\{e^{2t}\sin 3t\}$$

$\therefore$ $y = e^{2t}\sin 3t$ solves

$$y'' - 4y' + 13y = 0, \quad y'(0)=3, \; y(0)=0.$$

eg. $$y'' + y' - 2y = 9e^t, \quad y(0)=3, \; y'(0)=0$$

$$\rightarrow \bar{y}(s) = \frac{3s^2+6}{(s+2)(s-1)^2} = \frac{2}{s+2} + \frac{1}{s-1} + \frac{3}{(s-1)^2}$$

But $$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{t\} = \frac{1}{s^2}$$

$$\therefore \mathcal{L}\{e^t\} = \frac{1}{s-1}, \quad \mathcal{L}\{te^t\} = \frac{1}{(s-1)^2}$$

$$\therefore y(t) = 2e^{-2t} + e^t + 3te^t$$

Useful transforms: $$\mathcal{L}\{e^{\alpha t}\cos\beta t\} = \frac{s-\alpha}{(s-\alpha)^2+\beta^2}, \quad \mathcal{L}\{e^{\alpha t}\sin\beta t\} = \frac{\beta}{(s-\alpha)^2+\beta^2}$$

Second shifting theorem

Heaviside step funk: $H(t) = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases}$

or $H(t-t_0) = \begin{cases} 0, & t<t_0 \\ 1, & t>t_0 \end{cases}$

From def. $\mathcal{L}\{f(t-t_0)H(t-t_0)\} = e^{-st_0}\bar{f}(s)$

eg. $\mathcal{L}^{-1}\left\{\frac{e^{-sa}}{s}\right\} = H(t-a)$

$\mathcal{L}^{-1}\left\{\frac{\omega e^{-sa}}{s^2+\omega^2}\right\} = H(t-a)\sin\omega(t-a)$

eg. $y''+y = H(t)-H(t-1)$, $y(0)=y'(0)=0$

$$\bar{y}(s) = \frac{1}{s} - \frac{s}{s^2+1} - e^{-s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$$

$$\therefore y(t) = t - \cos t - H(t-1)\left[1-\cos(t-1)\right]$$

Delta-function: $\int_{-\infty}^{\infty} \delta(t-t_0)F(t)\,dt = F(t_0)$

$\rightarrow \mathcal{L}\{\delta(t)\} = 1, \quad \mathcal{L}^{-1}\{1\} = \delta(t)$

Note that $\int_{-\infty}^{A} \delta(t-t_0)\,dt = \begin{cases} 0 & \text{if } A<t_0 \\ 1 & \text{if } A>t_0 \end{cases}$

$\equiv H(A-t_0)$

$\therefore$ the derivative of a step is a δ-function