Convolutions and transfer functions

eg. $y'' + 4y = g(t)$, $y(0)=3$, $y'(0)=-1$

$$\rightarrow \bar{y}(s) = \frac{3s-1+\bar{g}(s)}{s^2+4}$$

$$\therefore y(t) = 3\cos 2t - \frac{1}{2}\sin 2t + \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{2}\frac{2}{s^2+4}\bar{g}(s)\right\}}_{\text{what is this more usefully?}}$$

Convolution integral : "$f * g$" $= \int_0^t f(t-\tau)g(\tau)\,d\tau$

Lap. transf. :
$$\mathcal{L}\{f*g\} = \bar{f}(s)\bar{g}(s)$$

Proof: $\mathcal{L}\{f*g\} = \int_0^\infty e^{-st}\left[\int_0^t f(t-\tau)g(\tau)\,d\tau\right]dt$

interchange order of integrals
$$= \int_0^\infty g(\tau)\left[\int_\tau^\infty f(t-\tau)e^{-st}\,dt\right]d\tau$$

put $u=t-\tau$, $du=dt$

$$= \int_0^\infty g(\tau)\int_0^\infty f(u)e^{-su}e^{-s\tau}\,du\,d\tau$$

$$= \underline{\bar{f}(s)\bar{g}(s)}$$

$$\therefore y(t) = 3\cos 2t - \frac{1}{2}\sin 2t + \int_0^t \underbrace{\frac{1}{2}\sin 2(t-\tau)}_{\text{"Transfer funk."}}\, g(\tau)\,d\tau$$

Thus the LT reduces any inhomog. problem to an integral over the transfer funk & inhomog. term.