KEY

1. C
2. D
3. C
4. A
5. E (any!)

1. $xy' + y^2 = 4, \quad y(1) = 0$

$$y' = \frac{4-y^2}{x} \qquad \int \frac{dy}{(2-y)(2+y)} = \hat{C} + \ln|x|$$

$$\frac{1}{4} \ln\left|\frac{y+2}{2-y}\right| = \hat{C} + \ln|x|$$

$$\longrightarrow \quad \frac{y+2}{2-y} = Ax^4 \qquad \& \quad y(1)=0 \Rightarrow A = 1$$

$$y+2 = (2-y)x^4$$

$$y = \frac{2(x^4-1)}{x^4+1}.$$

2. Eigenvalues: $\lambda = 0$ & $(\lambda+2)(\lambda+1) - 6 = 0$

$$\lambda^2 + 3\lambda - 4 = 0$$

$$(\lambda-1)(\lambda+4) = 0$$

$0, 1, -4$

3. $$\frac{-8}{s(s^2-4)} = \frac{-8}{4}\left[\frac{-1}{s} + \frac{s}{s^2-4}\right] = \frac{2}{s} - \frac{2s}{s^2-4}$$

$$= \frac{2}{s} - \left(\frac{1}{s-2} + \frac{1}{s+2}\right) \longrightarrow 2 - e^{2t} - e^{-2t}$$

4. $\sin(x-t)$!

II

1. $q'' + 2q' + 10q = 0$

$m^2 + 2m + 10 = (m+1)^2 + 9 \quad \therefore \quad m = -1 \pm 3i$

$\therefore \quad q = \frac{1}{3} A e^{-t} \sin 3t \qquad [q(0) = 0, \; q'(0) = A]$ 5

$d' + d = \frac{1}{3} A e^{-t} \sin 3t$

$I \equiv e^t$ int. fac.

$(e^t d)' = \frac{1}{3} A \sin 3t$ 2

$e^t d = -\frac{1}{3^2} A \cos 3t + \frac{1}{3^2} A$ 2 since $d(0) = 0$.

$d = \frac{A}{9} e^{-t} (1 - \cos 3t)$ 5

$d(\pi) = \frac{2A}{9} e^{-\pi} \quad \therefore \quad A > \frac{9e^{\pi}}{2}$. 2

2. $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t$ 1

E-values:

$(3-\lambda)^2 - 1 = 0 \rightarrow \lambda^2 - 6\lambda + 8 = 0$

$(\lambda - 4)(\lambda - 2) = 0 \qquad \lambda = 4 \text{ or } 2$ 1

$\left.\begin{array}{l} \lambda = 2: \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \underset{\sim}{v} = 0 \rightarrow \underset{\sim}{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \lambda = 4: \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \underset{\sim}{v} = 0 \rightarrow \underset{\sim}{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{array}\right\}$ e-vectors. 2

With $\begin{pmatrix} x \\ y \end{pmatrix} = \underset{\sim}{v} e^{mt}$ for hom. sols. $\Rightarrow m = \lambda = 2 \text{ or } 4$ 1 5

Part. Sol. try $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} e^t \Rightarrow$ 1

$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 3\alpha + \beta \\ \alpha + 3\beta \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \left.\begin{array}{l} 2\alpha + \beta + 1 = 0 \\ \alpha + 2\beta = 0 \end{array}\right\} \begin{array}{l} \alpha = -2\beta \\ \beta = 1/3 \\ \alpha = -2/3 \end{array}$ 2

Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = A_1 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + A_2 e^{\overset{+4t}{-2t}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -2/3 \\ 1/3 \end{pmatrix} e^{t}$$ 4

1

$x(0)=y(0)=0 \Rightarrow$

$$A_1 + A_2 = 2/3$$
$$-A_1 + A_2 = -1/3 \rightarrow A_2 = 1/6$$
$$2 \quad A_1 = 1/2 \quad 1$$

3

3. $$\mathcal{L}\{t\,\delta'(t-c)\} = \int_0^\infty t e^{-st}\,\delta'(t-c)\,dt$$
$$= \left[t e^{-st}\delta(t-c)\right]_0^\infty - \int_0^\infty (1-st)e^{-st}\,\delta(t-c)\,dt$$
$$= (sc-1)e^{-cs}$$ 2

$$\mathcal{L}\{\text{ODE}\} \rightarrow (s^2+2s+5)\bar{y} = (3s-1)e^{-3s}$$
$$\bar{y} = \frac{3s-1}{(s+1)^2+4}e^{-3s}$$
$$= \frac{3(s+1)-4}{(s+1)^2+4}e^{-3s}$$

$$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4} \qquad \mathcal{L}\{e^{-t}\cos 2t\} = \frac{s+1}{(s+1)^2+4}$$
$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}, \quad \mathcal{L}\{e^{-t}\sin 2t\} = \frac{2}{(s+1)^2+4}$$

$$\therefore\ y = \left[3e^{-(t-3)}\cos 2(t-3) - 2e^{-(t-3)}\sin 2(t-3)\right]$$
$$* \ H(t-3)$$

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Part I

1. Integr. fact $\exp\left(\int \cot x\,dx\right) = \exp\left(\int \frac{\cos x\,dx}{\sin x}\right)$

$= \sin x$!

$\therefore\ y'\sin x + y\cos x = 4\cos x$

$= (y\sin x)'$

$\therefore\ y\sin x = C + 4\sin x$

$y(\pi/2) = 2 \Rightarrow 2 = C + 4$

$\rightarrow C = -2$

$\therefore\ y = \frac{4\sin x - 2}{\sin x}$. (a)

2. Eigenvalues $\begin{vmatrix} -3-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ -5 & 0 & 3-\lambda \end{vmatrix} = 0$

$(-3-\lambda)(1-\lambda)(3-\lambda) + 5(1-\lambda) = 0$

$\therefore\ \lambda = 1$ or $5 + \lambda^2 - 9 = 0$

$\lambda = \pm 2$. (b)

3. $\frac{18}{s(s^2-9)} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s+3}$

↑ gives $A + Be^{3t} + Ce^{-3t}$

$A + B + C = 0$

$+3B - 3C = 0 \qquad B = C = 1$ (c)

$-9A = 18 \qquad A = -2$

4. (a) works! $\begin{pmatrix} u_{xxxx} = +u \\ u_{tt} = -u \end{pmatrix}$

Part II

$$q'' + 4q' + 40q = 0$$

$q = 0,\ q' = 1$ @ $t = 0$.

$$m^2 + 4m + 40 = 0 \rightarrow (m+2)^2 + 36 = 0$$

$$\rightarrow \quad m = -2 \pm 6i$$

$$q = (A\cos 6t + B\sin 6t)e^{-2t}$$

At $t=0$, $q = 0 \Rightarrow A = 0$

$$q' = 1 \Rightarrow 6B = 1, \quad B = \frac{1}{6}.$$

$$q = \frac{1}{6}\sin 6t\, e^{-2t}$$

$$d' + 2d = q \Rightarrow (de^{2t})' = e^{2t}q$$

(int. fac. e^{2t})

$$= \frac{1}{6}\sin 6t$$

$$\therefore d = \left(\frac{-\cos 6t}{36} + C\right)e^{-2t}$$

$d(0) = 0$

$$\rightarrow \quad d = \left(\frac{1 - \cos 6t}{36}\right)e^{-2t}$$

$$\frac{ff'}{1+f^2} = \frac{1}{6}\sin 6t\ e^{-2t}$$

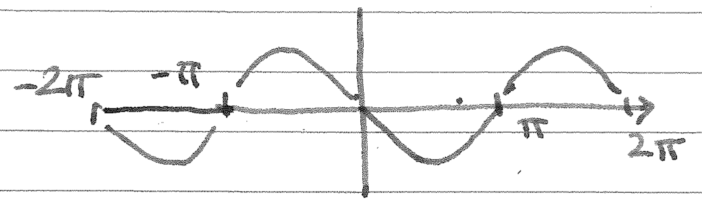
$$\therefore \quad \frac{1}{2}\ln(1+f^2) = \text{const} + \int \frac{1}{6}\sin 6t\, e^{-2t}\, dt$$

$f(0) = 0 \rightarrow$

$$\frac{1}{2}\ln(1+f^2) = \int_0^t \frac{1}{6}\sin 6t\, e^{-2t}\, dt$$

int. by parts ...

4. (a) $f(x) = \pi \sin^2 \frac{x}{2} - x \qquad 0 < x < \pi$



2

$f(x)$ is <u>odd</u>, $\therefore \quad a_0 = a_n = 0$

$$b_n = \frac{2}{\pi} \int_0^{\pi} \left(\pi \sin^2 \frac{x}{2} - x \right) \sin nx \, dx$$

$$\int_0^{\pi} \sin^2 \frac{x}{2} \sin nx \, dx = \frac{1}{2} \int_0^{\pi} (1 - \cos x) \sin nx \, dx$$

$$= \left[-\frac{\cos nx}{2n} \right]_0^{\pi} - \frac{1}{4} \int_0^{\pi} \left[\sin(n+1)x + \sin(n-1)x \right] dx$$

$$= \frac{\left[1 - (-1)^n \right]}{2n} + \frac{1}{4} \left[\frac{\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)} \right]_0^{\pi}$$

$$= \frac{1 - (-1)^n}{2n} - \frac{1}{4(n+1)} \left[1 - (-1)^{n+1} \right]$$

$$- \frac{1}{4(n-1)} \left[1 - (-1)^{n-1} \right]$$

only for $n \neq 1$

$$\cancel{b_n} = \begin{cases} \dfrac{1 - (-1)^n}{2n} - \dfrac{n}{2(n^2-1)} \left[1 + (-1)^n \right] & n \neq 1 \\ 1 \quad \text{if} & n = 1. \end{cases}$$

$$-\int_0^{\pi} x\sin nx\,dx = \left[\frac{x\cos nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\cos nx}{n}\,dx$$

$$= \frac{\pi}{n}(-1)^n - \cancel{\left[\frac{\sin nx}{n^2}\right]_0^{\pi}}$$

$$\therefore\ b_n = 2\begin{cases}\frac{1-(-1)^n}{2n} - \frac{n}{2(n^2-1)}\left[1+(-1)^n\right] \\ 1\end{cases}$$

$$+ \frac{2}{n}(-1)^n$$

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(b) $U = x$.

2

(c) $u_t = u_{xx}$, $u(0,t)=0$

$u(\pi,t)=\pi$

$u(x,0) = \pi\sin^2\frac{x}{2}$.

Put $u = U + v \Rightarrow v(0,t) = v(\pi,t) = 0$.

Then sep. of var: $v = X(x)T(t) \Rightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda$

$X = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

But $X(0) = X(\pi) = 0 \Rightarrow A = 0$

$\lambda = n^2$, $n = 1, 2 \ldots$

$$\therefore\ v = \sum_1 b_n e^{-n^2 t}\sin nx$$

But $v(x,0) = -x + u(x,0) = \pi\sin^2\frac{x}{2} - x$

$\therefore$ b_n is given by part (a)

7.