1 The Laplace transform

The Laplace transform is defined by the integral

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = \overline{f}(s) \tag{1}$$

The crucial feature of the transform from the perspective of differential equations is what it does to derivatives:

$$\mathcal{L}\{\dot{f}(t)\} = s\overline{f}(s) - f(0) \qquad \& \qquad \mathcal{L}\{\ddot{f}(t)\} = s^2\overline{f}(s) - sf(0) - \dot{f}(0) \tag{2}$$

1.1 Inverting the transform

The inverse Laplace transform is a messy contour integral on the complex $s-$plane:

$$f(t) = \mathcal{L}^{-1}\{\overline{y}(s)\} = \int_{\mathcal{C}_B} e^{st}\overline{f}(s)\frac{ds}{2\pi i}. \tag{3}$$

The path $\mathcal{C}_B$ is the "Bromwich contour", an infinite vertical line $s = s_r + is_i$ with s_r fixed and $-\infty < s_i < \infty$. The only restriction on the value of s_r is that the contour must lie to the right of any singularities or other unpleasantries contained in $\overline{f}(s)$. See figure 1(a).

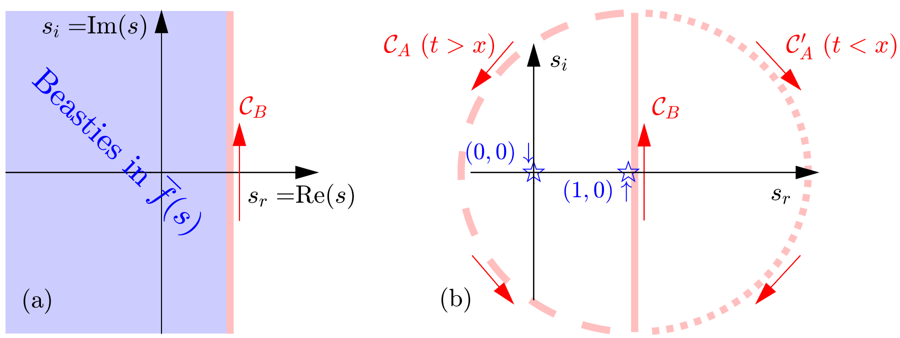

Figure 1: (a) The Bromwich contour. (b) Completion of the Bromwich contour for (4).

1.2 Useful Laplace Transforms:

Most of these relations can be derived by plugging $f(t)$ into the definition of the transform.

$f(t)$	$\to$	$\bar{f}(s)$
1	$\to$	$1/s$
$t^n, \quad n = 0, 1, 2, ...$	$\to$	$n!/s^{n+1}$
e^{at}	$\to$	$1/(s-a)$
$\sin at$	$\to$	$a/(s^2+a^2)$
$\cos at$	$\to$	$s/(s^2+a^2)$
$t \sin at$	$\to$	$2as/(s^2+a^2)^2$
$t \cos at$	$\to$	$(s^2-a^2)/(s^2+a^2)^2$
$y'(t)$	$\to$	$s\bar{y}(s) - y(0)$

$$\begin{aligned} y''(t) &\to s^2\bar{y}(s) - y'(0) - sy(0) \\ f(t)\delta(t-a) &\to f(a)e^{-as} \\ e^{at}f(t) &\to \bar{f}(s-a) \\ f(t-a)H(t-a) &\to e^{-as}\bar{f}(s) \end{aligned}$$

The last two relations are the two shifting theorems for the Laplace tranforms. Heaviside's step function is $H(t)$; Dirac's delta function is $\delta(t)$.

Note that

$$\bar{f}(0) = \int_0^\infty f(t)\,dt \quad \& \quad \bar{f}^{(n)}(0) = (-1)^n \int_0^\infty t^n f(t)\,dt,$$

which are sometimes useful.

2 Solution of PDEs with Laplace transforms

Our goal is to use the Laplace transform to solve a PDE. The transform is clearly suitable for an initial-value problem in time for a function $u(x,t)$ in which, when we zap the PDE with $\mathcal{L}\{...\}$, we emerge with an ODE in x for $\overline{u}(x,s)$. Note that, in view of (2), the Laplace transform will unceremoniously shove the initial conditions directly into the ODE, which needs to the solved subject to the Laplace-transformed boundary conditions in x.

e.g. Solve

$$u_t + u_x = xt, \qquad u(x,0) = e^{-x}, \qquad u(0,t) = 0.$$

First, let's zap everything in sight with the Laplace transform: since $\mathcal{L}\{u(x,t)\} = \overline{u}(x,s)$, $\mathcal{L}\{u_t\} = s\overline{u} - u(x,0)$, $\mathcal{L}\{u_x\} = \overline{u}_x$ and $\mathcal{L}\{xt\} = x/s^2$, we have

$$s\overline{u} - e^{-x} + \overline{u}_x = \frac{x}{s^2}, \qquad \text{or} \qquad (e^{sx}\overline{u})_x = \left(\frac{x}{s^2} + e^{-x}\right)e^{sx}, \qquad \text{with} \quad \overline{u}(0,s) = 0.$$

That is,

$$\overline{u}(x,s) = e^{-sx}\int_0^x \left(\frac{\tilde{x}}{s^2} + e^{-\tilde{x}}\right)e^{s\tilde{x}}d\tilde{x} = \frac{x}{s^3} - \frac{1}{s^4} + \frac{e^{-sx}}{s^4} + \frac{e^{-x}}{s-1} - \frac{e^{-sx}}{s-1}, \tag{4}$$

after intergrating by parts. Armed with the table on the previous page, we note that

$$\mathcal{L}\{t^2\} = \frac{2}{s^3}, \qquad \mathcal{L}\{t^3\} = \frac{6}{s^4}, \qquad \mathcal{L}\{(t-a)^3H(t-a)\} = \frac{6}{s^4}e^{-as},$$

$$\mathcal{L}\{e^t\} = \frac{1}{s-1}, \qquad \mathcal{L}\{e^{t-a}H(t-a)\} = \frac{1}{s-1}e^{-as}.$$

Hence,

$$u(x,t) = \frac{1}{2}xt^2 - \frac{1}{6}t^3 + \frac{1}{6}(t-x)^3H(t-x) + e^{t-x} - e^{t-x}H(t-x). \tag{5}$$

Let's check this: for $t < x$, the step functions disappear and

$$u = \frac{1}{2}xt^2 - \frac{1}{6}t^3 + e^{t-x}, \qquad u_t = xt - \frac{1}{2}t^2 + e^{t-x}, \qquad u_x = \frac{1}{2}t^2 - e^{t-x}, \qquad u(x,0) = e^{-x},$$

indicating that both the PDE and initial condition are satisfied. On the other hand, for $t > x$,

$$u = \frac{1}{2}tx^2 - \frac{1}{6}x^3, \qquad u_t = \frac{1}{2}x^2, \qquad u_x = tx - \frac{1}{2}x^2,$$

and again the PDE is satisfied.

2.1 Direct inversion using the Bromwich contour

We can also find the solution directly using the inverse transform, which is instructive in regard to the evaluation of such beasts more generally using techniques from contour integration. We first look at $\overline{u}(s,x)$ as a function of s. There are singularities at $s=0$ and $s=1$. Thus, the Bromwich contour is along the line $s=\varepsilon+is_i$ with $\varepsilon>1$ and $-\infty<s_i<\infty$.

The trick to evaluating the integral in the inverse transform is to add an arc to the Bromwich contour to turn the integral into one around a closed loop, for which Cauchy's Theorem and residue theory are available. For example, consider the first term x/s^3. We can complete a closed contour $\mathcal{C}$ from the Bromwich contour $\mathcal{C}_B$ by adding a semicircular arc, $\mathcal{C}_A$, of infinite radius enclosing the entire left-half plane; figure 1(b). *i.e.* we add the integral

$$\int_{\mathcal{C}_A} \frac{x}{s^3} e^{st} \frac{ds}{2\pi i} = \frac{x e^{\varepsilon t}}{2\pi} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{R \exp(i\theta + tR\cos\theta + itR\sin\theta)\, d\theta}{(\varepsilon + Re^{i\theta})^3} \to 0 \qquad (s=\varepsilon+Re^{i\theta};\ ds = iRe^{i\theta}d\theta),$$

since, along $\mathcal{C}_A$, we have $s\equiv\varepsilon+Re^{i\theta}$ with $\frac{1}{2}\pi\le\theta\le\frac{3}{2}\pi$ (implying $\cos\theta\le 0$), and we take the limit $R\to\infty$. The required inverse transform, $\mathcal{L}^{-1}\{x/s^3\}$ is therefore equal to the residue of the triple pole at $s=0$. *i.e.* $\mathcal{L}^{-1}\{x/s^3\}=\frac{1}{2}xt^2$. Similarly $\mathcal{L}^{-1}\{1/s^4\}=\frac{1}{6}t^3$ and $\mathcal{L}^{-1}\{1/(s-1)\}=xe^t$ (from the residues of the quartic pole at $s=0$ and the simple pole at $s=1$).

Note that the residue theorem says that, at a pole $z=z_P$ of the integrand,

$$\int_{\mathcal{C}} F(z)\frac{dz}{2\pi i} = \mathrm{Res}(F, z_P) \quad \text{(the residue of } F(z) \text{ at } z=z_P\text{)},$$

where the contour $\mathcal{C}$ is an anti-clockwise path encircling the pole. For a simple pole, $F(z)\to G(z)/(z-z_P)$ and the residue is $G(z_P)$. For a pole of order n, we have the generalization $F(z)\to G(z)/(z-z_P)^n$, with residue $G^{(n-1)}(z_P)/(n-1)!$. Here, $G(z)=e^{zt}$.

The third term in $\overline{u}(x,s)$ is more tricky, as it contains the additional exponential factor e^{-sx}. This complicates the evaluation of the integral, because this factor combines with the exponential in the inverse transform to give the integrand $s^{-4}e^{(t-x)s}$. Adding the integral along $\mathcal{C}_A$ now longer always works because

$$\int_{\mathcal{C}_A} \frac{1}{s^4} e^{(t-x)s} \frac{ds}{2\pi i} = \frac{e^{\varepsilon(t-x)}}{2\pi} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{R \exp[i\theta + (t-x)R\cos\theta + i(t-x)R\sin\theta]\, d\theta}{(\varepsilon + Re^{i\theta})^4}.$$

Only if $t>x$ does this integral converge to zero when $R\to\infty$ (recalling $\cos\theta\le 0$). For this situation (*i.e.* $t<x$), we proceed as before, adding the integral along $\mathcal{C}_A$ to complete the contour and thereby obtain $\mathcal{L}^{-1}\{s^{-4}e^{-sx}\}=\frac{1}{6}(t-x)^3$. But when $t<x$, the integral along $\mathcal{C}_A$ diverges and so the completion of the contour must be performed differently. In particular, instead of $\mathcal{C}_A$, we may add an integral along the alternative arc $\mathcal{C}'_A$, the semi-circular arc of infinite radius to the right of $\mathcal{C}_B$ enclosing the rest of the right-half plane; see figure 1(b). *i.e.* we add

$$\int_{\mathcal{C}'_A} \frac{1}{s^4} e^{(t-x)s} \frac{ds}{2\pi i} = -\frac{e^{\varepsilon(t-x)}}{2\pi R^3} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{R \exp[i\theta + (t-x)R\cos\theta + i(t-x)R\sin\theta]\, d\theta}{(\varepsilon + Re^{i\theta})^4} \to 0$$

(bearing in mind that the new integral must proceed from $\theta=\frac{1}{2}\pi$ (the top end of $\mathcal{C}_B$), to $\theta=-\frac{1}{2}\pi$ (the lower end of $\mathcal{C}_B$), with $\cos\theta\ge 0$). Although we now traverse the closed contour $\mathcal{C}=\mathcal{C}_B+\mathcal{C}'_A$ along a clockwise path (in violation of convention, and potentially incurring a sign error), the result does not care: the integrand is analytic within the contour as there are no singularities to the right of $\mathcal{C}_B$. Thus, $\mathcal{L}^{-1}\{s^{-4}e^{-sx}\} = 0$ if $t < x$. In summary, $\mathcal{L}^{-1}\{s^{-4}e^{-sx}\} = \frac{1}{6}(t-x)^3 H(t-x)$. A similar line of argument is needed for the final term in $\overline{u}(x,s)$, and we eventually emerge with (5). Obviously, using the table was easier...

e.g. Try

$$xu_t + u + u_x = 0, \qquad u(0,t) = \cos t, \qquad u(x,0) = 0.$$

After Laplace transforming and solving the ODE in x, you should find

$$\overline{u}(x,s) = \frac{se^{-x-sx^2/2}}{s^2+1}.$$

Invert the transform to find

$$u(x,t) = e^{-x}\cos(t - \tfrac{1}{2}x^2)\; H(t - \tfrac{1}{2}x^2).$$

Double check the solution by subsitution into the PDE and initial and boundary conditions

3 Signalling problems

Consider the wave equation on the half line with a prescribed source at $x = 0$:

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = u_t(x,0) = 0, \qquad u(0,t) = S(t), \qquad u \to 0 \text{ for } x \to \infty.$$

Applying the Laplace transform:

$$s^2\overline{u} = c^2\overline{u}_{xx}, \qquad \to \qquad \overline{u}(x,s) = A(s)e^{sx/c} + B(s)e^{-sx/c},$$

for two "constants" of integration, $A(s)$ and $B(s)$ (they may depend on s!). Of the two solutions here, we do not want $e^{sx/c}$ as $u \to 0$ for $x \to \infty$ and we must evaluate s along the Bromwich contour, which lies somewhere in the right half of the complex $s-$plane (*i.e.* $\mathrm{Re}(s) > 0$). Also, $\overline{u}(0,s) = \overline{S}(s)$ from the Laplace transform of the boundary condition at $x = 0$. Hence, $\overline{u}(x,s) = \overline{S}(s)e^{-sx/c}$. Inverting the transform immediately gives $u(x,t) = S(t - x/c)H(t - x/c)$.

The wave equation describes information propagating with speed c. Since $u(x,0) = 0$, the signal $S(t)$ from $x = 0$ only reaches the position x once $t > x/c$. And once the signal arrives, it corresponds to information delayed by a time x/c. Note that our solution is just a particular case of d'Alembert's general solution.

Now consider an impulsively excited oscillator shedding waves (such as a mass on a spring attached to a string); this example illustrates "radiative damping":

$$u_{tt} = c^2 u_{xx}, \qquad u(0,t) = X(t), \qquad u(x,0) = u_t(x,0) = 0, \qquad u \to 0 \text{ for } x \to \infty,$$
$$\ddot{X} + \omega^2 X = u_x(0,t), \qquad X(0) = 0, \qquad \dot{X}(0) = 1.$$

Laplace transforming everything in sight:

$$\overline{u}_{xx} = \frac{s^2}{c^2}\overline{u}, \qquad (\omega^2 + s^2)\overline{X} = 1 + \overline{u}_x(0,s), \qquad \overline{u}(0,s) = \overline{X}(s).$$

i.e. the signal $S(t)$ in our previous example is just $X(t)$. Hence

$$\overline{u}(x,s) = \overline{X}e^{-sx/c} \qquad \& \qquad \overline{X}(s) = \frac{1}{s^2 + s/c + \omega^2} = \left[\left(s + \frac{1}{2c}\right)^2 + \omega^2 - \frac{1}{4c^2}\right]^{-1}.$$

With the help of the transform table we see that $\mathcal{L}\{e^{at}\sin\Omega t\} = \Omega/[(s-a)^2+\Omega^2]$. Hence

$$X(t) = \frac{1}{\Omega}e^{-t/2c}\sin\Omega t, \qquad \Omega = \sqrt{\omega^2 - \frac{1}{4c^2}}.$$

Finally, $u(x,t) = X(t-x/c)H(t-x/c)$.

Even though the oscillator in this problem has no damping, it is coupled to the string and any oscillations excite waves that propagate away. This effect constitutes a loss of energy and therefore effective *(radiative)* damping with rate $(2c)^{-1}$. Note that the oscillator cannot oscillate if the damping is too strong ($\omega^2 < (4c^2)^{-1}$) and one needs a better way of writing the solution.

4 Connection with separation of variables

Finally, let's tie in the Laplace transform approach with separation of variables: consider

$$u_{tt} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right), \qquad u_t(\theta,0)=0, \qquad u(\theta,0)=f(\theta),$$

(the axisymmetrical wave equation for a spherical membrane). Laplace transforming and setting $x=\cos\theta$ and $F(x)=f(\theta)$:

$$[(1-x^2)\overline{u}_x]_x - s^2\overline{u} = -sF,$$

which is Legendre's equation with an inhomogeneous term. Expanding both $\overline{u}$ and F in terms of Legendre poynomials,

$$\overline{u}(x,s) = \sum_{n=0}^{\infty}\overline{u}_n(s)P_n(x) \qquad \& \qquad F(x) = \sum_{n=0}^{\infty}F_nP_n(x), \qquad \text{with } F_n = (n+\tfrac{1}{2})\int_{-1}^{1}F(x)P_n(s)dx,$$

now gives $\overline{u}_n(s) = sF_n/[s^2+n(n+1)]$. Hence

$$u = \mathcal{L}^{-1}\left\{\sum_{n=0}^{\infty}\frac{sF_nP_n(x)}{s^2+n(n+1)}\right\} = \sum_{n=0}^{\infty}F_nP_n(x)\cos[\sqrt{n(n+1)}t].$$

In other words, applying the Laplace transform and expanding using the relevant Sturm-Liouville eigenfunctions gives us the separation of variables solution.