Summer Research Experiences program:
Local to global principle for expected values over function fields

What is this about: This is an opportunity to do some undergraduate research project in the realm of number theory together with another student under the supervision of Dr. Schraven. The goal is for you to experience research in mathematics. This is part of the Summer Research Experiences program run by Prof. Walji. You should be ready to present your work to the other participants in some informal meeting. Communication skills are vital in research (and beyond).

Content: In mathematics we have several ways of quantify whether something is "big" or "small". In topology that might be that a set is dense or in real analysis that a set has small Lebesgue measure (for an interval $[a; b]$ the Lebesgue measure gives you $b - a$, i.e. the length of the interval. This can be done for much wilder sets). In our setting we have a set $A \subseteq \mathbb{Z}^d$ and we want to understand what proportion of $\mathbb{Z}^d$ is in the set A. For example for A being the even integers, we intuitively understand that half of the integers are in A. This proportion (if one can make sense of it) is called the natural density of A.

In general it is quite hard to compute the natural density of a given set. However, under some conditions this can be done using p-adic numbers. Intuitively, we get the p-adic numbers if we tweak the metric on the real numbers a bit (p-adic numbers are fun and we will learn about them in the beginning of the project). If you are familiar with probability theory, you can also think of the natural density as some kind of probability measure on the lattice $\mathbb{Z}^d$. Naturally we can ask whether we can push this analogy and define a notion of expected value. This is indeed possible and using similar ideas as in the density case one can get some effective tool to compute this expected value again in terms of p-adic numbers. In this talk `https://www.youtube.com/watch?v=dtE2Zv7bryE&t=1041s` you can hear me talk about this.

As some of my colleagues would say "be wise, generalize". If we can do expected values for $\mathbb{Z}^d$, can we do the same thing for more general rings? Certain things are already known. For example we can do the expected value over the ring of integers of number fields (see for example [3]) and the density argument carries over to function fields (see [1]). The goal of this project is to investigate the notion of expected value for function fields.

Prerequisites: Having taken (or currently taking) Math 319 & Math 323. Knowledge about measure theory or p-adic number is a plus, but not required.

Possible route: This paragraph should not scare you, in fact you do not even need to read it. This is just some outline of the approach I have in mind and it will make much more sense once you started working on the project. Research is like climbing a mountain. It looks scary from a distance, but you only need to make one step at a time and enjoy the journey!

The first step in this project is to familiarize ourselves with the existing papers [2] and

[3]. For this we will try to compute the higher moments of the rectangular unimodular matrices. This will allow to get you a better conceptual understanding of conditions that are checkable.

After this we are ready to think about the generalizations to function fields. In this step we will need to familiarize ourself with the setting of [1]. First we want to consider the special case of coprime pairs and see how things work out. This will hopefully give us some insight about the correct definition of the expected value.

Now things will get very interesting. We will try to find conditions that are checkable (should at least allow us deal with coprime n-tuples and Eisenstein polynomials) and allow us to compute the expected value. Once we can deal with the expected value, we can also try to tackle higher moments as done in [2] for number fields.

References

[1] Micheli, Giacomo. "A local to global principle for densities over function fields." arXiv preprint arXiv:1701.01178 (2017).

[2] Micheli, Giacomo, Severin Schraven, and Violetta Weger. "A local to global principle for expected values." Journal of Number Theory (2021).

[3] Micheli, Giacomo, et al. "Local to global principle over number fields for higher moments." arXiv preprint arXiv:2201.03751 (2022).