RESOLUTIONS FOR PRINCIPAL SERIES REPRESENTATIONS OF p-ADIC GL_n

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To Peter Schneider on the occasion of his birthday.

Abstract. Let $\mathfrak{F}$ be a nonarchimedean locally compact field with residue characteristic p and $\mathbf{G}(\mathfrak{F})$ the group of $\mathfrak{F}$-rational points of a connected reductive group. In [12], Schneider and Stuhler realize, in a functorial way, any smooth complex finitely generated representation of $\mathbf{G}(\mathfrak{F})$ as the 0-homology of a certain coefficient system on the semi-simple building of $\mathbf{G}$. It is known that this method does not apply in general for smooth mod p representations of $\mathbf{G}(\mathfrak{F})$, even when $\mathbf{G} = \mathrm{GL}_2$. However, we prove that a principal series representation of $\mathrm{GL}_n(\mathfrak{F})$ over a field with arbitrary characteristic can be realized as the 0-homology of the corresponding coefficient system as defined in [12].

Contents

1. Introduction

Let $\mathfrak{F}$ be a nonarchimedean locally compact field with residue characteristic p and $\mathbf{G}(\mathfrak{F})$ the group of $\mathfrak{F}$-rational points of a connected reductive group $\mathbf{G}$. By a result of Bernstein, the blocks of the category of smooth complex representations of $\mathbf{G}(\mathfrak{F})$ have finite global dimension. The $\mathbf{G}(\mathfrak{F})$-equivariant coefficient systems on the semisimple building $\mathscr{X}$ of $\mathbf{G}$ introduced in [12] allow Schneider and Stuhler to construct explicit projective resolutions for finitely generated representations in this category. One of the key ingredients for their result

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is the following fact, which is valid over an arbitrary field $\mathbf{k}$: consider the (level 0) universal representation $\mathbf{X} = \mathbf{k}[\mathrm{I}\backslash\mathbf{G}(\mathfrak{F})]$ where I is a fixed pro-p Iwahori subgroup of $\mathbf{G}(\mathfrak{F})$, then the attached coefficient system $\underline{\underline{\mathbf{X}}}$ on $\mathscr{X}$ gives the following *exact* augmented chain complex

$$0 \longrightarrow C_c^{or}(\mathscr{X}_{(d)}, \underline{\underline{\mathbf{X}}}) \xrightarrow{\partial} \dots \xrightarrow{\partial} C_c^{or}(\mathscr{X}_{(0)}, \underline{\underline{\mathbf{X}}}) \xrightarrow{\epsilon} \mathbf{X} \longrightarrow 0 \tag{1.1}$$

of $\mathbf{G}(\mathfrak{F})$-representations and of left $\mathrm{H} := \mathbf{k}[\mathrm{I}\backslash\mathbf{G}(\mathfrak{F})/\mathrm{I}]$-modules (see §3 below for the notation).

If $\mathbf{k}$ has characteristic p, it is no longer true that the category of smooth representations of $\mathbf{G}(\mathfrak{F})$ generated by their pro-p Iwahori fixed vectors has finite global dimension: in the case of $\mathrm{PGL}_2(\mathbb{Q}_p)$, this category is equivalent to the category of modules over H ([5]) and it is proved in [7] that the latter has infinite global dimension.

Still if $\mathbf{k}$ has characteristic p, it is also no longer true that any $\mathbf{G}(\mathfrak{F})$-representation $\mathbf{V}$ generated by its I-invariant subspace can be realized as the 0-homology of the coefficient system $\underline{\underline{\mathbf{V}}}$ defined as in [12]: it is true for the universal representation $\mathbf{X}$ as noted above, but [7, Remark 3.2] points out a counter-example when $\mathbf{V}$ is a supercuspidal representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. However, realizing any smooth irreducible $\mathbf{k}$-representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ as the 0-homology of some finite dimensional coefficient system is important in Colmez's construction of a functor yielding the p-adic local Langlands correspondence ([2]). As explained in [8], the resolutions in [2] can be retrieved in the following way: let $\mathbf{V}$ be a smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ with a central character and generated by its I-invariant subspace $\mathbf{V}^{\mathrm{I}}$, then by the equivalence of categories of [5], tensoring (1.1) by the H-module $\mathbf{V}^{\mathrm{I}}$ gives an exact resolution of $\mathbf{V} \simeq \mathbf{V}^{\mathrm{I}} \otimes_{\mathrm{H}} \mathbf{X}$. But the equivalence of categories does not hold in general. For arbitrary $\mathfrak{F}$, Hu attaches to any irreducible representation of $\mathrm{GL}_2(\mathfrak{F})$ with central character a coefficient system on the tree whose 0-homology is isomorphic to $\mathbf{V}$ ([4]). This coefficient system, although not finite dimensional in general, turns out to be finite dimensional when $\mathfrak{F} = \mathbb{Q}_p$ and one retrieves, once again, the resolutions of [2]. But if $\mathfrak{F}$ has positive characteristic (respectively if $\mathfrak{F}/\mathbb{Q}_p$ is a quadratic unramified extension), then for $\mathbf{V}$ supercuspidal, there is no finite dimensional coefficient system whose 0-homology is isomorphic to $\mathbf{V}$ as proved in [4] (respectively in [14]).

Most of the surprising phenomena occurring in the smooth mod p representation theory of $\mathbf{G}(\mathfrak{F})$ are related to the properties of the supercuspidal representations, whereas the behavior of the principal series representations is easier to analyze and is somewhat similar to what is observed in the setting of complex representations. To formalize this remark, Peter Schneider asked me the following question: the Hecke algebra H contains a copy $\mathcal{A}_{anti}$ of the $\mathbf{k}$-algebra of the semigroup of all (extended) antidominant cocharacters of a split torus of $\mathbf{G}$; is H free as a $\mathcal{A}_{anti}$-module when localized at a regular character of $\mathcal{A}_{anti}$? (see §2.3 and §4 for the definitions and Propositions 4.3 and 4.4 for the link between regular characters of $\mathcal{A}_{anti}$ and principal series representations). The answer is yes and the present note is largely inspired by this question. We prove the following theorem, where n is an integer ≥ 1.

Theorem 1.1. *Let* $\mathbf{k}$ *be an arbitrary field and* $\mathbf{V}$ *a smooth principal series representation of* $\mathrm{GL}_n(\mathfrak{F})$ *over* $\mathbf{k}$*. Let* $\underline{\underline{\mathbf{V}}}$ *be the coefficient system associated to* $\mathbf{V}$ *as in* [12]*. Then the following augmented chain complex*

$$0 \longrightarrow C_c^{or}(\mathscr{X}_{(n-1)}, \underline{\underline{\mathbf{V}}}) \xrightarrow{\partial} \dots \xrightarrow{\partial} C_c^{or}(\mathscr{X}_{(0)}, \underline{\underline{\mathbf{V}}}) \xrightarrow{\epsilon} \mathbf{V} \longrightarrow 0 \tag{1.2}$$

yields an exact resolution of $\mathbf{V}$ *as a representation of* $\mathrm{GL}_n(\mathfrak{F})$*.*

This theorem is proved in Section 5. In the previous sections, the arguments are written in the setting of a general split group. However, in Section 5, we need an extra geometric condition on the facets of the standard apartment to be able to fully use Iwasawa decomposition. Therefore, we restrict ourselves to the case of GL_n. We suspect that Theorem 1.1 is true in general.

A generalization of Colmez' functor to reductive groups over $\mathbb{Q}_p$ is proposed by Schneider and Vignéras in [13]. The first fundamental construction is the one of a universal δ-functor $V \mapsto D^i(V)$ for $i \geq 0$, from the category $\mathcal{M}_{o-tor}(\mathrm{B})$ to the category $\mathcal{M}_{et}(\Lambda(\mathrm{B}^+))$. The ring o is the ring of integers of a fixed finite extension of $\mathbb{Q}_p$ and $\mathcal{M}_{o-tor}(\mathrm{B})$ is the abelian category of smooth representations of B in o-torsion modules, where B is a fixed Borel subgroup in $\mathbf{G}(\mathbb{Q}_p)$. We do not describe the category $\mathcal{M}_{et}(\Lambda(\mathrm{B}^+))$ explicitly here. The restriction V to B of a principal series representation of $\mathbf{G}(\mathbb{Q}_p)$ over a field with characteristic p is an example of an object in $\mathcal{M}_{o-tor}(\mathrm{B})$ for a suitable ring o. Using Theorem 1.1, we prove the following result (§6):

Proposition 1.2. *Suppose that* $\mathbf{k}$ *is a field with characteristic* p*. Let* V *be the restriction to* B *of a principal series representation of* $\mathrm{GL}_n(\mathbb{Q}_p)$ *over* $\mathbf{k}$*. Then* $D^i(V) = 0$ *for* $i \geq n-1$*.*

2. Notations and preliminaries

2.1. From now on we suppose that $\mathbf{G}$ is $\mathfrak{F}$-split and we denote $\mathbf{G}(\mathfrak{F})$ by G. We fix a uniformizer ϖ for $\mathfrak{F}$ and choose the valuation $val_{\mathfrak{F}}$ on $\mathfrak{F}$ normalized by $val_{\mathfrak{F}}(\varpi) = 1$. The ring of integers of $\mathfrak{F}$ is denoted by $\mathfrak{O}$ and its residue field with cardinality $q = p^f$ by $\mathbb{F}_q$.

In the semisimple building $\mathscr{X}$ of G, we choose the chamber C corresponding to the Iwahori subgroup I' that contains the pro-p subgroup I. This choice is unique up to conjugacy by an element of G. Since G is split, C has at least one hyperspecial vertex x_0 and we denote by K the associated maximal compact subgroup of G. We fix a maximal $\mathfrak{F}$-split torus T in G such that the corresponding apartment $\mathscr{A}$ in $\mathscr{X}$ contains C.

Let $\mathbf{G}_{x_0}$ and $\mathbf{G}_C$ denote the Bruhat-Tits group schemes over $\mathfrak{O}$ whose $\mathfrak{O}$-valued points are K and I′ respectively. Their reductions over the residue field $\mathbb{F}_q$ are denoted by $\overline{\mathbf{G}}_{x_0}$ and $\overline{\mathbf{G}}_C$. Note that $\mathrm{G} = \mathbf{G}_{x_0}(\mathfrak{F}) = \mathbf{G}_C(\mathfrak{F})$. By [16, 3.4.2, 3.7 and 3.8], $\overline{\mathbf{G}}_{x_0}$ is connected reductive and $\mathbb{F}_q$-split. Therefore we have $\mathbf{G}_C^\circ(\mathfrak{O}) = \mathbf{G}_C(\mathfrak{O}) = \mathrm{I}'$ and $\mathbf{G}_{x_0}^\circ(\mathfrak{O}) = \mathbf{G}_{x_0}(\mathfrak{O}) = \mathrm{K}$. Denote by K_1 the pro-unipotent radical of K. More generally we consider the fundamental system of open normal subgroups

$$\mathrm{K}_m := \ker\left(\mathbf{G}_{x_0}(\mathfrak{O}) \xrightarrow{\mathrm{pr}} \mathbf{G}_{x_0}(\mathfrak{O}/\varpi^m\mathfrak{O})\right) \qquad \text{for } m \geq 1$$

of K. The quotient K/K_1 is isomorphic to $\overline{\mathbf{G}}_{x_0}(\mathbb{F}_q)$. The Iwahori subgroup I′ is the preimage in K of the $\mathbb{F}_q$-rational points of a Borel subgroup $\overline{\mathbf{B}}$ with Levi decomposition $\overline{\mathbf{B}} = \overline{\mathbf{T}}\,\overline{\mathbf{N}}$. The pro-$p$ Iwahori subgroup I is the preimage in I′ of $\overline{\mathbf{N}}(\mathbb{F}_q)$. The preimage of $\overline{\mathbf{T}}(\mathbb{F}_q)$ is the the maximal compact subgroup T^0 of T. Note that $\mathrm{T}^0/\mathrm{T}^1 = \mathrm{I}'/\mathrm{I} = \overline{\mathbf{T}}(\mathbb{F}_q)$ where $\mathrm{T}^1 := \mathrm{T}^0 \cap \mathrm{I}$.

To the choice of T is attached the root datum $(\Phi, \mathrm{X}^*(\mathrm{T}), \check{\Phi}, \mathrm{X}_*(\mathrm{T}))$. This root system is reduced because the group $\mathbf{G}$ is $\mathfrak{F}$-split. We denote by $\mathfrak{W}$ the finite Weyl group $N_{\mathrm{G}}(\mathrm{T})/\mathrm{T}$, quotient by T of the normalizer of T. Let $\langle\,.\,,\,.\,\rangle$ denote the perfect pairing $\mathrm{X}_*(\mathrm{T})\times\mathrm{X}^*(\mathrm{T}) \to \mathbb{Z}$. The elements in $\mathrm{X}_*(\mathrm{T})$ are the cocharacters of T and we will call them the coweights. We identify the set $\mathrm{X}_*(\mathrm{T})$ with the subgroup T/T^0 of the extended Weyl group $\mathrm{W} = N_{\mathrm{G}}(\mathrm{T})/\mathrm{T}^0$ as in [16, I.1] and [12, I.1]: to an element $g \in \mathrm{T}$ corresponds a vector $\nu(g) \in \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(\mathrm{T})$ defined by

$$\langle \nu(g),\, \chi \rangle = -\, val_{\mathfrak{F}}(\chi(g)) \qquad \text{for any } \chi \in \mathrm{X}^*(\mathrm{T}). \tag{2.1}$$

and ν induces the required isomorphism $\mathrm{T}/\mathrm{T}^0 \cong \mathrm{X}_*(\mathrm{T})$. Recall that $\mathscr{A}$ denotes the apartment of the semisimple building attached to T ([16] and [12, I.1]). The group T/T^0 acts by translation on $\mathscr{A}$ via ν. The actions of $\mathfrak{W}$ and T/T^0 combine into an action of W on $\mathscr{A}$ as recalled in [12, page 102]. Since x_0 is a special vertex of the building, W is isomorphic to the semidirect product $\mathrm{X}_*(\mathrm{T}) \rtimes \mathfrak{W}$ where we see $\mathfrak{W}$ as the fixator in W of any point in the extended apartment lifting x_0 ([16, 1.9]). A coweight λ will sometimes be denoted by e^λ to underline that we see it as an element in W, meaning as a translation on the semisimple apartment $\mathscr{A}$.

We see the roots Φ as the set of affine roots taking value zero at x_0. Then Φ^+ is the set of roots in Φ taking non negative values on C. The set of dominant coweights $\mathrm{X}_*^+(\mathrm{T})$ is the set of all $\lambda \in \mathrm{X}_*(\mathrm{T})$ such that $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Phi^+$. A coweight is called antidominant if its opposite is dominant. A coweight λ such that $\langle \lambda, \alpha \rangle < 0$ for all $\alpha \in \Phi^+$ is called strongly antidominant.

2.2. We fix a lift $\hat{w} \in N_{\mathrm{G}}(\mathrm{T})$ for any $w \in \mathrm{W}$. By Bruhat decomposition, G is the disjoint union of all $\mathrm{I}'\hat{w}\mathrm{I}'$ for $w \in \mathrm{W}$. Recall that T^1 is the pro-p Sylow subgroup of T^0. We denote by $\tilde{\mathrm{W}}$ the quotient of $N_{\mathrm{G}}(\mathrm{T})$ by T^1 and obtain the exact sequence

$$0 \to \mathrm{T}^0/\mathrm{T}^1 \to \tilde{\mathrm{W}} \to \mathrm{W} \to 0.$$

The group $\tilde{\mathrm{W}}$ parametrizes the double cosets of G modulo I. We fix a lift $\hat{w} \in N_{\mathrm{G}}(\mathrm{T})$ for any $w \in \tilde{\mathrm{W}}$. For Y a subset of W, we denote by $\tilde{Y}$ its preimage in $\tilde{\mathrm{W}}$. In particular, we have the preimage $\tilde{\mathrm{X}}_*(\mathrm{T})$ of $\mathrm{X}_*(\mathrm{T})$. As well as those of $\mathrm{X}_*(\mathrm{T})$, its elements will be denoted by λ or e^λ and called coweights. Note that a system of representatives of T/T^1 is given by the set of all $\widehat{e^\lambda}$ for $\lambda \in \tilde{\mathrm{X}}_*(\mathrm{T})$. In fact, we recall that the map

$$\lambda \in \mathrm{X}_*(\mathrm{T}) \to [\lambda(\varpi^{-1}) \bmod \mathrm{T}^1] \in \tilde{\mathrm{X}}_*(\mathrm{T}) \tag{2.2}$$

is a $\mathfrak{W}$-equivariant splitting for the exact sequence of abelian groups

$$0 \longrightarrow \mathrm{T}^0/\mathrm{T}^1 \longrightarrow \tilde{\mathrm{X}}_*(\mathrm{T}) \longrightarrow \mathrm{X}_*(\mathrm{T}) \longrightarrow 0. \tag{2.3}$$

We will identify $\mathrm{X}_*(\mathrm{T})$ with its image in $\tilde{\mathrm{X}}_*(\mathrm{T})$ *via* (2.2).

For $\alpha \in \Phi$, we inflate the function $\langle\, .\, , \alpha\rangle$ defined on $\mathrm{X}_*(\mathrm{T})$ to $\tilde{\mathrm{X}}_*(\mathrm{T})$. We still call *dominant coweights* (resp. *antidominant coweights*) the elements in the preimage $\tilde{\mathrm{X}}_*^+(\mathrm{T})$ (resp. $\tilde{\mathrm{X}}_*^-(\mathrm{T})$) of $\mathrm{X}_*^+(\mathrm{T})$ (resp. $\mathrm{X}_*^-(\mathrm{T})$).

The group $\tilde{\mathrm{W}}$ is equipped with a length function $\ell : \tilde{\mathrm{W}} \to \mathbb{N}$ that inflates the length function on W ([18, Proposition 1]).

2.3. Let $\mathbf{k}$ be an arbitrary field. We consider the pro-p Iwahori-Hecke algebra

$$\mathrm{H} = \mathbf{k}[\mathrm{I}\backslash\mathrm{G}/\mathrm{I}]$$

of $\mathbf{k}$-valued functions with compact support in $\mathrm{I}\backslash\mathrm{G}/\mathrm{I}$ under convolution. For $w \in \tilde{\mathrm{W}}$, denote by τ_w the characteristic function of the double coset $\mathrm{I}\hat{w}\mathrm{I}$. The set of all $(\tau_w)_{w\in\tilde{\mathrm{W}}}$ is a $\mathbf{k}$-basis for H. For $g \in \mathrm{G}$, we will also use the notation τ_g for the characteristic function of the double coset $\mathrm{I}g\mathrm{I}$. In H we have the following relation, for w, w' in $\tilde{\mathrm{W}}$ ([18, Theorem 50]):

$$\tau_w\tau_{w'} = \tau_{ww'} \text{ if } \ell(w) + \ell(w') = \ell(ww'). \tag{2.4}$$

It implies in particular that in H we have, for λ and λ' in $\tilde{\mathrm{X}}_*(\mathrm{T})$:

$$\tau_{e^\lambda}\tau_{e^{\lambda'}} = \tau_{e^{\lambda+\lambda'}} \text{ if } \lambda \text{ and } \lambda' \text{ are both antidominant.} \tag{2.5}$$

We denote by $\mathcal{A}_{anti}$ the commutative sub-$\mathbf{k}$-algebra of H with $\mathbf{k}$-basis the set of all $\{\tau_{e^\lambda},\, \lambda \in \tilde{\mathrm{X}}_*^-(\mathrm{T})\}$.

2.4. Let U be the unipotent subgroup of G generated by all the root subgroups U_α for $\alpha \in \Phi^+$ and B the Borel subgroup with Levi decomposition $\mathrm{B} = \mathrm{TU}$. Recall that we have $\mathrm{G} = \mathrm{BK}$ since x_0 is a special vertex. Furthermore, $\mathrm{B} \cap \mathrm{K} = \mathrm{I}' \cap \mathrm{B}$.

Let U^- denote the opposite unipotent subgroup of G generated by all the root subgroups U_α for $-\alpha \in \Phi^+$. The pro-p Iwahori subgroup I has the following decomposition:

$$\mathrm{I} = \mathrm{I}^+ \, \mathrm{I}^0 \, \mathrm{I}^- \text{ where } \mathrm{I}^+ := \mathrm{I} \cap \mathrm{U},\ \mathrm{I}^0 := \mathrm{I} \cap \mathrm{T} = \mathrm{T}^1,\ \mathrm{I}^- := \mathrm{I} \cap \mathrm{U}^-.$$

An element $t \in \mathrm{T}$ contracts I^+ and dilates I^- if it satisfies the conditions (see [1, (6.5)]):

$$t\, \mathrm{I}^+ t^{-1} \subseteq \mathrm{I}^+, \qquad t^{-1} \mathrm{I}^- t \subseteq \mathrm{I}^-. \tag{2.6}$$

Denote by T^{++} the semigroup of such $t \in \mathrm{T}$.

Lemma 2.1. *We have* $\mathrm{T}^{++} = \coprod_{\lambda \in \tilde{\mathrm{X}}_*^-(\mathrm{T})} \mathrm{T}^1 \widehat{e^\lambda}$.

Proof. Let $\lambda \in \tilde{\mathrm{X}}_*(\mathrm{T})$. It is proved in [9, Lemma 5.20] that the element $\widehat{e^\lambda}$ satisfies (2.6) if and only if λ is antidominant.

□

2.5. We consider the **k**-basis $(E(w))_{w \in \tilde{\mathrm{W}}}$ for H as introduced in [18]. Recall that $E(e^\lambda) = \tau_{e^\lambda}$ for all $\lambda \in \tilde{\mathrm{X}}_*^-(\mathrm{T})$. For $w \in \tilde{\mathrm{W}}$, there is $\lambda_0 \in \tilde{\mathrm{X}}_*(\mathrm{T})$ and $w_0 \in \tilde{\mathfrak{W}}$ such that $w = e^{\lambda_0} w_0$. Let $\lambda \in \tilde{\mathrm{X}}_*^-(\mathrm{T})$ such that $\lambda + \lambda_0 \in \tilde{\mathrm{X}}_*^-(\mathrm{T})$. We claim that

$$\tau_{e^\lambda}\, E(w) = q^{(\ell(w) + \ell(e^\lambda) - \ell(e^\lambda w))/2} E(e^{\lambda_0 + \lambda}) \tau_{w_0} \in \mathcal{A}_{anti}\, \tau_{w_0}. \tag{2.7}$$

The proof of this equality given in the case of GL_n in [6, Proposition 4.8] works in the general case with no modification.

2.6. Let $t \in \mathrm{T}$ such that the double class $\mathrm{I}\, t\, \mathrm{I}$ corresponds to a strongly antidominant element in $\mathrm{X}_*^-(\mathrm{T})$. The following lemma proved in [11, Proposition 8, p.78] is valid in the case of a general split reductive group.

Lemma 2.2. *An open compact subset of* $\mathrm{B}\backslash\mathrm{G}$ *decomposes into a finite disjoint union of subsets of the form* $\mathrm{BI}t^n k = \mathrm{BI}^- t^n k$ *for n large enough, where k ranges over a finite subset of* K.

Lemma 2.3. *A system of neighborhoods of the identity in* U^- *is given by the set of all* $t^{-m}\mathrm{I}^- t^m$ *for* $m \in \mathbb{N}$.

Proof. A system of neighborhoods of the identity in U^- is given by the set of all $\mathrm{K}_m \cap \mathrm{U}^-$ and one checks that $t^{-m}\mathrm{I}^- t^m \subseteq \mathrm{K}_{m+1} \cap \mathrm{U}^-$ for all $m \in \mathbb{N}$.

□

3. Resolution of the level 0 universal representation of $\mathbf{G}(\mathfrak{F})$

We gather here results from [12] and use the notations of [7]. We recall (cf. [12] I.1-2 for a brief overview) that the semi-simple building $\mathscr{X}$ is (the topological realization of) a G-equivariant polysimplicial complex of dimension equal to the semisimple rank d of G. The (open) polysimplices are called facets and the d-dimensional, resp. zero dimensional, facets chambers, resp. vertices. For $i \in \{0, ..., d\}$, we denote by $\mathscr{X}_{(i)}$ the set of oriented facets of dimension i. Associated with each facet F is, in a G-equivariant way, a smooth affine $\mathfrak{O}$-group scheme $\mathbf{G}_F$ whose general fiber is $\mathbf{G}$ and such that $\mathbf{G}_F(\mathfrak{O})$ is the pointwise stabilizer in G of the preimage of F in the extended building of G. Its neutral component is denoted by $\mathbf{G}_F^\circ$ so that the reduction $\overline{\mathbf{G}}_F^\circ$ over $\mathbb{F}_q$ is a connected smooth algebraic group. The subgroup $\mathbf{G}_F^\circ(\mathfrak{O})$ of G is compact open. Let

$$\mathrm{I}_F := \{g \in \mathbf{G}_F^\circ(\mathfrak{O}) : (g \mod \varpi) \in \text{ unipotent radical of } \overline{\mathbf{G}}_F^\circ\}.$$

The I_F are compact open pro-p subgroups in G which satisfy $\mathrm{I}_C = \mathrm{I}$, $\mathrm{I}_{x_0} = \mathrm{K}_1$,

$$g\mathrm{I}_F g^{-1} = \mathrm{I}_{gF} \qquad \text{for any } g \in \mathrm{G}, \tag{3.1}$$

and

$$\mathrm{I}_{F'} \subseteq \mathrm{I}_F \qquad \text{whenever } F' \subseteq \overline{F}. \tag{3.2}$$

For any smooth $\mathbf{k}$-representation $\mathbf{V}$ of G, the family $\{\mathbf{V}^{\mathrm{I}_F}\}_F$ of subspaces of I_F-fixed vectors in $\mathbf{V}$ forms a G-equivariant coefficient system on $\mathscr{X}$ which we will denote by $\underline{\underline{\mathbf{V}}}$ ([12] II.2). Let $\mathbf{X}$ be the space $\mathbf{k}[\mathrm{I}\backslash\mathrm{G}]$ of $\mathbf{k}$-valued functions with finite support in $\mathrm{I}\backslash\mathrm{G}$. It is a natural left H-module. Let $\underline{\underline{\mathbf{X}}}$ be the associated coefficient system. The corresponding augmented oriented chain complex

$$0 \longrightarrow C_c^{or}(\mathscr{X}_{(d)}, \underline{\underline{\mathbf{X}}}) \xrightarrow{\partial} \dots \xrightarrow{\partial} C_c^{or}(\mathscr{X}_{(0)}, \underline{\underline{\mathbf{X}}}) \xrightarrow{\epsilon} \mathbf{X} \longrightarrow 0 \tag{3.3}$$

is a complex of G-representations and of left H-modules.

As noticed in [7, Remark 3.2], the following result is contained in the proof of [12, Theorem II.3.1]:

Theorem 3.1 ([12] Thm. II.3.1). *The complex* (3.3) *is exact.*

Let F be a facet in $\overline{C}$. Extending functions on $\mathbf{G}_F^\circ(\mathfrak{O})$ by zero to G induces a $\mathbf{G}_F^\circ(\mathfrak{O})$-equivariant embedding

$$\mathbf{X}_F := \mathbf{k}[\mathrm{I}\backslash\mathbf{G}_F^\circ(\mathfrak{O})] \hookrightarrow \mathbf{X}$$

and we consider the subalgebra

$$\mathfrak{H}_F := \mathbf{k}[\mathrm{I}\backslash\mathbf{G}_F^\circ(\mathfrak{O})/\mathrm{I}]$$

of the functions in H with support in $\mathbf{G}_F^\circ(\mathfrak{O})$.

Lemma 3.2. *The natural maps of respectively* $(\mathbf{G}_F^\circ(\mathfrak{O}), \mathfrak{H}_{x_0}^{opp})$*-bimodules and* $(\mathbf{G}_F^\circ(\mathfrak{O}), \mathrm{H}^{opp})$*-bimodules*

$$\mathfrak{H}_{x_0} \otimes_{\mathfrak{H}_F} \mathbf{X}_F \to \mathbf{X}_{x_0}^{\mathrm{I}_F} \tag{3.4}$$

$$\mathrm{H} \otimes_{\mathfrak{H}_F} \mathbf{X}_F \to \mathbf{X}^{\mathrm{I}_F} \tag{3.5}$$

are bijective.

Proof. The isomorphism (3.5) is proved in [7, Proposition 4.25]. The proof of the bijectivity of (3.4), is obtained similarly as follows. Let Φ_F^+ denote the set of positive roots that take value zero on F and $\mathcal{D}_F$ the subset of all elements d in $\mathfrak{W}$ such that $d\Phi_F^+ \subseteq \Phi^+$. Choose a lift $\tilde{d} \in \tilde{\mathfrak{W}}$ for each such d. Then it is classical to establish that $\mathfrak{H}_{x_0}$ is a free right $\mathfrak{H}_F$-module with basis $\{\tau_{\tilde{d}}\}_{d\in\mathcal{D}_F}$. Since $\mathfrak{H}_F$ is Frobenius ([10, Thm. 2.4] and [15, Prop 3.7]), it is self-injective: this implies that the $\mathfrak{H}_F$-module $\mathfrak{H}_{x_0}$ is a direct summand of H and the composition $\mathfrak{H}_{x_0} \otimes_{\mathfrak{H}_F} \mathbf{X}_F \to \mathrm{H} \otimes_{\mathfrak{H}_F} \mathbf{X}_F$ is an injective map inducing the natural injection

$$\mathfrak{H}_{x_0} \otimes_{\mathfrak{H}_F} \mathbf{X}_F \to \mathbf{X}_{x_0}^{\mathrm{I}_F}.$$

To prove that it is surjective, we argue (again as in [7, Proposition 4.25]) using the fact that the set of all $\tilde{d}$ for $d \in \mathcal{D}_F$ yields a system of representatives for the double cosets $\mathrm{I}\backslash\mathrm{K}/\mathbf{G}_F^\circ(\mathfrak{O})$ and that $\mathrm{I}\tilde{d}\mathrm{I}_F = \mathrm{I}\tilde{d}\mathrm{I}$. □

We define $\mathcal{P}_F^\dagger$ to be the stabilizer of F in G. For $g \in \mathcal{P}_F^\dagger$, set $\epsilon_F(g) = +1$, respectively -1, if g preserves, respectively reverses, a given orientation of F. For any representation V of $\mathcal{P}_F^\dagger$, we denote by $\mathrm{V} \otimes \epsilon_F$ the space V endowed with the structure of a representation of $\mathcal{P}_F^\dagger$ where the action of $\mathcal{P}_F^\dagger$ is twisted by the character ϵ_F.

For $i \in \{0, ..., d\}$, we fix a (finite) set of representatives $\mathscr{F}_i$ for the G-orbits in $\mathscr{X}_i$ such that every member in $\mathscr{F}_i$ is contained in $\overline{C}$. As explained in [7, 3.3.2]:

Proposition 3.3. *Let* $i \in \{0, ..., d\}$.

i. *The* $(\mathrm{G}, \mathrm{H}^{opp})$*-bimodule* $C_c^{or}(\mathscr{X}_{(i)}, \underline{\underline{\mathbf{X}}})$ *is isomorphic to the direct sum*

$$\bigoplus_{F\in\mathscr{F}_i} \mathrm{ind}_{\mathcal{P}_F^\dagger}^{\mathrm{G}}(\mathbf{X}^{\mathrm{I}_F} \otimes \epsilon_F).$$

ii. *In particular, as a left* $\mathcal{A}_{anti}$*-module, it is isomorphic to a direct sum of modules of the form* $\mathbf{X}^{\mathrm{I}_F}$ *for* $F \in \mathscr{F}_i$.

4. Principal series representations over a ring

Let R be a commutative **k**-algebra. Given a topological group H, we consider R-representations of H that is to say R-modules endowed with a R-linear action of H. If the stabilizers of the points are open in H, then such a representation is called smooth. Let $\mathrm{R}^\times$ be the group of invertible elements in R. A morphism of **k**-algebras $\mathcal{A}_{anti} \to \mathrm{R}$ is called a character. If the image of every element τ_{e^λ}, $\lambda \in \tilde{\mathrm{X}}_*^-(\mathrm{T})$ lies in $\mathrm{R}^\times$, then the character is called regular.

Lemma 4.1. *There is a bijection $\phi \mapsto \overline{\phi}$ from the set of morphisms $\mathrm{T}/\mathrm{T}^1 \mapsto \mathrm{R}^\times$ into the set of regular characters $\mathcal{A}_{anti} \to \mathrm{R}$ such that*

$$\overline{\phi}(\tau_{e^\lambda}) := \phi(\widehat{e^{-\lambda}}) \text{ for all } \lambda \in \tilde{\mathrm{X}}_*^-(\mathrm{T}).$$

We denote the inverse map by $\psi \mapsto \underline{\psi}$.

Proof. We use (2.5) repeatedly to justify the following arguments. The formula given for $\overline{\phi}$ defines a regular character $\mathcal{A}_{anti} \to \mathrm{R}$. Now consider $\psi : \mathcal{A}_{anti} \to \mathrm{R}$ a regular character. Let $t \in \mathrm{T}$ and denote by λ the element in $\tilde{\mathrm{X}}^*(\mathrm{T})$ such that $\mathrm{I}t\mathrm{I} = \mathrm{I}\widehat{e^{\lambda}}\mathrm{I}$. There are $\lambda_1, \lambda_2 \in \tilde{\mathrm{X}}_*^-(\mathrm{T})$ such that $\lambda = \lambda_1 - \lambda_2$ and we set

$$\underline{\psi}(t) := \psi(\tau_{e^{\lambda_1}})\psi(\tau_{e^{\lambda_2}})^{-1}$$

which is well defined because ψ is regular. Furthermore, one checks that it defines a morphism $\mathrm{T} \to \mathrm{R}^\times$ which is trivial on T^1.

□

Consider a regular R-character $\xi : \mathcal{A}_{anti} \to \mathrm{R}$ and the corresponding morphism $\underline{\xi}$ which we see as a map $\mathrm{T} \to \mathrm{R}^\times$ trivial on T^1. Inflating $\underline{\xi}$ to a character of the Borel B, we consider the R-module of the functions $f : \mathrm{G} \to \mathrm{R}$ satisfying $f(bg) = \underline{\xi}(b)f(g)$ for all $g \in \mathrm{G}$, $b \in \mathrm{B}$. It is endowed with a R-linear action of G by right translations namely $(g, f) \mapsto f(\,.\,g)$. We denote by

$$\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi})$$

its smooth part and obtain a smooth R-representation of G.

Lemma 4.2. *Let Ω be a pro-p subgroup of* K*. The space of Ω-invariant functions*

$$(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^\Omega$$

is a free R*-module of finite rank equal to $|\mathrm{B}\backslash\mathrm{G}/\Omega|$.*

Proof. The morphism $\underline{\xi}$ can be seen as a $\mathbf{k}$-representation of T over the $\mathbf{k}$-vector space R and therefore, by the classical theory of $\mathbf{k}$-representations of G, if Ω is a compact open subgroup of G, we have a $\mathbf{k}$-linear isomorphism

$$(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\Omega} \cong \prod_{k \in \mathrm{B}\backslash \mathrm{G}/\Omega} \mathrm{R}^{\mathrm{B}\cap k\Omega k^{-1}}$$

given by the evaluation of $f \in (\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\Omega}$ at all k in a chosen system of representatives of $\mathrm{B}\backslash\mathrm{G}/\Omega$. If Ω is a pro-p subgroup of K then, by Cartan decomposition, one can choose $k \in \mathrm{K}$ and then $\mathrm{B} \cap k\Omega k^{-1} \subseteq \mathrm{B} \cap \mathrm{K} = \mathrm{B} \cap \mathrm{I}'$. But $\mathrm{B} \cap k\Omega k^{-1}$ is a pro-p group so it is contained in $\mathrm{B} \cap \mathrm{I}$ on which $\underline{\xi}$ is trivial. We therefore have a $\mathbf{k}$-linear isomorphism

$$(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\Omega} \cong \prod_{k \in \mathrm{B}\backslash \mathrm{G}/\Omega} \mathrm{R}$$

given by the evaluation of $f \in (\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\Omega}$ at all k in a chosen system of representatives of $\mathrm{B}\backslash\mathrm{G}/\Omega$. This map being obviously R-equivariant, we have proved that $(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\Omega}$ is a free R-module of rank $|\mathrm{B}\backslash\mathrm{G}/\Omega|$.

□

Proposition 4.3. *We have an isomorphism of* R*-representations of* G

$$\xi \otimes_{\mathcal{A}_{anti}} \mathbf{X} \cong \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}).$$

Proof. The proof follows closely the strategy of [11, Prop 11, p.80] which considers the case of the principal series representation induced by the trivial character with values in $\mathbb{Z}$ in the case of $\mathbf{G} = \mathrm{GL}_n$. In the case of unramified principal series representations of GL_n over a ring, and respectively, for more general comparison between compact and parabolic induction over an algebraically closed field with characteristic p, [17, 4.5] and [3, Theorem 3.1, Corollary 3.6] use similar techniques inspired by [11].

Denote by f_1 the I-invariant function in $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi})$ with support BI and value 1_{R} at 1_{G}. Since $\underline{\xi}$ is trivial on $\mathrm{B} \cap \mathrm{I}$, it is well defined by the formula $f_1(bu) = \underline{\xi}(b)$ for all $b \in \mathrm{B}$ and $u \in \mathrm{I}$.

1/ We consider the morphism of $\mathbf{k}$-representations of G

$$\Phi : \mathbf{X} \longrightarrow \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi})$$

sending the characteristic function $\mathrm{char}_{\mathrm{I}}$ of I onto f_1. Let $\lambda \in \tilde{\mathrm{X}}_*^-(\mathrm{T})$. We compute $f_1\tau_{e^\lambda}$. Decompose $\mathrm{I}\widehat{e^\lambda}\mathrm{I}$ into simple right cosets mod I. By Lemma 2.1, one can find such a decomposition $\mathrm{I}\widehat{e^\lambda}\mathrm{I} = \coprod_k \mathrm{I}\widehat{e^\lambda}k$ with k ranging over some finite subset of I^-. Now $f_1\tau_{e^\lambda}$ is I-invariant with support in $\mathrm{BI}^-\widehat{e^\lambda}\mathrm{I}^- = \mathrm{BI}$. To compute its value at 1, one checks that for $k \in \mathrm{I}^-$, we have $1 \in \mathrm{BI}^-\widehat{e^\lambda}k$ if and only if $\mathrm{I}\widehat{e^\lambda}k = \mathrm{I}\widehat{e^\lambda}$ and therefore

$$f_1\tau_{e^\lambda}(1) = [\widehat{e^\lambda}^{-1}.f_1](1) = \underline{\xi}(\widehat{e^{-\lambda}}) = \xi(\tau_{e^\lambda}).$$

We have proved that $\Phi(\tau_{e^\lambda}) = \xi(\tau_{e^\lambda})\Phi(\mathrm{char_I})$. It proves that Φ induces a morphism of R-representations of G

$$\Phi' : \xi \otimes_{\mathcal{A}_{anti}} \mathbf{X} \longrightarrow \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}).$$

2/ We show that f_1 generates $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi})$ as a R-representation of G. Let $f \in \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi})$. Its support is open and compact in $\mathrm{B}\backslash\mathrm{G}$ and by Lemma 2.2, we can suppose (after restricting and translating) that f has support in BU^-. The restriction $f|_{\mathrm{U}^-}$ is locally constant and we can suppose (after restricting the support more) that $f|_{\mathrm{U}^-}$ is constant on some compact open subset $\mathcal{C}$. By Lemma 2.3, this set $\mathcal{C}$ is the finite union of subsets of the form $t^{-n}\mathrm{I}^-t^nu$ for n large enough and $u \in \mathrm{U}^-$, where t is defined in §2.6. Restricting again (and translating), one can suppose that $f|_{\mathrm{U}^-}$ has support $t^{-n}\mathrm{I}^-t^n$ and is constant with value $r \in \mathrm{R}$ on this subset. Now for all $(b,u) \in \mathrm{B}\times\mathrm{I}$, write $u = u^+u_0u^- \in \mathrm{I}^+\mathrm{I}^0\mathrm{I}^-$ and recall that $\xi(u^+u^0) = 1$. We have $(t^nf)(bu) = f(bu^+u_0t^nt^{-n}u^-t^n) = \underline{\xi}(bu^+u_0t^n)r = \underline{\xi}(b)\underline{\xi}(t^n)r$. Therefore, $f = \underline{\xi}(t^n)r\ (t^{-n}.f_1)$ lies in the sub-R-representation generated by f_1. This proves that Φ' is surjective.

3/ To prove that Φ' is injective we follow the strategy of [11, pp.80 & 81]. For $n \in \mathbb{N}$, denote by $\mathbf{Y}_n$ the subspace of $\mathbf{X}$ of the functions with support in $\mathrm{I}t^n\mathrm{K}$.

Fact i. *Consider an element in $\xi \otimes_{\mathcal{A}_{anti}} \mathbf{X}$. There is $n \in \mathbb{N}$ such that it can be written as a sum of elements of the form $r \otimes f$ where $r \in \mathrm{R}$ and $f \in \mathbf{Y}_n$.*

Fact ii. *For $k \in \mathrm{K}$ and $n \in \mathbb{N}$, we have $\mathrm{BI}t^nk \cap \mathrm{BI}t^n \neq \emptyset$ if and only if $\mathrm{I}t^nk = \mathrm{I}t^n$.*

The facts together prove the injectivity of Φ'.

Proof of the facts. The proof of Fact ii in the case of $\mathbf{G} = \mathrm{GL}_n$ given in [11, p. 81] and [11, Proposition 7, p.77] is the same in the general case of a split group. For Fact i, we first notice that the statement of [11, Lemma 12, p.80] holds in the case of a general split group since $(\mathrm{G}, \mathrm{I}', N_G(\mathrm{T}))$ is a generalized Tits system. Therefore, for any $g \in \mathrm{G}$, there is $y \in \mathrm{T}^{++}$ such that $\mathrm{I}y\mathrm{I}g \subseteq \mathrm{IT}^{++}\mathrm{K}$. The element $1 \otimes \mathrm{char}_{\mathrm{I}g}$ can be written $\xi(\tau_y)^{-1} \otimes \mathrm{char}_{\mathrm{I}y\mathrm{I}g}$. Therefore, an element in $\xi \otimes_{\mathcal{A}_{anti}} \mathbf{X}$ can be written as a sum of elements of the form $r' \otimes f'$ where $r' \in \mathrm{R}$ and f' has support in $\mathrm{IT}^{++}\mathrm{K}$. Now let $y' \in \mathrm{T}^{++}$ and $k \in \mathrm{K}$. One can find $n \in \mathbb{N}$ large enough such that $t^ny'^{-1} = y'' \in \mathrm{T}^{++}$. Hence the element $r \otimes \mathrm{char}_{\mathrm{I}y'k}$ can be written $r\xi(\tau_{y''})^{-1} \otimes \mathrm{char}_{\mathrm{I}y''\mathrm{I}y'k}$ and by (2.5) we have $\mathrm{I}y''\mathrm{I}y'k \subseteq \mathrm{I}y''\mathrm{I}y'\mathrm{I}k = \mathrm{I}t^n\mathrm{I}k \subseteq \mathrm{I}t^n\mathrm{K}$.

□

□

Proposition 4.4. *As a right* $\mathrm{R}\otimes_{\mathbf{k}}\mathrm{H}$*-module,* $(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\mathrm{I}}$ *is isomorphic to* $\xi\otimes_{\mathcal{A}_{anti}}\mathrm{H}$.

Proof. By Lemma 4.2, the R-module $(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\mathrm{I}}$ is free of rank $|\mathrm{B}\backslash\mathrm{G}/\mathrm{I}|=|\mathfrak{W}|$. More precisely, for any $w\in\mathfrak{W}$, fix lifts $\tilde{w}\in\tilde{\mathfrak{W}}$ and $\hat{\tilde{w}}\in N_{\mathrm{G}}(\mathrm{T})$ for w, and denote by f_w the function in $(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\mathrm{I}}$ with support $\mathrm{B}\hat{\tilde{w}}\mathrm{I}$ and value 1_{R} at $\hat{\tilde{w}}$. The family $(f_w)_{w\in\mathfrak{W}}$ is a basis for the free R-module $(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\mathrm{I}}$ (see for example [9, 5.5.1] for more detail). By [9, Proposition 5.16], the composition

$$\xi\otimes_{\mathcal{A}_{anti}}\mathrm{H}\longrightarrow(\xi\otimes_{\mathcal{A}_{anti}}\mathbf{X})^{\mathrm{I}}\xrightarrow{\Phi'}(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\mathrm{I}}\tag{4.1}$$

is a surjective morphism of $\mathrm{R}\otimes_{\mathbf{k}}\mathrm{H}$-modules since the image of $1_{\mathrm{R}}\otimes\tau_{\tilde{w}}$ is equal to f_w for all $w\in\mathfrak{W}$. From (2.7) and since ξ is regular, we deduce that $\xi\otimes_{\mathcal{A}_{anti}}\mathrm{H}$ is generated as an R-module by the set of all $1_{\mathrm{R}}\otimes\tau_{\tilde{w}}$ for $w\in\mathfrak{W}$. This is enough to prove that (4.1) is injective. □

By Propositions 4.3 and 4.4, there are natural isomorphisms of R-representations of G

$$\xi\otimes_{\mathcal{A}_{anti}}\mathbf{X}\cong(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\mathrm{I}}\otimes_{\mathrm{H}}\mathbf{X}\cong\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}).\tag{4.2}$$

For any facet F of C containing x_0 in its closure, they induce morphisms of R-representations of $\mathcal{P}_F^{\dagger}$:

$$\xi\otimes_{\mathcal{A}_{anti}}\mathbf{X}^{\mathrm{I}_F}\cong(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\mathrm{I}}\otimes_{\mathrm{H}}\mathbf{X}^{\mathrm{I}_F}\longrightarrow(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\mathrm{I}_F}.\tag{4.3}$$

We identify $\mathbf{k}[\mathrm{T}^0/\mathrm{T}^1]$ with its image in $\mathcal{A}_{anti}$ via $t\mapsto\tau_{t^{-1}}$. The $\mathcal{A}_{anti}$-module R therefore inherits a structure of $\mathbf{k}[\mathrm{T}^0/\mathrm{T}^1]$-module and this structure is given by the restriction of $\underline{\xi}$ to $\mathrm{T}^0/\mathrm{T}^1$. Below, we also consider $\underline{\xi}$ (or rather its restriction to $\mathrm{T}^0/\mathrm{T}^1$) as a character of I' trivial on I.

Lemma 4.5. *Let F be a facet of C containing the hyperspecial vertex x_0 in its closure. There is a natural isomorphism of* $\mathrm{R}[[\mathrm{K}]]$*-modules*

$$\mathrm{R}\otimes_{\mathbf{k}[\mathrm{T}^0/\mathrm{T}^1]}\mathbf{X}_{x_0}\cong\mathrm{Ind}_{\mathrm{I}'}^{\mathrm{K}}(\underline{\xi}).$$

It induces an isomorphism of $\mathrm{R}[[\mathbf{G}_F^{\circ}(\mathfrak{O})]]$*-modules*

$$\mathrm{R}\otimes_{\mathbf{k}[\mathrm{T}^0/\mathrm{T}^1]}\mathbf{X}_{x_0}^{\mathrm{I}_F}\cong(\mathrm{Ind}_{\mathrm{I}'}^{\mathrm{K}}(\underline{\xi}))^{\mathrm{I}_F}.$$

Proof. The first abstract isomorphism is clear because, as representations of K, we have $\mathbf{X}_{x_0}\cong\mathrm{Ind}_{\mathrm{I}'}^{\mathrm{K}}\mathbf{k}[\mathrm{T}^0/\mathrm{T}^1]$ and the tensor product commutes with compact induction. We describe this isomorphism explicitly in order to deduce the second one. Denote by φ the function in $\mathrm{Ind}_{\mathrm{I}'}^{\mathrm{K}}(\underline{\xi})$ with support I' and value 1_{R} at 1_{K}. It is I-invariant.

The following well defined map realizes the first isomorphism of R[[K]]-modules:

$$\begin{array}{ccc} \mathrm{R}\otimes_{\mathbf{k}[\mathrm{T}^0/\mathrm{T}^1]}\mathbf{X}_{x_0} & \longrightarrow & \mathrm{Ind}_{\mathrm{I}'}^{\mathrm{K}}(\underline{\xi}). \\ r\otimes \mathrm{char}_{\mathrm{I}} & \longmapsto & r\varphi. \end{array}$$

Consider the $\mathbf{k}[\mathrm{T}^0/\mathrm{T}^1]$-module $\mathbf{X}_{x_0}^{\mathrm{I}_F}$. It is free with basis the set of all $\mathrm{char}_{\mathrm{I}x\mathrm{I}_F}$ for x ranging over a system of representatives of $\mathrm{I}'\backslash\mathrm{K}/\mathrm{I}_F$. This can be seen by noticing that $\mathrm{I}'x\mathrm{I}_F$ is the disjoint union of all $\mathrm{I}tx\mathrm{I}_F$ for $t\in\mathrm{T}^0/\mathrm{T}^1$. In particular, $\mathbf{X}_{x_0}^{\mathrm{I}_F}$ is projective and therefore injective over the Frobenius algebra $\mathbf{k}[\mathrm{T}^0/\mathrm{T}^1]$: it is a direct summand of $\mathbf{X}_{x_0}$ and we have an injective morphism of $\mathrm{R}[[\mathbf{G}_F^\circ(\mathfrak{O})]]$-modules

$$\mathrm{R}\otimes_{\mathbf{k}[\mathrm{T}^0/\mathrm{T}^1]}\mathbf{X}_{x_0}^{\mathrm{I}_F}\hookrightarrow(\mathrm{R}\otimes_{\mathbf{k}[\mathrm{T}^0/\mathrm{T}^1]}\mathbf{X}_{x_0})^{\mathrm{I}_F}\cong(\mathrm{Ind}_{\mathrm{I}'}^{\mathrm{K}}(\underline{\xi}))^{\mathrm{I}_F}. \tag{4.4}$$

For $x\in\mathrm{K}$, the I_F-invariant function in $(\mathrm{Ind}_{\mathrm{I}'}^{\mathrm{K}}(\underline{\xi}))^{\mathrm{I}_F}$ with support $\mathrm{I}'x\mathrm{I}_F$ and value $r\in\mathrm{R}$ at x is the image by (4.4) of $r\otimes\mathrm{char}_{\mathrm{I}x\mathrm{I}_F}$. Therefore (4.4) is surjective.

□

Proposition 4.6. *If F is a facet of C containing x_0 in its closure, then (4.3) is a chain of isomorphisms*

$$\xi\otimes_{\mathcal{A}_{anti}}\mathbf{X}^{\mathrm{I}_F}\cong(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\mathrm{I}}\otimes_{\mathrm{H}}\mathbf{X}^{\mathrm{I}_F}\cong(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\mathrm{I}_F}. \tag{4.5}$$

of R*-representations of* $\mathcal{P}_F^\dagger$*. In particular, the* R*-module* $\xi\otimes_{\mathcal{A}_{anti}}\mathbf{X}^{\mathrm{I}_F}$ *is free.*

Proof. There is a well defined morphism of R-representations of K

$$\mathrm{Ind}_{\mathrm{I}'}^{\mathrm{K}}(\underline{\xi})\longrightarrow(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\mathrm{K}_1} \tag{4.6}$$

defined by sending the function φ onto the function $f_1\in(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\mathrm{I}}$ (notations of the proof of Lemma 4.5 and Proposition 4.3). It is injective since for $k\in\mathrm{K}$, the equality $\mathrm{BI}k\cap\mathrm{BI}\neq\emptyset$ implies $k\in\mathrm{I}'$. By Iwasawa decomposition and since K_1 is normal in K, the R[[K]]-module $(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\mathrm{K}_1}$ is generated by the K_1-invariant function with support in $\mathrm{BK}_1=\mathrm{BI}$ and value 1_{R} at 1_{G}. This function is in fact equal to f_1 because $\underline{\xi}$ is trivial on I^+. Therefore (4.6) is an isomorphism.

We want to show that the natural morphism of R-representations of $\mathcal{P}_F^\dagger$

$$(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\mathrm{I}}\otimes_{\mathrm{H}}\mathbf{X}^{\mathrm{I}_F}\longrightarrow(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\mathrm{I}_F} \tag{4.7}$$

is bijective. By (3.5), it is enough to show that the natural morphism of $\mathrm{R}[[\mathbf{G}_F^\circ(\mathfrak{O})]]$-modules

$$(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\mathrm{I}}\otimes_{\mathfrak{H}_F}\mathbf{X}_F\longrightarrow(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\mathrm{I}_F} \tag{4.8}$$

is bijective. Since x_0 is in the closure of F, passing to I-invariant vectors in (4.6) yields an isomorphism of right $\mathrm{R}\otimes_{\mathbf{k}}\mathfrak{H}_{x_0}$-modules and therefore of right $\mathrm{R}\otimes_{\mathbf{k}}\mathfrak{H}_F$-modules. Likewise, passing to I_F-invariant vectors yields an isomophism of $\mathrm{R}[[\mathbf{G}_F^\circ(\mathfrak{O})]]$-modules. Therefore we want to show that the natural morphism of $\mathrm{R}[[\mathbf{G}_F^\circ(\mathfrak{O})]]$-modules

$$(\mathrm{Ind}_{\mathrm{I}'}^{\mathrm{K}}(\underline{\xi}))^{\mathrm{I}} \otimes_{\mathfrak{H}_F} \mathbf{X}_F \longrightarrow (\mathrm{Ind}_{\mathrm{I}'}^{\mathrm{K}}(\underline{\xi}))^{\mathrm{I}_F} \tag{4.9}$$

is bijective. Now by Lemma 4.5 and using (3.4), we check that (4.9) can be decomposed into the following chain of isomorphisms

$$(\mathrm{Ind}_{\mathrm{I}'}^{\mathrm{K}}(\underline{\xi}))^{\mathrm{I}} \otimes_{\mathfrak{H}_F} \mathbf{X}_F \simeq \mathrm{R} \otimes_{\mathbf{k}[\mathrm{T}^0/\mathrm{T}^1]} \mathfrak{H}_{x_0} \otimes_{\mathfrak{H}_F} \mathbf{X}_F \cong \mathrm{R} \otimes_{\mathbf{k}[\mathrm{T}^0/\mathrm{T}^1]} \mathbf{X}_{x_0}^{\mathrm{I}_F} \cong (\mathrm{Ind}_{\mathrm{I}'}^{\mathrm{K}}(\underline{\xi}))^{\mathrm{I}_F}.$$

□

Proposition 4.7. *The* R*-module* $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}) \simeq \xi \otimes_{\mathcal{A}_{anti}} \mathbf{X}$ *is free.*

Proof. As an R-module, $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi})$ is the inductive limit of the family $((\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\mathrm{K}_m})_{m\geq 0}$ where we set $\mathrm{K}_0 = \mathrm{I}$. We prove the proposition by first invoking Lemma 4.2 which ensures that $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi})^{\mathrm{K}_0}$ is a free (finitely generated) R-module, and then by proving that for all $m \geq 0$, the quotient $(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\mathrm{K}_{m+1}}/(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\mathrm{K}_m}$ is a free (finitely generated) R-module. For this, let $m \geq 0$. For $g \in \mathrm{G}$, denote by $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{B}g\mathrm{K}_m}(\underline{\xi})$ the subspace of the functions in $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi})$ with support in $\mathrm{B}g\mathrm{K}_m$ and decompose the latter into a finite disjoint union $\mathrm{B}g\mathrm{K}_m = \coprod_{i=1}^{s} \mathrm{B}gk_i\mathrm{K}_{m+1}$. By Lemma 4.2, the map

$$\begin{aligned} (\mathrm{Ind}_{\mathrm{B}}^{\mathrm{B}g\mathrm{K}_m}(\underline{\xi}))^{\mathrm{K}_{m+1}} &\longrightarrow \mathrm{R}^s \\ f &\longmapsto (f(gk_i))_{\mathbf{1}\leq i\leq s} \end{aligned} \tag{4.10}$$

is a R-linear isomorphism. A function $f \in (\mathrm{Ind}_{\mathrm{B}}^{\mathrm{B}g\mathrm{K}_m}(\underline{\xi}))^{\mathrm{K}_{m+1}}$ is K_m-invariant if and only if its image by (4.10) lies in the submodule D of R^s generated by $(1)_{\mathbf{1}\leq i\leq s}$. Since R^s/D is a free R-module and $(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\mathrm{K}_{m+1}}/(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\underline{\xi}))^{\mathrm{K}_m}$ is isomorphic to the direct sum of all $(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{B}g\mathrm{K}_m}(\underline{\xi}))^{\mathrm{K}_{m+1}}/(\mathrm{Ind}_{\mathrm{B}}^{\mathrm{B}g\mathrm{K}_m}(\underline{\xi}))^{\mathrm{K}_m}$ for g in the (finite) set $\mathrm{B}\backslash\mathrm{G}/\mathrm{K}_m$, we obtain the expected result.

□

5. Resolutions for principal series representations of $\mathrm{GL}_n(\mathfrak{F})$ in arbitrary characteristic

Let $\chi : \mathrm{T} \to \mathbf{k}^\times$ a morphism of groups which we suppose to be trivial on T^1. We are interested in the principal series $\mathbf{k}$-representation of G

$$\mathbf{V} = \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi).$$

and the associated coefficient system $\underline{\underline{\mathbf{V}}}$ defined in Section 3. As in Section 4, we consider the sub-$\mathbf{k}$-algebra $\mathcal{A}_{anti}$ of H and we attach to χ the regular $\mathbf{k}$-character $\overline{\chi} : \mathcal{A}_{anti} \to k$ as in Lemma 4.1. Define R to be the localization of $\mathcal{A}_{anti}$ at the kernel of $\overline{\chi}$ and $\xi : \mathcal{A}_{anti} \to \mathrm{R}$ to be the natural morphism of localization. It is a regular character of $\mathcal{A}_{anti}$. There is a $\mathbf{k}$-character

$\overline{\overline{\chi}} : \mathrm{R} \to \mathbf{k}$ satisfying $\overline{\overline{\chi}} \circ \xi = \overline{\chi}$. Since R is a flat $\mathcal{A}_{anti}$-module, tensoring the complex of left $\mathcal{A}_{anti}$-modules (3.3) by R yields an exact sequence of R-representations of G:

$$0 \longrightarrow \xi \otimes_{\mathcal{A}_{anti}} C_c^{or}(\mathscr{X}_{(d)}, \underline{\underline{\mathbf{X}}}) \to \ldots \to \xi \otimes_{\mathcal{A}_{anti}} C_c^{or}(\mathscr{X}_{(0)}, \underline{\underline{\mathbf{X}}}) \to \xi \otimes_{\mathcal{A}_{anti}} \mathbf{X} \longrightarrow 0 \tag{5.1}$$

Suppose that $\mathbf{G} = \mathrm{GL}_n$ for $n \geq 1$. Then for any $i \in \{0, \ldots, d\}$, we can choose the facets in $\mathscr{F}_i$ to contain x_0 in their closure. Therefore, by Propositions 3.3, 4.6 and 4.7, all the terms of the exact complex (5.1) are free R-modules. The complex splits as a complex of R-modules and it remains exact after tensoring by the $\mathbf{k}$-character $\overline{\overline{\chi}}$ of R. But $\overline{\overline{\chi}} \otimes_{\mathrm{R}} \xi$ is isomorphic to the space $\mathbf{k}$ endowed with the structure of $\mathcal{A}_{anti}$-module given by $\overline{\chi} : \mathcal{A}_{anti} \to \mathbf{k}$. By Proposition 4.3, this gives a G-equivariant resolution of $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi)$:

$$0 \longrightarrow \overline{\chi} \otimes_{\mathcal{A}_{anti}} C_c^{or}(\mathscr{X}_{(d)}, \underline{\underline{\mathbf{X}}}) \to \ldots \to \overline{\chi} \otimes_{\mathcal{A}_{anti}} C_c^{or}(\mathscr{X}_{(0)}, \underline{\underline{\mathbf{X}}}) \to \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi) \longrightarrow 0 \tag{5.2}$$

This complex is isomorphic to the augmented complex associated to the coefficient system on $\mathscr{X}$ denoted by $\overline{\chi} \otimes_{\mathcal{A}_{anti}} \underline{\underline{\mathbf{X}}}$ and defined by $F \longmapsto \overline{\chi} \otimes_{\mathcal{A}_{anti}} \mathbf{X}^{\mathrm{I}_F}$ for any facet in $\mathscr{X}$. By Proposition 4.6, $\overline{\chi} \otimes_{\mathcal{A}_{anti}} \underline{\underline{\mathbf{X}}}$ is isomorphic to $\underline{\underline{\mathbf{V}}}$. Therefore, the exact complex (5.2) is isomorphic to the complex (1.2) and we have proved Theorem 1.1. Note that by Proposition 3.3, the exact resolution (1.2) is of the form:

$$0 \longrightarrow \bigoplus_{F \in \mathscr{F}_d} \mathrm{ind}_{\mathcal{P}_F^\dagger}^{\mathrm{G}}((\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}} \chi)^{\mathrm{I}_F} \otimes \epsilon_F) \to \ldots \to \bigoplus_{F \in \mathscr{F}_0} \mathrm{ind}_{\mathcal{P}_F^\dagger}^{\mathrm{G}}((\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi))^{\mathrm{I}_F} \otimes \epsilon_F) \to \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi) \longrightarrow 0. \tag{5.3}$$

Here since $\mathbf{G} = \mathrm{GL}_n$, the semisimple rank is $d = n - 1$.

6. A remark about the Schneider-Vignéras functor

Assume that $\mathrm{G} = \mathrm{GL}_n(\mathbb{Q}_p)$ with $n \geq 2$, and denote by Z its center. We set $\mathrm{B}_0 := \mathrm{B} \cap \mathrm{K}$. It is a subgroup of I'.

Lemma 6.1. *Let F be a facet of the standard apartment $\mathscr{A}$ containing x_0 in its closure. We have $\mathcal{P}_F^\dagger \cap \mathrm{B} \subset \mathrm{B}_0 Z$.*

Proof. Any vertex in the closure of F is of the form $\widehat{e^\lambda} x_0$ for some $\lambda \in \mathrm{X}_*(\mathrm{T})$ and this vertex coincides with x_0 if and only if $\widehat{e^\lambda}$ is in the center Z that is to say if $\lambda \in \mathrm{X}_*(Z)$. Let $b \in \mathcal{P}_F^\dagger \cap \mathrm{B}$. There is $\lambda_1 \in \mathrm{X}_*(\mathrm{T})$ such that $\widehat{e^{\lambda_1}} x_0 \in F$ and $b x_0 = \widehat{e^{\lambda_1}} x_0$. Therefore $b \in \widehat{e^{\lambda_1}} \mathrm{K} Z \cap \mathrm{B} = \widehat{e^{\lambda_1}} \mathrm{B}_0 Z$. Write $b = \widehat{e^{\lambda_1}} u z$ with $u \in \mathrm{B}_0$ and $z \in Z$. Inductively, we construct a sequence $(\lambda_m)_{m \geq 1}$ in $\mathrm{X}_*(\mathrm{T})$ such that $\widehat{e^{\lambda_m}} x_0 \in F$ and $b \widehat{e^{\lambda_m}} x_0 = \widehat{e^{\lambda_{m+1}}} x_0$. It implies $\widehat{e^{\lambda_1 - \lambda_{m+1}}} u \widehat{e^{\lambda_m}} \in \mathrm{K} Z$. Looking at the diagonal of this element, we find $\lambda_{m+1} = \lambda_1 + \lambda_m \mod \mathrm{X}_*(Z)$ and therefore $\lambda_m = m \lambda_1$

mod $X_*(Z)$ for any $m \geq 1$. If $\lambda_1 \notin X_*(Z)$, then the family of all $\widehat{e^{\lambda_m}} x_0$ is infinite: we obtain a contradiction. Therefore $b \in B_0 Z$.

$\square$

We can identify $\mathfrak{W}$ with a subgroup of G and $\mathfrak{W}$ yields a system of representatives of the double cosets $I \backslash G/B$. For any $i \in \{0, ..., n-1\}$, choose the facets in $\mathscr{F}_i$ to contain x_0 in their closure. For $F \in \mathscr{F}_i$, we choose a system of representatives of $\mathcal{P}_F^\dagger \backslash G/B$ in $\mathfrak{W}$. For $w \in \mathfrak{W}$, we can apply Lemma 6.1. to the facet $w^{-1}F$ of $\mathscr{A}$.

Let $\chi : T \to \mathbf{k}^\times$ a morphism of groups which is trivial of T^1. Restricting (5.3) to a complex of $\mathbf{k}$-representations of B, we obtain an exact complex:

$$0 \to \bigoplus_{\substack{F \in \mathscr{F}_{n-1},\\ w \in \mathcal{P}_F^\dagger \backslash G/B}} \mathrm{ind}^{B}_{w^{-1}\mathcal{P}_F^\dagger w \cap B} (w \star ((\mathrm{Ind}_B^G(\chi))^{I_F} \otimes \epsilon_F)) \to \dots$$

$$\dots \to \bigoplus_{\substack{F \in \mathscr{F}_{0},\\ w \in \mathcal{P}_F^\dagger \backslash G/B}} \mathrm{ind}^{B}_{w^{-1}\mathcal{P}_F^\dagger w \cap B} (w \star ((\mathrm{Ind}_B^G(\chi))^{I_F} \otimes \epsilon_F)) \to \mathrm{Ind}_B^G(\chi)|_B \to 0$$

where $w \star ((\mathrm{Ind}_B^G(\chi))^{I_F} \otimes \epsilon_F)$ denotes the space $(\mathrm{Ind}_B^G(\chi))^{I_F} \otimes \epsilon_F$ with the group $w^{-1}\mathcal{P}_F^\dagger w \cap B$ acting through the homomorphism $w^{-1}\mathcal{P}_F^\dagger w \cap B \xrightarrow{w\,.\,w^{-1}} \mathcal{P}_F^\dagger$. Therefore, applying Lemma 6.1, there exist smooth $\mathbf{k}$-representations V_0, ..., V_{n-1} of $B_0 Z$ and an exact resolution of the restriction to B of $\mathrm{Ind}_B^G(\chi)$ of the form:

$$\mathcal{I}_\bullet : 0 \longrightarrow \mathrm{ind}^B_{B_0 Z}(V_{n-1}) \xrightarrow{\partial_{n-1}} \dots \longrightarrow \mathrm{ind}^B_{B_0 Z}(V_0) \xrightarrow{\partial_0} \mathrm{Ind}_B^G(\chi)|_B \longrightarrow 0 \tag{6.1}$$

From now on, $\mathbf{k}$ has characteristic p. As noted by Zàbràdi in [19, §4], the argument of [13, Lemma 11.8] generalizes to the case of $\mathrm{GL}_n(\mathbb{Q}_p)$. Therefore, we can compute the image of $\mathrm{Ind}_B^G(\chi)|_B$ by the universal δ-functor $V \mapsto D^i(V)$, $i \geq 0$, defined in [13] using the cohomology of the complex $D(\mathcal{I}_\bullet)$: for $i \geq 0$

$$D^i(\mathrm{Ind}_B^G(\chi)|_B) = h^i(D(\mathrm{ind}^B_{B_0 Z}(V_0)) \xrightarrow{D(\partial_0)} D(\mathrm{ind}^B_{B_0 Z}(V_1)) \to ... \xrightarrow{D(\partial_{n-1})} D(\mathrm{ind}^B_{B_0 Z}(V_{n-1})) \to 0 \to 0...)$$

By [13, Remark 2.4, i], the map $D(\partial_{n-1})$ is surjective. Therefore, we have proved that

$$D^i(\mathrm{Ind}_B^G(\chi)|_B) = 0 \text{ for all } i \geq n-1.$$

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