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Appl. Comput. Harmon. Anal. 14 (2003) 107–132
Applied and Computational Harmonic Analysis
www.elsevier.com/locate/acha

Coarse quantization of highly redundant time–frequency representations of square-integrable functions

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Received 27 December 2001; revised 27 February 2003; accepted 7 March 2003
Communicated by Dominique Picard

Abstract

We introduce a family of coarse quantization algorithms for heavily oversampled Gabor expansions of certain classes of functions in $L^2(\mathbb{R})$. These algorithms, which we call the TF$\Sigma\Delta$ quantization algorithms, are inspired by sigma–delta modulation, a widely implemented coarse quantization scheme for oversampled bandlimited functions. We show that the TF$\Sigma\Delta$ algorithms produce weak type approximations where modulation spaces $M_m^{1,1}$ with suitable weight functions m are the appropriate test function spaces. We also show that the TF$\Sigma\Delta$ algorithms are translation invariant up to some uniform correction.

Keywords: Coarse quantization; Redundant time–frequency representations; Oversampling; Weyl–Heisenberg frames; Gabor frames; Windowed Fourier frames; Sigma–delta quantization

1. Introduction

In this paper we introduce a family of algorithms to 'coarsely quantize' redundant time–frequency representations of certain classes of functions in $L^2(\mathbb{R})$. By *quantization* we understand the reduction of the continuous range of the coefficients to a discrete, possibly finite set. More precisely, given an expansion of the form

$$f = \sum_{\lambda\in\Lambda} f_\lambda \varphi_\lambda, \tag{1.1}$$

where $f_\lambda \in \mathbb{C}$ and Λ is a countable set, a quantization algorithm will produce a sequence $(q_\lambda)_{\lambda\in\Lambda}$ that takes values in some discrete set D such that $\tilde{f} = \sum_{\lambda\in\Lambda} q_\lambda \varphi_\lambda$ is an approximation to the function f in some suitable norm.

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1063-5203/03/$ – see front matter
doi:10.1016/S1063-5203(03)00020-4

There are two different approaches to quantization: *fine quantization* and *coarse quantization*. Given an expansion as in (1.1), one way to quantize the coefficients f_λ is to replace f_λ^R and f_λ^I, the real and imaginary parts of f_λ, respectively, by $q_\lambda^R := \delta\,\mathrm{round}(f_\lambda^R/\delta)$ and $q_\lambda^I := \delta\,\mathrm{round}(f_\lambda^I/\delta)$. Here δ is the *step size* of the quantizer. In this case, setting $q_\lambda = q_\lambda^R + i q_\lambda^I$, we have $\sup|f_\lambda - q_\lambda| = \sqrt{2}\delta$. Therefore by decreasing the step size, one can make $|f_\lambda - q_\lambda|$ arbitrarily small, and thus the approximation error diminishes as δ approaches zero. Such algorithms are usually called *fine quantization algorithms*.

An alternative approach exists if the expansion is highly redundant. In this case one can replace the coefficients f_λ with coarsely quantized values q_λ, i.e., $q_\lambda \in D$ where D has just a few elements, and still have a good approximation. Instead of controlling the individual differences $|f_\lambda - q_\lambda|$, such an algorithm aims to produce q_λ so that the approximation error $\|f - \sum_{\lambda\in\Lambda} q_\lambda \varphi_\lambda\|$ is small. Moreover, the algorithm is constructed such that the approximation error diminishes as the redundancy of the expansion increases. Such algorithms are called *coarse quantization algorithms*. Note that a coarse quantization algorithm exploits the redundancy of the expansion to compensate for the coarseness of the quantization.

An important property of coarse quantization algorithms is that they are more efficient in utilizing the redundancy of an expansion. For example, consider a function, f, that is sufficiently well localized in both time and frequency. A heuristic argument in [2] shows that quantizing the Gabor frame expansion of f using a fine quantization algorithm with a fixed step size δ yields an approximation $\tilde{f}$ with $\|f - \tilde{f}\| = \mathrm{O}(A^{-1/2})$. Here A is the frame bound of the (tight) Gabor frame (and thus a measure of the redundancy of the expansion). In [11] it is shown that the asymptotic behavior of the approximation error is $\mathrm{O}(A^{-1})$ for tight Gabor frames if the frame bound A is an integer. In this paper we introduce a family of coarse quantization algorithms which yield weak-type approximations, where the approximation error is $\mathrm{O}(A^{-k})$ for a kth-order scheme.

One may of course argue that instead of increasing the redundancy of the expansion, one can increase the resolution of the quantizer, i.e., decrease the step size, δ, to obtain a better approximation. Like increasing redundancy, this would correspond to using more bits per critical sampling interval (or rectangle in the case of Gabor frames). Indeed, it can be easily shown that fine quantization algorithms achieve exponential precision, i.e., the approximation error decays exponentially as the bit rate—the number of bits used to quantize each sample—increases. This is usually not the case for coarse quantization algorithms. Despite this shortcoming, coarse quantization algorithms are widely implemented to quantize oversampled bandlimited functions (functions with compactly supported Fourier transforms) mainly because of their superior robustness properties. Detailed discussions about robustness properties of particular coarse quantization schemes can be found in [3,9,13]. On contrary, [12] shows the strong dependence of the numerical stability of fine quantization algorithms to computational accuracy in the case of discrete windowed Fourier expansions. In this paper we do not discuss robustness properties of $\mathrm{TF}\Sigma\Delta$ schemes in detail; however we should note that these algorithms exhibit similar robustness properties to sigma–delta schemes by construction.

Throughout the paper we will be discussing methods to coarsely quantize Weyl–Heisenberg frame[1] expansions of functions in $L^2(\mathbb{R})$. *Weyl–Heisenberg frames* are frames of $L^2(\mathbb{R})$ that are generated by

[1] These frames are also called *Gabor frames* and *windowed Fourier frames*.

shifting a fixed function $\varphi \in L^2(\mathbb{R})$ along a lattice $\Gamma = \tau_0\mathbb{Z} \times \xi_0\mathbb{Z}$ in the time–frequency plane: For $\varphi_{n,m}(t) := \varphi(t - n\tau_0)\mathrm{e}^{im\xi_0 t}$, the $\{\varphi_{n,m}\colon n, m \in \mathbb{Z}\}$ constitute a frame in $L^2(\mathbb{R})$; in other words

$$A\|f\|^2 \leqslant \sum_{n,m}\bigl|\langle f, \varphi_{n,m}\rangle\bigr|^2 \leqslant B\|f\|^2$$

for all $f \in L^2(\mathbb{R})$, where the frame bounds $A > 0$, $B < \infty$ are independent from f. (Here $\langle f, \varphi_{n,m}\rangle := \int f(t)\overline{\varphi_{n,m}(t)}$.) For a detailed discussion, consult [2,5,6,10]. For the sake of convenience we denote by (φ, τ_0, ξ_0) the collection $\{\varphi_{n,m}\}_{(n,m)\in\mathbb{Z}^2}$ with $\varphi_{n,m}(t)$ as defined above. As is well known, if (φ, τ_0, ξ_0) is a Weyl–Heisenberg frame, the function $\tilde{\varphi} := U^{-1}\varphi$, where $Uf := \sum_{n,m}\langle f, \varphi_{n,m}\rangle\varphi_{n,m}$, also generates a Weyl–Heisenberg frame $(\tilde{\varphi}, \tau_0, \xi_0)$ with frame bounds B^{-1} and A^{-1}, and one has $f = \sum_{n,m}\langle f, \varphi_{n,m}\rangle\tilde{\varphi}_{n,m}$. The frame $(\tilde{\varphi}, \tau_0, \xi_0)$ is called the *dual* of (φ, τ_0, ξ_0). If (φ, τ_0, ξ_0) is a tight[2] frame with frame bound A, $U = \mathrm{Id}\,A$, thus $\tilde{\varphi} = A^{-1}\varphi$ and we have

$$f = \frac{1}{A}\sum\langle f, \varphi_{n,m}\rangle\varphi_{n,m}, \tag{1.2}$$

where equality is in the sense of L^2.

Suppose (φ, τ_0, ξ_0) is a tight Weyl–Heisenberg frame of $L^2(\mathbb{R})$ with the frame bound A where φ is a smooth and well-localized function that is normalized in L^2, $t\varphi \in L^2$, and $\xi\hat{\varphi} \in L^2$. Then it is a standard result [4] that $A > 1$ (necessary to have a frame) and $A = (2\pi)/(\tau_0\xi_0)$.

One can define also the continuous windowed Fourier transform of f with respect to φ by $V_\varphi f(\tau, \xi) := \langle f, \varphi_{\tau,\xi}\rangle$, where $\varphi_{\tau,\xi} = \varphi(t - \tau)\mathrm{e}^{i\xi t}$. Combining this with (1.2) implies

$$V_\varphi f(\tau, \xi) = \frac{1}{A}\sum_{n,m}\langle f, \varphi_{n,m}\rangle\langle\varphi_{n,m}, \varphi_{\tau,\xi}\rangle, \tag{1.3}$$

where the convergence is pointwise as well as in L^2.

Note that (1.2) essentially tells us how to reconstruct f from its frame coefficients $\langle f, \varphi_{n,m}\rangle$. Our goal, as discussed above, is to devise an algorithm to replace the $\langle f, \varphi_{n,m}\rangle$ by some $q_{n,m} \in \{d_1, d_2, \ldots, d_K\}$, with $d_i \in \mathbb{C}$, (i.e., to quantize $c_{n,m}$) such that

$$\tilde{f}_A = \frac{1}{A}\sum q_{n,m}\varphi_{n,m} \tag{1.4}$$

is a 'good' approximation of f in some norm, preferably in L^2-norm.

The algorithms that we consider throughout the paper are inspired by sigma–delta quantization algorithms that are commonly used to coarsely quantize oversampled bandlimited functions [1]. Consider a function f that is bandlimited with bandwidth π, i.e., $\operatorname{supp}\hat{f} \subset [-\pi, \pi]$, and that satisfies $\|f\|_{L^\infty} < 1$. It is well known that f can be stably reconstructed from its sample values $f(n/\lambda)$ where $\lambda > 1$ is fixed; in particular, with g satisfying $\hat{g} \in C^\infty$, $\hat{g}(\xi) = 1/\sqrt{2\pi}$ for $\xi \in [-\pi, \pi]$, and $\hat{g}(\xi) = 0$ for $|\xi| > \lambda\pi$, one has

$$f(t) = \frac{1}{\lambda}\sum_{n\in\mathbb{Z}} f\left(\frac{n}{\lambda}\right)g\left(t - \frac{n}{\lambda}\right). \tag{1.5}$$

[2] A frame with frame bounds A and B is called *tight* if $A = B$.

Sigma–delta algorithms generate sequences $(q_n)_{n\in\mathbb{Z}}$, $q_n \in \{-1, 1\}$, such that replacing the sample value $f(n/\lambda)$ in (1.5) by q_n gives an L^∞-approximation of f. This is achieved by constructing q_n such that the running sums of q_n track the running sums of the sample values $f(n/\lambda)$ uniformly. Many different schemes exist; typically the q_n are constructed recursively. For example, a first-order sigma–delta quantizer generates the q_n via the following recursion:

$$v_n - v_{n-1} = f_n^\lambda - q_n^\lambda, \quad q_n^\lambda = \operatorname{sign}\left(v_{n-1} + f_n^\lambda\right). \tag{1.6}$$

In this case, one can show that [3]:

$$|v_n| < 1 \quad \text{for all } n, \text{ if } v_0 \in (-1, 1), \tag{1.7}$$

$$\|f - \tilde{f}\|_{L^\infty} \leqslant \frac{1}{\lambda}\|g'\|_{L^1}. \tag{1.8}$$

In fact, this bound can be improved; [7] contains a proof that the error can be bounded pointwise by $C\lambda^{-4/3+\eta}$ where C depends on η and on the value of the derivative of the original function at the corresponding point.

A kth-order sigma–delta quantizer can be defined replacing the first-order backward difference operator in (1.6) by a kth-order backward difference operator and adjusting the rule that determines q_n such that the $|v_n|$ stay uniformly bounded. In this case, the kth-order running sums of q_n track the kth-order running sums of $f(n/\lambda)$ uniformly, i.e.,

$$\left|\sum_{m_{k-1}=N_k}^{M_k} \cdots \sum_{m_1=N_2}^{m_2} \sum_{n=N_1}^{m_1} f\left(\frac{n}{\lambda}\right) - \sum_{m_{k-1}=N_k}^{M_k} \cdots \sum_{m_1=N_2}^{m_2} \sum_{n=N_1}^{m_1} q_n\right| < C,$$

where the value of the constant C does not depend on $N_1, \ldots, N_k$, M_k, or $f(n/\lambda)$. Thus one can prove that the L^∞ approximation error is $\mathrm{O}(\lambda^{-k})$. Detailed discussions of higher-order schemes can be found in [3,13].

In Section 2, we introduce a coarse quantization algorithm for tight Weyl–Heisenberg expansions, called the TF$\Sigma\Delta$ *quantization algorithm*. Given the frame coefficients $\langle f, \varphi_{n,m}\rangle$ of a function f, this algorithm produces $q_{n,m} \in \{q^R + iq^I\colon\ q^R, q^I \in \{-3, -1, 1, 3\}\}$. When (φ, τ_0, ξ_0) is a tight Weyl–Heisenberg frame with frame bound A, we show that for functions f that satisfy $|V_\varphi f| \leqslant 1$,

$$\tilde{f} = A^{-1}\sum_{n,m} q_{n,m}\varphi_{n,m} \tag{1.9}$$

yields a weak-type approximation where the modulation spaces $M_m^{1,1}$ with suitable weight functions m are the natural test function spaces. Moreover, we show that the resulting approximation error is $\mathrm{O}(A^{-1})$. Like the case with the sigma–delta schemes, this is achieved by producing $q_{n,m}$ such that the running sums of $q_{n,m}$ track the running sums of $\langle f, \varphi_{n,m}\rangle$ uniformly.

In Section 3, we show that the TF$\Sigma\Delta$ quantization algorithm is translation invariant up to some uniform adjustment. In Section 5, we define the higher-order TF$\Sigma\Delta$ schemes, and show that the approximation error is $\mathrm{O}(A^{-k})$ if the approximation is produced by a kth-order scheme (where k is a positive integer). Sections 4 and 6 present numerical experiments for the first-order and second-order TF$\Sigma\Delta$ schemes, respectively.

2. The time–frequency sigma–delta (TF$\Sigma\Delta$) quantization algorithm

Let (φ, τ_0, ξ_0) be a tight Weyl–Heisenberg frame with frame bound A. We will consider functions $f \in L^2(\mathbb{R})$ that satisfy $|\langle f, \varphi_{n,m}\rangle| \leqslant 1$ for all integers n and m. Denote the collection of such functions by $\mathcal{B}^\varphi$. Let $c^R_{n,m}$ and $c^I_{n,m}$ be the real and imaginary parts of the frame coefficients $c_{n,m} := \langle f, \varphi_{n,m}\rangle$, respectively. In this paper we consider algorithms to quantize the frame expansions of certain functions. The frame coefficients are generally complex numbers and the algorithms quantize real and imaginary parts of these numbers separately; moreover, the algorithms that we consider are recursive and the recursion relations that are used to quantize the real and imaginary parts of the frame coefficients are identical. Thus, to simplify the notation, we will use the superscript S whenever we have an equation, a system of equations, or an expression that is valid for both $S = $ "R" and $S = $ "I".

Now consider the recursions:

$$\begin{aligned} u^S_{n,m} - u^S_{n-1,m} &= c^S_{n,m} - p^S_{n,m}, \quad p^S_{n,m} = \operatorname{sign}\bigl(u^S_{n-1,m} + c^S_{n,m}\bigr),\\ v^S_{n,m} - v^S_{n,m-1} &= u^S_{n,m} - r^S_{n,m}, \quad r^S_{n,m} = \operatorname{sign}\bigl(v^S_{n,m-1} + u^S_{n,m}\bigr), \end{aligned} \tag{2.10}$$

where

$$\operatorname{sign}(x) = \begin{cases} 1 & x > 0,\\ -1 & x \leqslant 0. \end{cases}$$

The difference equations given in (2.10) will be used to quantize the real part ($S = $ "R") and imaginary part ($S = $ "I") of the frame coefficients $c_{n,m}$. Denote the sequences $(u^S_{n,m})$, $(v^S_{n,m})$ by u^S and v^S, respectively. Similarly p^S and r^S will denote $(p^S_{n,m})$ and $(r^S_{n,m})$, respectively. Note that

$$\bigl(\Delta_1\Delta_2 v^R\bigr)_{n,m} = c^R_{n,m} - \bigl(p^R_{n,m} + \bigl(\Delta_1 r^R\bigr)_{n,m}\bigr), \tag{2.11}$$

and

$$\bigl(\Delta_1\Delta_2 v^I\bigr)_{n,m} = c^I_{n,m} - \bigl(p^I_{n,m} + \bigl(\Delta_1 r^I\bigr)_{n,m}\bigr), \tag{2.12}$$

where $(\Delta_1 v)_{n,m} := v_{n,m} - v_{n-1,m}$ and $(\Delta_2 v)_{n,m} := v_{n,m} - v_{n,m-1}$. We will define the sequences q^R and q^I by $q^R_{n,m} := p^R_{n,m} + (\Delta_1 r^R)_{n,m}$ and $q^I_{n,m} := p^I_{n,m} + (\Delta_1 r^I)_{n,m}$, respectively. Let $c := (c_{n,m})_{(n,m)\in\mathbb{Z}^2}$ and define the mapping T_{TF} from $l^2(\mathbb{C})$ to $\mathcal{Q}$ by

$$T_{\mathrm{TF}}(c) = q := q^R + iq^I, \tag{2.13}$$

where $\mathcal{Q}$ denotes the collection of all sequences $(x_{n,m} + iy_{n,m})$ where both $x_{n,m}$ and $y_{n,m}$ take values in $\{-3, -1, 1, 3\}$.

Theorem 1. *Let (φ, τ_0, ξ_0) be a tight Weyl–Heisenberg frame of $L^2(\mathbb{R})$ with frame bound A. Let f be in $\mathcal{B}^\varphi$ and set $q = T_{\mathrm{TF}}(c)$ where $c_{n,m} = \langle f, \varphi_{n,m}\rangle$. Define*

$$\widetilde{F}_A(\tau, \xi) := \frac{1}{A}\sum_{n,m} q_{n,m}\langle \varphi_{n,m}, \varphi_{\tau,\xi}\rangle. \tag{2.14}$$

Suppose φ is chosen such that $(1 + |\xi| + |\tau\xi|)\Phi(\tau, \xi)$, $(1 + |\tau|)\partial_1\Phi(\tau, \xi)$, $\xi\partial_2\Phi(\tau, \xi)$, and $\partial_1\partial_2\Phi(\tau, \xi)$ are in $L^1(\mathbb{R}^2)$, where $\Phi(\tau, \xi) := \langle \varphi, \varphi_{\tau,\xi}\rangle$ and $\partial_i\Phi$ is the ith partial derivative of Φ. Then

$$\bigl|V_\varphi f(\tau, \xi) - \widetilde{F}_A(\tau, \xi)\bigr| \leqslant \frac{1}{A}\bigl(C_{\varphi,1} + |\tau| C_{\varphi,2}\bigr), \tag{2.15}$$

where $C_{\varphi,1}$ and $C_{\varphi,2}$ depend only on φ. We will call $\widetilde{F}_A$ the time–frequency sigma–delta approximation of $V_\varphi f$.

Before we proceed to prove this theorem we observe that (1.7) implies:

Lemma 1. *For each u^R, v^R, u^I, v^I, defined as in* (2.10) *the l_∞-norm is bounded by* 1.

Proof. Note that u^S (for both $S =$ "R" and $S =$ "I") is the state variable of a first-order sigma–delta quantizer, described in (1.6), where the sequence $(c^S_{n,m})$ is the input and the sigma–delta quantization is over the index n. Since $f \in \mathcal{B}^\varphi$, $|c^S_{n,m}|$ is bounded by 1. Then by (1.7) $u^S_{n,m}$ is bounded by 1. Similarly, $v^S_{n,m}$ are the state variables of a first-order sigma–delta quantizer with the input $(u^S_{n,m})$, where sigma–delta quantization is over m; again since $u^S_{n,m}$ is bounded by 1, so is $v^S_{n,m}$. $\square$

Now, we are ready to prove Theorem 1.

Proof of Theorem 1. Let us write the error term

$$V_\varphi f(\tau,\xi) - \widetilde{F}_A(\tau,\xi) = \frac{1}{A}\sum_{n,m}(c_{n,m} - q_{n,m})\langle\varphi_{n,m},\varphi_{\tau,\xi}\rangle, \tag{2.16}$$

$$= \frac{1}{A}\sum_{n,m}(\Delta_1\Delta_2 v)_{n,m}\langle\varphi_{n,m},\varphi_{\tau,\xi}\rangle, \tag{2.17}$$

$$= \frac{1}{A}\sum_{n,m} v_{n,m}\big(\bar{\Delta}_2\bar{\Delta}_1\langle\varphi_{n,m},\varphi_{\tau,\xi}\rangle\big), \tag{2.18}$$

where, for any $x = (x_{n,m})$, $(\bar{\Delta}_1 x)_{n,m} := x_{n,m} - x_{n+1,m}$, and $(\bar{\Delta}_2 x)_{n,m} := x_{n,m} - x_{n,m+1}$. (To avoid unnecessarily complicated notation, sometimes we will write $(\Delta_i x_{n,m})$ instead of $(\Delta_i x)_{n,m}$, and $(\bar{\Delta}_i x_{n,m})$ instead of $(\bar{\Delta}_i x)_{n,m}$.) The first equality is obvious, the second comes directly from the quantization algorithm by setting

$$v_{n,m} = v^R_{n,m} + i v^I_{n,m}. \tag{2.19}$$

The third equality is the result of summing (2.17) by parts; note that the boundary values disappear since $\langle\varphi_{n,m},\varphi_{\tau,\xi}\rangle = \mathrm{e}^{-in\tau_0(\xi-m\xi)}\Phi(\tau - n\tau_0, \xi - m\xi_0)$ vanishes as n and/or m tends to infinity for any τ,ξ. Let us define I by $I := \bar{\Delta}_2\bar{\Delta}_1\langle\varphi_{n,m},\varphi_{\tau,\xi}\rangle$. Then

$$I = \bar{\Delta}_2\bar{\Delta}_1\big(\mathrm{e}^{-in\tau_0(\xi-m\xi)}\Phi(\tau - n\tau_0, \xi - m\xi_0)\big), \tag{2.20}$$

$$= \mathrm{e}^{-i\tau\xi}\,\bar{\Delta}_2\bar{\Delta}_1\Omega_{\tau,\xi}(\tau - n\tau_0, m\xi_0) \tag{2.21}$$

after defining $\Omega_{\tau,\xi}(t,z) := \mathrm{e}^{iz\tau}\mathrm{e}^{it(\xi-z)}\Phi(t,\xi - z)$. Since $\Omega_{\tau,\xi}$ is smooth, we can rewrite (2.21) as

$$I = \mathrm{e}^{-i\tau\xi}\left(\bar{\Delta}_2\int_{\tau-(n+1)\tau_0}^{\tau-n\tau_0}\partial_1\Omega_{\tau,\xi}(t,m\xi_0)\,\mathrm{d}t\right)$$

$$= \mathrm{e}^{-i\tau\xi}\int_{\tau-(n+1)\tau_0}^{\tau-n\tau_0}\big[\partial_1\Omega_{\tau,\xi}(t,m\xi_0) - \partial_1\Omega_{\tau,\xi}\big(t,(m+1)\xi_0\big)\big]\,\mathrm{d}t$$

$$= \mathrm{e}^{-i\tau\xi} \int_{\tau-(n+1)\tau_0}^{\tau-n\tau_0} \int_{(m+1)\xi_0}^{m\xi_0} \partial_2\partial_1 \Omega_{\tau,\xi}(t,z)\,\mathrm{d}t\,\mathrm{d}z. \tag{2.22}$$

Substituting (2.22) into (2.18) we obtain

$$V_\varphi f(\tau,\xi) - \widetilde{F}_A(\tau,\xi) = \frac{1}{A}\sum_{n,m} v_{n,m}\mathrm{e}^{-i\tau\xi} \int_{(\tau-n+1)\tau_0}^{\tau-n\tau_0} \int_{(m+1)\xi_0}^{m\xi_0} \partial_2\partial_1 \Omega_{\tau,\xi}(t,z)\,\mathrm{d}t\,\mathrm{d}z, \tag{2.23}$$

which yields

$$\begin{aligned}\left|V_\varphi f(\tau,\xi) - \widetilde{F}_A(\tau,\xi)\right| &\leqslant \frac{1}{A}\sum_{n,m}\left|v_{n,m}\mathrm{e}^{-i\tau\xi}\right| \int_{(\tau-n+1)\tau_0}^{\tau-n\tau_0} \int_{(m+1)\xi_0}^{m\xi_0} \left|\partial_2\partial_1 \Omega_{\tau,\xi}(t,z)\right|\mathrm{d}t\,\mathrm{d}z \\ &\leqslant \frac{\sqrt{2}}{A}\left\|\partial_2\partial_1\Omega_{\tau,\xi}(t,z)\right\|_{L^1(\mathbb{R}^2)}. \end{aligned} \tag{2.24}$$

Note that in the second inequality we used Lemma 1 to bound $\|v\|_{l^\infty}$ by $\sqrt{2}$. We complete the proof by estimating the L^1-norm of $\partial_2\partial_1\Omega_{\tau,\xi}(t,z)$: For the sake of convenience, define $\Gamma(t,z) := \mathrm{e}^{itz}\Phi(t,z)$, and note that $\Omega_{\tau,\xi}(t,z) = \mathrm{e}^{iz\tau}\Gamma(t,\xi - z)$. We then observe

$$\left\|\partial_2\partial_1\Omega_{\tau,\xi}(t,z)\right\|_{L^1(\mathbb{R}^2)} \leqslant \|\partial_2\partial_1\Gamma\|_{L^1(\mathbb{R}^2)} + |\tau|\|\partial_1\Gamma\|_{L^1(\mathbb{R}^2)},$$

which yields the desired bound by setting

$$C_{\varphi,1} := \sqrt{2}\|\partial_2\partial_1\Gamma\|_{L^1(\mathbb{R}^2)} \tag{2.25}$$

and

$$C_{\varphi,2} := \sqrt{2}\|\partial_1\Gamma\|_{L^1(\mathbb{R}^2)}. \qquad \square \tag{2.26}$$

Remark 1. Note that (2.15) still holds up to some small correction term if the frame (φ,τ_0,ξ_0) is "almost tight." A frame is said to be *almost tight* if the ratio of the frame bounds is close to 1. Suppose (φ,τ_0,ξ_0) is a frame with frame bounds A and B. If we denote the quantity $B/A - 1$ by r, the windowed Fourier transform $V_\varphi f$ of any function $f \in L^2(\mathbb{R})$ can be written as

$$V_\varphi f(\tau,\xi) = \frac{2}{(2+r)A}\sum\langle f,\varphi_{n,m}\rangle\langle\varphi_{n,m},\varphi_{\tau,\xi}\rangle + \langle Rf,\varphi_{\tau,\xi}\rangle, \tag{2.27}$$

where $\|R\| \leqslant r/(2+r)$. In this case, after defining

$$\widetilde{F}_A(\tau,\xi) := \frac{2}{(2+r)A}\sum q_{n,m}\langle\varphi_{n,m}\varphi_{\tau,\xi}\rangle \tag{2.28}$$

we can apply the proof of Theorem 1 to show that

$$\left|V_\varphi f(\tau,\xi) - \widetilde{F}_A(\tau,\xi)\right| \leqslant \frac{2}{(2+r)A}\left(C_{\varphi,1} + |\tau|C_{\varphi,2}\right) + \frac{r}{2+r}. \tag{2.29}$$

Note that to obtain (2.29), we used the fact that $|\langle Rf,\varphi_{\tau,\xi}\rangle| \leqslant r/(2+r)$. Thus, the approximation error $|V_\varphi f(\tau,\xi) - \widetilde{F}_A(\tau,\xi)|$ still has the same asymptotic behavior when $r \approx 0$.

Remark 2. A sufficient condition for $\Phi = V_\varphi\varphi$ to satisfy the smoothness and decay conditions listed in Theorem 1 is that the function φ is in the Schwartz space $\mathcal{S}(\mathbb{R})$.

Remark 3. A natural question to ask is whether the second recursion in (2.10) is essential from a practical point of view, i.e., whether we obtain an approximation using only $p^S_{n,m}$ in (2.10). Our numerical experiments suggest that if the function f is well localized in both time and frequency, then we get a weak type approximation $\tilde{f}$, using only $p^S_{n,m}$, for which the approximation error is $\mathrm{O}(A^{-1/2})$. Determining conditions that f has to satisfy for this approximation to exist is an open problem.

Now we want to raise the question of whether we can approximate f using $\widetilde{F}_A$, and if yes, in what sense. Fix the weight function $m(\tau,\xi) := 1 + |\tau|$ and consider the modulation space $M^{1,1}_m$, i.e.,[3]

$$M^{1,1}_m = \bigl\{g \in L^2(\mathbb{R})\colon\ \bigl(1+|\tau|\bigr)V_\varphi g(\tau,\xi) \in L^1\bigl(\mathbb{R}^2\bigr)\bigr\}. \tag{2.30}$$

Clearly any function $f \in L^2(\mathbb{R})$ defines a linear functional L_f on $M^{1,1}_m$ by $L_f g := \langle f, g\rangle$. By the Parseval identity we also have $L_f g = (2\pi)^{-1}\langle V_\varphi f, V_\varphi g\rangle$. Let $\widetilde{F}_A$ be as above and define $\langle \widetilde{F}_A, V_\varphi g\rangle$ as

$$\langle \widetilde{F}_A, V_\varphi g\rangle := \int \widetilde{F}_A(\tau,\xi)\overline{V_\varphi g(\tau,\xi)}\,\mathrm{d}\tau\,\mathrm{d}\xi. \tag{2.31}$$

Note that (2.31) makes sense since

$$\begin{aligned}\left|\int \widetilde{F}_A(\tau,\xi)\overline{V_\varphi g(\tau,\xi)}\,\mathrm{d}\tau\,\mathrm{d}\xi\right| &\leqslant \bigl|\langle V_\varphi f, V_\varphi g\rangle\bigr| + \left|\int (\widetilde{F}_A - V_\varphi f)(\tau,\xi)\overline{V_\varphi g(\tau,\xi)}\,\mathrm{d}\tau\,\mathrm{d}\xi\right| \\ &\leqslant \bigl|\langle V_\varphi f, V_\varphi g\rangle\bigr| + \frac{C_{\varphi,1}}{A}\|V_\varphi g\|_{L^1} + \frac{C_{\varphi,2}}{A}\bigl\|\tau V_\varphi g(\tau,\xi)\bigr\|_{L^1} \\ &< \infty. \end{aligned} \tag{2.32}$$

This suggests that we define $\tilde{f}_A$ as the linear functional that maps $g \in M^{1,1}_m$ to $(2\pi)^{-1}\langle \widetilde{F}_A, V_\varphi g\rangle$. Thus we have

Theorem 2. *Let $\tilde{f}_A$ be defined as above, i.e.,*

$$\tilde{f}_A\colon\quad g \in M^{1,1}_m \to \langle \tilde{f}_A, g\rangle := (2\pi)^{-1}\langle \widetilde{F}_A, V_\varphi g\rangle. \tag{2.33}$$

Then $\tilde{f}_A$ converges to f on $M^{1,1}_m$ as A tends to infinity, in the sense that for all $g \in M^{1,1}_m$

$$\bigl|\langle \tilde{f}_A, g\rangle - \langle f, g\rangle\bigr| \leqslant \frac{1}{2\pi A}\bigl(C_{\varphi,1}\|V_\varphi g\|_{L^1} + C_{\varphi,2}\bigl\|\tau V_\varphi g(\tau,\xi)\bigr\|_{L^1}\bigr). \tag{2.34}$$

Remark 4. Note that $A = (2\pi)/(\tau_0\xi_0)$; thus increasing A means decreasing the time and/or frequency translation steps, τ_0 and ξ_0, so increasing the redundancy of the expansion.

[3] Note that the modulation space $M^{1,1}_m$ is independent from the window φ we used in (2.30). In other words, $\|V_{\varphi_1} f\|_{L^1}$ and $\|V_{\varphi_2} f\|_{L^1}$ define equivalent norms on $M^{1,1}_m$ for sufficiently nice windows φ_1 and φ_2. A proof of this as well as an extensive discussion on modulation spaces can be found in [6].

Proof. Let $g \in M_m^{1,1}$ be arbitrary. Then

$$\langle \tilde{f}_A, g\rangle = (2\pi)^{-1} \int \widetilde{F}_A(\tau,\xi)\overline{V_\varphi g(\tau,\xi)}\,\mathrm{d}\tau\,\mathrm{d}\xi, \tag{2.35}$$

$$\langle f, g\rangle = (2\pi)^{-1} \int V_\varphi f(\tau,\xi)\overline{V_\varphi g(\tau,\xi)}\,\mathrm{d}\tau\,\mathrm{d}\xi, \tag{2.36}$$

where (2.35) is by definition true, and (2.36) follows from the Parseval identity for windowed Fourier transform. Thus

$$\begin{aligned}\bigl|\langle \tilde{f}_A, g\rangle - \langle f, g\rangle\bigr| &= (2\pi)^{-1}\left|\int (\widetilde{F}_A - V_\varphi f)(\tau,\xi)\overline{V_\varphi g(\tau,\xi)}\,\mathrm{d}\tau\,\mathrm{d}\xi\right|, &(2.37)\\ &\leqslant (2\pi)^{-1}\int |\widetilde{F}_A - V_\varphi f|(\tau,\xi)|V_\varphi g|(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi, &(2.38)\\ &\leqslant \frac{1}{2\pi A}\bigl(C_{\varphi,1}\|V_\varphi g\|_{L^1} + C_{\varphi,2}\bigl\|\tau V_\varphi g(\tau,\xi)\bigr\|_{L^1}\bigr), &(2.39)\end{aligned}$$

where to obtain (2.39) we use Theorem 1. □

Now we have a way of approximating f using the discrete sequence $(q_{n,m})$; of course the approximation is in the above described sense and we do not even know whether $\tilde{f}_A$ is a function. However, one can observe that this way of approximation is particularly useful for 'comparing' two functions (thus leading to applications such as pattern recognition); next we will show how one can 'compare' two functions in L^2 using their approximations which are obtained via this time–frequency sigma–delta quantization algorithm.

First let us focus on how to calculate the inner product $\langle \widetilde{F}_A, V_\varphi g\rangle$; note that

$$\begin{aligned}\langle \widetilde{F}_A, V_\varphi g\rangle &= \left\langle \frac{1}{A}\sum_{n,m} q_{n,m} V_\varphi \varphi_{n,m}(\tau,\xi), V_\varphi g(\tau,\xi)\right\rangle, &(2.40)\\ &= \frac{1}{A}\sum_{n,m} q_{n,m}\bigl\langle V_\varphi \varphi_{n,m}(\tau,\xi), V_\varphi g(\tau,\xi)\bigr\rangle. &(2.41)\end{aligned}$$

But by the Parseval identity for windowed Fourier transform,

$$\bigl\langle V_\varphi \varphi_{n,m}(\tau,\xi), V_\varphi g(\tau,\xi)\bigr\rangle = 2\pi\,\langle \varphi_{n,m}, g\rangle. \tag{2.42}$$

Let us denote the frame coefficients $\langle g, \varphi_{n,m}\rangle$ of g by $d_{n,m}$. After substituting (2.42) in (2.41), we get

$$\langle \widetilde{F}_A, V_\varphi g\rangle = \frac{2\pi}{A}\sum_{n,m} q_{n,m}\overline{d_{n,m}}. \tag{2.43}$$

Hence we have proved

Theorem 3. *Let $f \in \mathcal{B}^\varphi$, $g \in M_m^{1,1}$ with $m(\tau,\xi) = 1 + |\tau|$. Let (φ, τ_0, ξ_0) be a tight Weyl–Heisenberg frame of $L^2(\mathbb{R})$ for some fixed τ_0 and ξ_0. Suppose that φ fulfills the assumptions of Theorem* 1. *Then $\widetilde{F}_A$, the time–frequency sigma–delta approximation of $V_\varphi f$, satisfies*

$$\langle \widetilde{F}_A, V_\varphi g\rangle = \frac{2\pi}{A}\sum_{n,m} q_{n,m}\overline{d_{n,m}}, \tag{2.44}$$

where $d_{n,m} = \langle g, \varphi_{n,m}\rangle$. *Moreover, since for* $g \in M_m^{1,1}$, *the sequence* $(\langle g, \varphi_{n,m}\rangle)_{(n,m)\in\mathbb{Z}^2}$ *is absolutely summable, we have*:

$$\langle V_\varphi f - \widetilde{F}_A, V_\varphi g\rangle = \frac{2\pi}{A}\sum_{n,m}(c_{n,m} - q_{n,m})\overline{d_{n,m}}, \tag{2.45}$$

where $c_{n,m} = \langle f, \varphi_{n,m}\rangle$, $d_{n,m} = \langle g, \varphi_{n,m}\rangle$ *and the sequence q is given by* $q = T_{\mathrm{TF}}(c)$; *and*

$$\left\langle \widetilde{F}_A^1 - \widetilde{F}_A^2, V_\varphi g\right\rangle = \frac{2\pi}{A}\sum_{n,m}\left(q_{n,m}^1 - q_{n,m}^2\right)\overline{d_{n,m}}, \tag{2.46}$$

where $\widetilde{F}_A^j$ *is the time–frequency sigma–delta approximation of* $V_\varphi f_j = \langle f_j, \varphi_{\tau,\xi}\rangle$ *for some* f_j *in* $\mathcal{B}^\varphi$ *and* $q^j = T_{\mathrm{TF}}(c^j)$ *with* $c_{n,m}^j = \langle f_j, \varphi_{n,m}\rangle$.

Remark 5. Note that (2.44) is an explicit formula to calculate the inner product $\langle \widetilde{F}_A, V_\varphi g\rangle$; the only terms in (2.44) that do depend on the function f are the $q_{n,m}$. In other words, one can calculate the $d_{n,m}$ just once and store them in memory.

Remark 6. The second part of the theorem, in particular (2.46), specifies a simple way of determining how ‘similar’ two functions are by using only the corresponding bit sequences; next we shall make clear what we mean by ‘similar.’

Theorem 4. *Let* f_1, f_2 *be in* $\mathcal{B}^\varphi$, $V_\varphi f_j = \langle f_j, \varphi_{\tau,\xi}\rangle$ *for* $j = 1, 2$. *Suppose* $\widetilde{F}_A^j$ *is the time–frequency sigma–delta approximation of* $V_\varphi f_j$. *Then*

$$\left|\langle V_\varphi f_1 - V_\varphi f_2, V_\varphi g\rangle - \left\langle \widetilde{F}_A^1 - \widetilde{F}_A^2, V_\varphi g\right\rangle\right| \leqslant \frac{4\pi}{A}\left(C_{\varphi,1}\|V_\varphi g\|_{L^1} + C_{\varphi,2}\left\|\tau V_\varphi g(\tau,\xi)\right\|_{L^1}\right), \tag{2.47}$$

where $C_{\varphi,i}$, $i = 1, 2$, *is defined as in* (2.25) *and* (2.26), *respectively.*

Proof. Note that

$$\langle V_\varphi f_1 - V_\varphi f_2, V_\varphi g\rangle - \left\langle \widetilde{F}_A^1 - \widetilde{F}_A^2, V_\varphi g\right\rangle = \left\langle V_\varphi f_1 - \widetilde{F}_A^1, V_\varphi g\right\rangle - \left\langle V_\varphi f_2 - \widetilde{F}_A^2, V_\varphi g\right\rangle. \tag{2.48}$$

Thus,

$$\begin{aligned}\left|\langle V_\varphi f_1 - V_\varphi f_2, V_\varphi g\rangle - \left\langle \widetilde{F}_A^1 - \widetilde{F}_A^2, V_\varphi g\right\rangle\right| &\leqslant \left|\left\langle V_\varphi f_1 - \widetilde{F}_A^1, V_\varphi g\right\rangle\right| + \left|\left\langle V_\varphi f_2 - \widetilde{F}_A^2, V_\varphi g\right\rangle\right| \\ &\leqslant \frac{4\pi}{A}\left(C_{\varphi,1}\|V_\varphi g\|_{L^1} + C_{\varphi,2}\left\|\tau V_\varphi g(\tau,\xi)\right\|_{L^1}\right),\end{aligned} \tag{2.49}$$

where the second inequality is due to Theorem 2. □

Theorem 4 clearly shows that $\langle \widetilde{F}_A^1 - \widetilde{F}_A^2, V_\varphi g\rangle$ is an estimate of $f_1 - f_2$ in the direction of g. In other words, our measure of similarity of f_1 and f_2, i.e., $\langle \widetilde{F}_A^1 - \widetilde{F}_A^2, V_\varphi g\rangle$, is completely insensitive to functions that are orthogonal to g. However, if two functions are close to each other in L^2, clearly $\langle \widetilde{F}_A^1 - \widetilde{F}_A^2, V_\varphi g\rangle$ will also be small.

Corollary 1. *Let g be in $M_m^{1,1}$ with $m(\tau,\xi)=1+|\tau|$, define $G:=V_\varphi g$, and suppose that f_1, f_2 are in $\mathcal{B}^\varphi$. Then*

$$\left|\left\langle \widetilde{F}_A^1-\widetilde{F}_A^2,G\right\rangle\right| \leqslant 2\pi\|f_1-f_2\|_{L^2}\|g\|_{L^2}+\frac{4\pi}{A}\left(C_{\varphi,1}\|G\|_{L^1}+C_{\varphi,2}\left\|\tau G(\tau,\xi)\right\|_{L^1}\right),$$
$$\left|\langle V_\varphi f_1-V_\varphi f_2,G\rangle\right| \leqslant \left|\left\langle \widetilde{F}_A^1-\widetilde{F}_A^2,G\right\rangle\right|+\frac{4\pi}{A}\left(C_{\varphi,1}\|G\|_{L^1}+C_{\varphi,2}\left\|\tau G(\tau,\xi)\right\|_{L^1}\right),$$

where $\widetilde{F}_A^j$ is the time–frequency sigma–delta approximation of f_j, and $C_{\varphi,i}$, $i=1,2$, is defined as in (2.25) *and* (2.26)*, respectively.*

We now generalize the above discussion in the following way.

Theorem 5. *Let $g_1,\dots,g_K$ be functions in $M_m^{1,1}$ with $m(\tau,\xi)=1+|\tau|$ such that $\|g_j\|_{L^2}=1$ and $\langle g_i,g_j\rangle=\delta_{i,j}$. On $\mathcal{B}^\varphi$ define the projection operator P by*

$$P(F)=\sum_{j=1}^{K}\langle F,G_i\rangle G_i, \tag{2.50}$$

where $G_i:=V_\varphi g_i$ and $F:=V_\varphi f$ for $f\in\mathcal{B}^\varphi$. Let c be the sequence $(\langle f,\varphi_{n,m}\rangle)$ and $q=T_{\mathrm{TF}}(c)$. Suppose $\widetilde{F}_A$ is the time–frequency sigma–delta approximation of F. Then

$$\left\|P(F-\widetilde{F}_A)\right\|^2=\frac{4\pi^2}{A^2}\sum_{n,m,n',m'}(c_{n,m}-q_{n,m})\overline{(c_{n',m'}-q_{n',m'})}\langle\widetilde{P}\varphi_{n,m},\varphi_{n',m'}\rangle, \tag{2.51}$$

where $\widetilde{P}$ is defined by $\widetilde{P}(f):=\sum_{i=1}^K\langle f,g_i\rangle g_i$ for $f\in\mathcal{B}^\varphi$.

Proof. By (2.32), $P(\widetilde{F}_A)$ is well defined and thus it is in the span of $\{G_1,\dots,G_K\}$. Then we can write

$$\begin{aligned}
\left\|P(F-\widetilde{F}_A)\right\|^2&=\sum_{i=1}^{K}\left|\langle F-\widetilde{F}_A,G_i\rangle\right|^2\\
&=\frac{4\pi^2}{A^2}\sum_{i=1}^{K}\left(\sum_{n,m}(c_{n,m}-q_{n,m})\overline{d_{n,m}^i}\right)\left(\sum_{n',m'}\overline{(c_{n',m'}-q_{n',m'})}d_{n,m}^i\right)\\
&=\frac{4\pi^2}{A^2}\sum_{n,m,n',m'}(c_{n,m}-q_{n,m})\overline{(c_{n',m'}-q_{n',m'})}\sum_{i=1}^{K}\langle\varphi_{n,m},g_i\rangle\langle g_i,\varphi_{n',m'}\rangle\\
&=\sum_{n,m,n',m'}(c_{n,m}-q_{n,m})\overline{(c_{n',m'}-q_{n',m'})}\langle\widetilde{P}\varphi_{n,m},\varphi_{n',m'}\rangle,
\end{aligned} \tag{2.52}$$

where $d_{n,m}^i:=\langle g_i,\varphi_{n,m}\rangle$. The first equality is due to the definition of P; the second equality follows from Theorem 3; the third and fourth equalities are obvious. □

Remark 7. Let F^1 and F^2 be the windowed Fourier transforms of two functions f^1 and f^2 in $\mathcal{B}^\varphi$. Denote the sequence $(\langle f^i,\varphi_{n,m}\rangle)$ by c^i and let $q^i=T_{\mathrm{TF}}(c^i)$. Suppose $\widetilde{F}_A^1$ and $\widetilde{F}_A^2$ are the time–frequency sigma–

delta approximations of F^1 and F^2, respectively. Then replacing F and $\widetilde{F}_A$ in the proof of the previous theorem by $\widetilde{F}_A^1$ and $\widetilde{F}_A^2$, respectively, yields

$$\left\| P\left(\widetilde{F}_A^1 - \widetilde{F}_A^2\right)\right\|^2 = \frac{4\pi^2}{A^2} \sum_{n,m,n',m'} \left(q_{n,m}^1 - q_{n,m}^2\right)\overline{\left(q_{n',m'}^1 - q_{n',m'}^2\right)}\langle \widetilde{P}\varphi_{n,m}, \varphi_{n',m'}\rangle. \tag{2.53}$$

Remark 8. By Corollary 1 we have

$$\left\| P\left(\widetilde{F}_A^1 - \widetilde{F}_A^2\right)\right\| \leqslant \left\| f^1 - f^2\right\|_{L^2} \sum_{i=1}^{K} \|g_i\|_{L^2} + \frac{4\pi}{A}\left(C_{\varphi,1}\sum_{i=1}^{K} \|G_i\|_{L^1} + C_{\varphi,2}\sum_{i=1}^{K} \left\|\tau G_i(\tau,\xi)\right\|_{L^1}\right). \tag{2.54}$$

3. Translation invariance

As mentioned before, one possible application area for the time–frequency sigma–delta quantization scheme described in this section is pattern recognition. We have shown above that we can measure how similar two functions f_1 and f_2 are by calculating $\langle \widetilde{F}_A^1 - \widetilde{F}_A^2, G\rangle$. The next important question is whether the quantization scheme is robust with respect to translation in both arguments; in this section we shall investigate how shifts in the bit-sequence affect the approximation.

For $\alpha, \beta \in \mathbb{R}$, define the operators $T_\alpha f := f(\cdot + \alpha)$ and $M_\beta f := \mathrm{e}^{i\beta\cdot} f$, the time–shift and modulation operators, respectively. Let (φ, τ_0, ξ_0) be a tight Weyl–Heisenberg frame and note that

$$\langle T_{N\tau_0} f, \varphi_{n,m}\rangle = \mathrm{e}^{imN(2\pi)/A}\langle f, \varphi_{n+N,m}\rangle, \tag{3.55}$$

where $A = (2\pi)/(\tau_0\xi_0)$ is the frame bound. Let us denote $\langle f, \varphi_{n,m}\rangle$ by $c_{n,m}$ and $\mathrm{e}^{iN(2\pi)/A}$ by γ_N and rewrite (3.55) as

$$\langle T_{N\tau_0} f, \varphi_{n,m}\rangle = (\gamma_N)^m c_{n+N,m}. \tag{3.56}$$

Thus we conclude

$$T_{N\tau_0} f = \sum_{n,m} (\gamma_N)^m c_{n+N,m}\varphi_{n,m}. \tag{3.57}$$

From the previous section we know that

$$\widetilde{F}_A = \frac{1}{A}\sum q_{n,m} V_\varphi \varphi_{n,m} \tag{3.58}$$

approximates $V_\varphi f$ as in (2.15). In (3.58) $q = (q_{n,m}) = T_{\mathrm{TF}}(c)$ with $c = (c_{n,m}) = (\langle f, \varphi_{n,m}\rangle)$. We also know by (3.57) that the windowed Fourier transform of $T_{N\tau_0} f$ is given by

$$V_\varphi T_{N\tau_0} f = \frac{1}{A}\sum_{n,m} (\gamma_N)^m c_{n+N,m} V_\varphi \varphi_{n,m}. \tag{3.59}$$

One important question to ask is whether

$$\widetilde{H}_A := \frac{1}{A}\sum_{n,m} (\gamma_N)^m q_{n+N,m} V_\varphi \varphi_{n,m} \tag{3.60}$$

which is obtained by replacing $c_{n+N,m}$ in (3.59) with $q_{n+N,m}$, approximates $V_\varphi T_{N\tau_0} f$ in a way similar to the unshifted (2.15), i.e., whether $|V_\varphi T_{N\tau_0} f(\tau,\xi) - \widetilde{H}_A(\tau,\xi)| \leqslant (\widetilde{C}_{\varphi,1})/A + |\tau|(\widetilde{C}_{\varphi,2})/A$ for some $\widetilde{C}_{\varphi,1}$ and $\widetilde{C}_{\varphi,2}$. The next theorem shows that the answer to this question is affirmative.

Theorem 6. *Let $q = T_{\mathrm{TF}}(c)$, where $c = (c_{n,m})$ with $c_{n,m} = \langle f, \varphi_{n,m}\rangle$ for some f in $\mathcal{B}^\varphi$. Suppose $\widetilde{H}_A$ is defined as in* (3.60). *Then*

$$\left|V_\varphi T_{N\tau_0} f(\tau,\xi) - \widetilde{H}_A(\tau,\xi)\right| \leqslant \frac{\widetilde{C}_{\varphi,1}}{A} + |\tau|\frac{\widetilde{C}_{\varphi,2}}{A} \tag{3.61}$$

with $\widetilde{C}_{\varphi,1} = \sqrt{2}\|\partial_2\partial_1\Gamma\|_{L^1(\mathbb{R}^2)} + N\tau_0\|\partial_1\Gamma\|_{L^1(\mathbb{R}^2)}$ and $\widetilde{C}_{\varphi,2} = \sqrt{2}\|\partial_1\Gamma\|_{L^1(\mathbb{R}^2)}$, where $\Gamma(t,z) := \mathrm{e}^{itz}\Phi(t,z)$.

Proof. We want to show that

$$\frac{1}{A}\left|\sum_{n,m}(\gamma_N)^m q_{n+N,m} V_\varphi\varphi_{n,m}(\tau,\xi) - \sum_{n,m}(\gamma_N)^m c_{n+N,m} V_\varphi\varphi_{n,m}(\tau,\xi)\right|, \tag{3.62}$$

$$= \left|\frac{1}{A}\sum_{n,m}(\gamma_N)^m (\Delta_1\Delta_2 v)_{n+N,m} V_\varphi\varphi_{n,m}(\tau,\xi)\right|, \tag{3.63}$$

$$\leqslant \frac{\widetilde{C}_{\varphi,1}}{A} + |\tau|\frac{\widetilde{C}_{\varphi,2}}{A}, \tag{3.64}$$

for some $\widetilde{C}_{\varphi,1}$ and $\widetilde{C}_{\varphi,2}$ where $v_{n,m}$ is as in (2.19). Define

$$D := \frac{1}{A}\sum_{n,m}(\Delta_1\Delta_2 v)_{n+N,m}(\gamma_N)^m V_\varphi\varphi_{n,m}(\tau,\xi).$$

Then since $V_\varphi\varphi_{n,m}(\tau,\xi) = \mathrm{e}^{-in\tau_0(\xi - m\xi_0)}\Phi(\tau - n\tau_0, \xi - m\xi_0)$, we have

$$D = \frac{1}{A}\sum_{n,m}(\Delta_1\Delta_2 v)_{n+N,m}\mathrm{e}^{-i\tau\xi}\,\Omega_{N,\tau,\xi}(\tau - n\tau_0, m\xi_0), \tag{3.65}$$

where $\Omega_{N,\tau,\xi}(t,z) = \mathrm{e}^{iz(N\tau_0+\tau)}\Gamma(t, \xi - z)$. After summing the left-hand side of (3.65) by parts we get

$$D = \frac{1}{A}\sum_{n,m} v_{n+N,m}\mathrm{e}^{-i\tau\xi}\,\bar{\Delta}_1\bar{\Delta}_2\Omega_{N,\tau,\xi}(\tau - n\tau_0, m\xi_0). \tag{3.66}$$

Since $\Omega_{N,\tau,\xi}$ is smooth, we have

$$D = \frac{1}{A}\sum_{n,m} v_{n+N,m}\mathrm{e}^{-i\tau\xi}\int_{(\tau-n+1)\tau_0}^{\tau-n\tau_0}\int_{(m+1)\xi_0}^{m\xi_0}\partial_2\partial_1\Omega_{N,\tau,\xi}(t,z)\,\mathrm{d}t\,\mathrm{d}z, \tag{3.67}$$

which yields

$$|D| \leqslant \frac{\sqrt{2}}{A}\sum_{n,m}\int_{(\tau-n+1)\tau_0}^{\tau-n\tau_0}\int_{(m+1)\xi_0}^{m\xi_0}\left|\partial_2\partial_1\Omega_{N,\tau,\xi}(t,z)\right|\mathrm{d}t\,\mathrm{d}z, \leqslant \frac{\sqrt{2}}{A}\|\partial_2\partial_1\Omega_{N,\tau,\xi}\|_{L^1(\mathbb{R}^2)}. \tag{3.68}$$

Finally, after estimating $\|\partial_2\partial_1\Omega_{N,\tau,\xi}\|_{L^1(\mathbb{R}^2)}$ we get

$$\left|V_\varphi T_{N\tau_0}(\tau,\xi)-\widetilde{H}_A(\tau,\xi)\right|\leqslant\frac{1}{A}\left(\widetilde{C}_{\varphi,1}+|\tau|\widetilde{C}_{\varphi,2}\right) \tag{3.69}$$

with

$$\widetilde{C}_{\varphi,1}=\sqrt{2}\|\partial_2\partial_1\Gamma\|_{L^1(\mathbb{R}^2)}+\sqrt{2}N\tau_0\|\partial_1\Gamma\|_{L^1(\mathbb{R}^2)}, \tag{3.70}$$

and

$$\widetilde{C}_{\varphi,2}=\sqrt{2}\|\partial_1\Gamma\|_{L^1(\mathbb{R}^2)}, \tag{3.71}$$

where $\Gamma(t,z):=\mathrm{e}^{itz}\Phi(t,z)$. □

Remark 9. Combining Theorem 6 with Theorem 4, we can conclude that

$$\left|\sum_{n,m}\left((\gamma_N)^m q_{n+N,m}-\bar{q}_{n,m}\right)d_{n,m}\right|\leqslant\frac{\widetilde{C}_{\varphi,1}}{A}+|\tau|\frac{\widetilde{C}_{\varphi,2}}{A}, \tag{3.72}$$

where $\bar{q}:=(\bar{q}_{n,m})=T_{\mathrm{TF}}(\bar{c})$ with $\bar{c}:=(\langle T_{N\tau_0}f,\varphi_{n,m}\rangle)$.

Remark 10. Note that the constant $\widetilde{C}_{\varphi,2}$ given in (3.71) is the same as $C_{\varphi,2}$ given in (2.26); $\widetilde{C}_{\varphi,1}$, given in (3.70), has an extra summand proportional to N, the amount of translation, and τ_0, the time translation step, when compared to $C_{\varphi,1}$, given in (2.26). Thus, for $N=0$, i.e., when there is no shift in the quantizer output $(q_{n,m})$, both estimates yield the same upper bound on the approximation error.

Remark 11. The time–frequency sigma–delta quantization scheme is translation invariant up to the adjustment factor $(\gamma_N)^m$; the approximation of $T_{N\tau_0}f$ obtained using $((\gamma_N)^m q_{n+N,m})$ is (almost) as good as that obtained by quantizing the translated version separately.

Next, let us investigate shifts in the other index of the bit sequence produced by the time–frequency sigma–delta scheme.

Theorem 7. *Let f be in $\mathcal{B}^\varphi$, $c=(\langle f,\varphi_{n,m}\rangle)$ and $q=(q_{n,m})=T_{\mathrm{TF}}(c)$. Define*

$$\widetilde{H}_A=\frac{1}{A}\sum_{n,m}q_{n,m-M}V_\varphi\varphi_{n,m}. \tag{3.73}$$

Then

$$\left|V_\varphi M_{M\xi_0}f(\tau,\xi)-\widetilde{H}_A(\tau,\xi)\right|\leqslant\frac{C_{\varphi,1}}{A}+|\tau|\frac{C_{\varphi,2}}{A}, \tag{3.74}$$

where $C_{\varphi,1}$ and $C_{\varphi,2}$ are as in (2.25) *and* (2.26), *respectively.*

Proof. Note that

$$\left\langle\mathrm{e}^{iM\xi_0\cdot}f(\cdot),\varphi_{n,m}\right\rangle=\int f(t)\varphi(t-n\tau_0)\mathrm{e}^{-i(m-M)\xi_0 t}\,\mathrm{d}t=\langle f,\varphi_{n,m-M}\rangle, \tag{3.75}$$

which yields

$$V_\varphi M_{M\xi_0}f=\frac{1}{A}\sum_{n,m}c_{n,m-M}V_\varphi\varphi_{n,m}. \tag{3.76}$$

Then

$$V_\varphi M_{M\xi_0} f(\tau,\xi) - \widetilde{H}_A(\tau,\xi) = \frac{1}{A}\sum_{n,m}(c_{n,m-M} - q_{n,m-M}) V_\varphi \varphi_{n,m}(\tau,\xi)$$
$$= \frac{1}{A}\sum_{n,m}(\Delta_1\Delta_2 \upsilon_{n,m-M}) V_\varphi \varphi_{n,m}(\tau,\xi), \tag{3.77}$$

where $\upsilon_{n,m}$ is as in (2.19). As in the proof of Theorem 1 summing by parts yields the result. □

Now we can combine these two results: Let (φ,τ_0,ξ_0) be a tight Weyl–Heisenberg frame of L^2 with frame bound A, $c = (\langle f,\varphi_{n,m}\rangle)$ for some $f \in \mathcal{B}^\varphi$, and $q = T_{\mathrm{TF}}(c)$. Then the windowed Fourier transform of $M_{M\xi_0} T_{N\tau_0} f = \mathrm{e}^{iM\xi_0 \cdot} f(\cdot + N\tau_0)$ is given by

$$V_\varphi M_{M\xi_0} T_{N\tau_0} f = \frac{1}{A}\sum_{n,m} \gamma_N^{m-M} c_{n+N,m-M} V_\varphi \varphi_{n,m}(\tau,\xi). \tag{3.78}$$

Similarly, the windowed Fourier transform of $T_{N\tau_0} M_{M\xi_0} f$ is

$$V_\varphi T_{N\tau_0} M_{M\xi_0} f = \frac{1}{A}\sum_{n,m} \gamma_N^{m} c_{n+N,m-M} V_\varphi \varphi_{n,m}(\tau,\xi). \tag{3.79}$$

Now define

$$\widetilde{H}_A^1(\tau,\xi) := \frac{1}{A}\sum_{n,m} \gamma_N^{m-M} q_{n+N,m-M} V_\varphi \varphi_{n,m}(\tau,\xi), \tag{3.80}$$

and

$$\widetilde{H}_A^2(\tau,\xi) := \frac{1}{A}\sum_{n,m} \gamma_N^{m} q_{n+N,m-M} V_\varphi \varphi_{n,m}(\tau,\xi). \tag{3.81}$$

Note that $V_\varphi M_{M\xi_0} f = (1/A)\sum_{n,m} c_{n,m-M} V_\varphi \varphi_{n,m}(\tau,\xi)$. We then have by Theorem 6,

$$\left|\sum_{n,m} (\gamma_N)^m q_{n+N,m-M} V_\varphi \varphi_{n,m}(\tau,\xi) - T_{N\tau_0} M_{M\xi_0} f, \varphi_{\tau,\xi}\right| \leqslant \frac{\widetilde{C}_{\varphi,1}}{A} + |\tau|\frac{\widetilde{C}_{\varphi,2}}{A}, \tag{3.82}$$

where $\widetilde{C}_{\varphi,1}$ and $\widetilde{C}_{\varphi,2}$ are as in (3.70) and (3.71), respectively. Moreover, since $|\gamma_N| = 1$, we can also write

$$\left|\sum_{n,m} (\gamma_N)^{(m-M)} q_{n+N,m-M} V_\varphi \varphi_{n,m}(\tau,\xi) - M_{M\xi_0} T_{N\tau_0} f(\tau,\xi)\right| \leqslant \frac{\widetilde{C}_{\varphi,1}}{A} + |\tau|\frac{\widetilde{C}_{\varphi,2}}{A}. \tag{3.83}$$

Thus we proved

Theorem 8. *Let* $\widetilde{H}_A^1$ *and* $\widetilde{H}_A^2$ *be as in* (3.80) *and* (3.81), *respectively. Then we have*:

(i) $|V_\varphi M_{M\xi_0} T_{N\tau_0} f(\tau,\xi) - \widetilde{H}_A^1(\tau,\xi)| \leqslant \widetilde{C}_{\varphi,1}/A + |\tau|\widetilde{C}_{\varphi,2}/A$, *for all* τ,ξ, *and*
(ii) $|V_\varphi T_{N\tau_0} M_{M\xi_0} f(\tau,\xi) - \widetilde{H}_A^2(\tau,\xi)| \leqslant \widetilde{C}_{\varphi,1}/A + |\tau|\widetilde{C}_{\varphi,2}/A$, *for all* τ,ξ,

where $\widetilde{C}_{\varphi,1}$ *and* $\widetilde{C}_{\varphi,2}$ *are as in* (3.70) *and* (3.71), *respectively.*

4. Numerical experiment

In this section, we will present some experimental results: We will fix a Weyl–Heisenberg frame and quantize the frame expansions of a function f using the algorithm TF$\Sigma\Delta$-I. We choose

$$\varphi(t) = \pi^{1/4} e^{-t^2/2}. \tag{4.84}$$

One can show that (φ, τ_0, ξ_0) is a frame of $L^2(\mathbb{R})$ if τ_0 and ξ_0 are sufficiently small. Moreover, the frame is almost tight[4] (with both frame bounds approximately equal to $(2\pi)/(\tau_0\xi_0)$) if one chooses sufficiently small τ_0 and ξ_0 such that $\tau_0 \approx \xi_0$.

Let us now consider the function[5]

$$f(t) = 0.5 e^{-i0.1t^3} e^{-0.05t^2}. \tag{4.85}$$

First we compute the frame coefficients of f, $\langle f, \varphi_{n,m}\rangle$, for different values of τ_0 and ξ_0. We use an FFT-based algorithm to compute the frame coefficients using the samples of f: Let τ_1 be the period at which we sample f. (It is convenient to choose $\tau_1 = \tau_0$.) We will use the sequence $(f(k\tau_1))_{k=-K}^{K}$ for some sufficiently large K to compute the frame coefficients of f. Of course K has to be finite for all practical purposes; however that does not introduce a large error if both f and φ are well localized in time and frequency, which is true for our example. Figure 1 shows the windowed Fourier transform, F, of f for φ given in (4.84); clearly $F(n\tau_0, m\xi_0)$ for integer n, m are the frame coefficients of f.

In Fig. 2, we show the quantized values of the frame coefficients of f, obtained via the time–frequency sigma–delta quantization scheme. Next, we consider the frame expansions of f with frames (φ, τ_0, ξ_0)

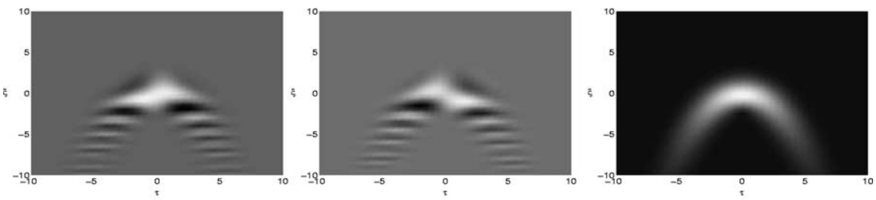


Fig. 1. The continuous windowed Fourier transform F of f, i.e., $F(\tau, \xi) = \langle f, \varphi_{\tau,\xi}\rangle$. Figure 1a shows the real part of F—black and white correspond to -0.49 and 0.75, respectively; Fig. 1b shows the imaginary part of F—black and white correspond to -0.57 and 0.69, respectively. Figure 1c shows the absolute value of F. In this graph, black corresponds to 0 and white corresponds to 0.86.

[4] As discussed in Remark 1, a frame is called "almost tight" if the ratio of the frame bounds is close to 1. Suppose (φ, τ_0, ξ_0) is a frame with frame bounds A and B. If we denote the quantity $B/A - 1$ by r, then any function $f \in L^2$ can be written as $f = 2/(A(2+r)) \sum \langle f, \varphi_{n,m}\rangle \varphi_{n,m} + Rf$ where $\|R\| \leqslant r/(2+r)$ [2]. Hence reconstructing f by (1.2) (with $(A(2+r))/2$ instead of A) introduces an error which is bounded in L^2 by $r/(2+r)\|f\|_{L^2}$. Therefore, if $r \approx 0$, we can assume the frame is tight and reconstruct f using (1.2). For all the frames we will use in this section $|r|$ is smaller than the arithmetical precision of the computer.

[5] The function f is clearly in $\mathcal{B}^{\varphi}$.

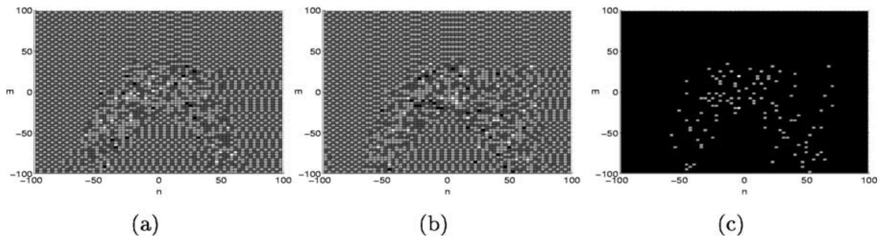


Fig. 2. The quantized frame coefficients $\langle f, \varphi_{n,m}\rangle$ for the frame $(\varphi, 0.1, 0.1)$. Figure 2a shows the real part of the quantized coefficients; Fig. 2b shows the imaginary parts of the quantized coefficients; Fig. 2c shows the absolute value of the quantized coefficients. In Fig. 2, a and b, black and white correspond to -3 and 3, respectively. In Fig. 2c black corresponds to $\sqrt{2}$ and white corresponds $3\sqrt{2}$.

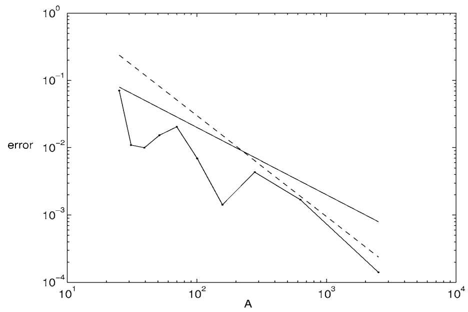


Fig. 3. The 'approximation error' $|\langle F - \widetilde{F}_A, G_{\text{tot}}\rangle|$ vs. the frame bound A. Both axes are logarithmic. The solid line seen in the figure is the graph $\{(A, 2A^{-1})\colon 25 < A < 1258\}$; the dashed line is the graph $\{(A, 30A^{-3/2})\colon 25 < A < 1258\}$.

where τ_0 and ξ_0 take values between 0.05 and 0.5; thus the frame bound A ranges from approximately 25.13 to 1256.64. We fix $G(\tau, \xi) = \mathrm{e}^{-0.2(\tau^2+\xi^2)}$ and we use

$$G_{\text{tot}} = \sum_{k=-2}^{2} \sum_{l=-2}^{2} T_{l,k} G, \tag{4.86}$$

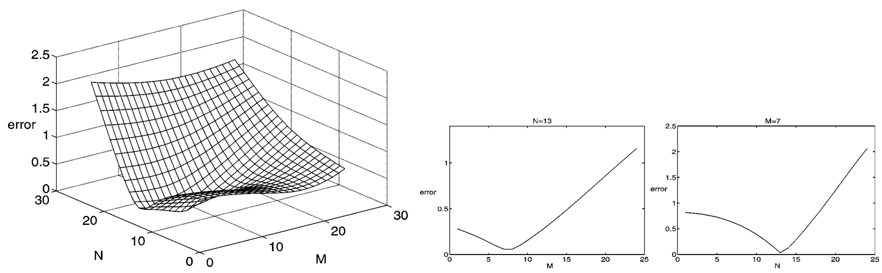
where $T_{l,k}G := G(\cdot + l, \cdot + k)$, as our test function. Clearly the inverse windowed Fourier transform of G_{tot} is in $M_m^{1,1}$.

Next, we compute $\langle F - \widetilde{F}_A, G_{\text{tot}}\rangle$ via (2.45). Figure 3 shows the value of this inner product as the frame bound increases. Theorem 1 bounds the decay of $|\langle F - \widetilde{F}_A, G_{\text{tot}}\rangle|$ by A^{-1}; however experimental evidence, e.g., Fig. 3, suggests a faster decay rate. This is similar to the first-order standard sigma–delta scheme for which the analogous estimate yields a bound of $\mathrm{O}(\lambda^{-1})$ [3] (λ is the oversampling ratio) whereas the empirically expected decay rate is $\lambda^{-3/2}$. In [7], S. Güntürk proved that the error can be bounded pointwise by $C\lambda^{-4/3+\eta}$ where C depends on η and on the value of the derivative of the original function at the corresponding point; the conjecture is that the error can be bounded pointwise by $C\lambda^{-3/2+\eta}$. (A detailed discussion of various types of improved estimates can be found in [8].) Whether there is a similar theorem for our case is an open problem; Fig. 3 suggests there may well be.

Now, we want to observe the translation invariance of our algorithm. Let f be as in (4.85). Fix the frame $(\varphi, 0.1, 0.1)$ and compute $q = T_{\text{TF}}(c)$ where $c_{n,m} = \langle f, \varphi_{n,m}\rangle$. Now, define $f_{T,\Omega}$ by $f_{T,\Omega}(t) := M_{-\Omega}T_T f$. Let $c_{T,\Omega}$ be the sequence $(\langle f_{T,\Omega}, \varphi_{n,m}\rangle)$ and $q_{T,\Omega} := T_{\text{TF}}(c_{T,\Omega})$. Using q as a template, we will estimate T and Ω when we are only given the sequence $q_{T,\Omega}$. To accomplish this, we will compare $\widetilde{F}_{T,\Omega,A} := \sum (q_{T,\Omega})_{n,m} \langle \varphi_{n,m}, \varphi_{\tau,\xi} \rangle$ with $I_{N,M} := \sum (\gamma_N)^{m+M} q_{n+N,m+M} \langle \varphi_{n,m}, \varphi_{\tau,\xi} \rangle$ for various N and M by comparing the inner products $\langle \widetilde{F}_{T,\Omega,A} - I_{N,M}, G_{\text{tot}} \rangle$. We will calculate these inner products via (2.46). Since the frame constant A is large ($A \approx 628$ in this case), we expect according to Theorem 8, although it is not guaranteed, to have $T \approx 0.1\overline{N}$ and $\Omega \approx 0.1\overline{M}$ where $(\overline{N}, \overline{M}) = \arg\inf_{(N,M)\in\mathbb{Z}^2} \langle \widetilde{F}_{T,\Omega,A} - I_{N,M}, G_{\text{tot}} \rangle$ if T and Ω are integer multiples of $\tau_0 = 0.1$ and $\xi_0 = 0.1$, respectively.

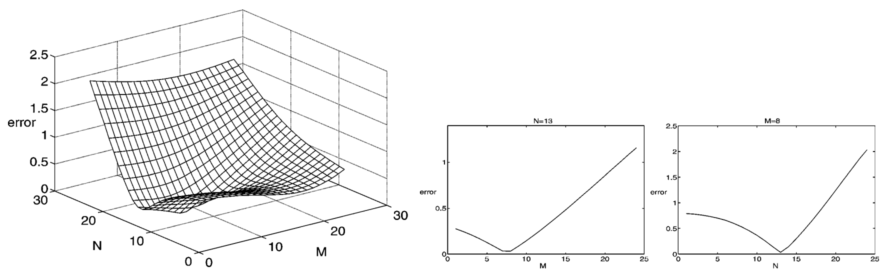
For $T = 1.2 = 12\tau_0$ and $\Omega = 0.9 = 9\tau_0$, we observe in Fig. 4 that the minimum is attained at $(N, M) = (13, 7)$. In other words, we estimate the amount translation T with an error of 0.1 and we make an error of 0.2 when we estimate Ω, the amount of modulation. Figure 5 shows the value of $\langle \widetilde{F}_{T,\Omega,A} - I_{N,M}, G_{\text{tot}} \rangle$ as a function of N and M for $T = 1.17$ and $\Omega = 0.93$. In this case $\langle \widetilde{F}_{T,\Omega,A} - I_{N,M}, G_{\text{tot}} \rangle$ attains its minimum at $N = 13$ and $M = 8$, i.e., the estimated values of T and Ω are 1.3 and 0.8, respectively. This indicates that even the original function is translated and modulated by amounts that are noninteger multiples of the time and frequency translation steps τ_0 and ξ_0 (both equal to 0.1 in this example), the algorithm can still estimate these amounts (with the resolution of integer multiples of τ_0 and ξ_0).

Finally, we want to observe the effects of noise. We consider the case where $f_{T,\Omega}$ is defined as above with $T = 1.2$ and $\Omega = 0.9$. We will add independent identically distributed Gaussian random



(a) (b) (c)

Fig. 4. The value $\langle \widetilde{F}_{T,\Omega,A} - I_{N,M}, G_{\text{tot}} \rangle$ vs. N and M for $T = 1.2$ and $\Omega = 0.9$; the minimum is obtained at $N = 13$ and $M = 7$, which means that the algorithm predicts $T = 1.3$ and $\Omega = 0.7$. Figure 4b shows $\langle \widetilde{F}_{T,\Omega,A} - I_{13,M}, G_{\text{tot}} \rangle$ vs. M; Fig. 4c shows $\langle \widetilde{F}_{T,\Omega,A} - I_{N,7}, G_{\text{tot}} \rangle$ vs. N.



(a) (b) (c)

Fig. 5. The value $\langle \widetilde{F}_{T,\Omega,A} - I_{N,M}, G_{\text{tot}} \rangle$ vs. N and M for $T = 1.17$ and $\Omega = 0.93$; the minimum is obtained at $N = 13$ and $M = 8$, which means that the algorithm predicts $T = 1.3$ and $\Omega = 0.8$. Figure 5b shows $\langle \widetilde{F}_{T,\Omega,A} - I_{13,M}, G_{\text{tot}} \rangle$ vs. M; Fig. 5c shows $\langle \widetilde{F}_{T,\Omega,A} - I_{N,8}, G_{\text{tot}} \rangle$ vs. N.

variables ν_k to each sample of $f_{T,\Omega}(k\tau_1)$ (τ_1 is the period at which $f_{T,\Omega}$ is sampled; we choose $\tau_1 = \tau_0$) before computing the frame coefficients. We then compute the frame coefficients $\tilde{c}_{n,m}$ using $(f_{T,\Omega}(k\tau_1) + \nu_k)_{k=-K}^{K}$ and via the time–frequency sigma–delta scheme we quantize $\tilde{c}_{n,m}$ to obtain $\widetilde{F}_{T,\Omega}^{\nu}$. Let us define the *signal-to-noise ratio* (SNR) as

$$\mathrm{SNR} = 10\log\frac{\sum_{k=-K}^{K}|f_{T,\Omega}(k\tau_1)|^2}{(2K+1)\sigma^2}\ \mathrm{dB}, \tag{4.87}$$

where σ^2 is the variance of ν_k; $2K+1$ samples $f_{T,\Omega}$ is used to compute the frame coefficients. In an experiment with $\mathrm{SNR} = 16$ dB, $\langle \widetilde{F}_{T,\Omega,A}^{\nu} - I_{N,M}, G_{\mathrm{tot}}\rangle$ attains its minimum at $N = 13$ and $M = 8$, i.e., the estimated values of T and Ω are 1.3 and 0.8, respectively. We repeat the same experiment using inputs with $\mathrm{SNR} = 8.5$ dB and $\mathrm{SNR} = 0$ dB. In the case where the $\mathrm{SNR} = 8.5$ dB, the parameters T and Ω are estimated as 1.4 and 0.6, respectively. For the input with $\mathrm{SNR} = 0$ dB the corresponding estimates are 1.4 and 0.2, respectively. We observe that the algorithm does reasonably well for the two cases where the signal-to-noise ratio is larger; however for $\mathrm{SNR} = 0$ dB, the minimum value of $\langle \widetilde{F}_{T,\Omega,A} - I_{N,M}, G_{\mathrm{tot}}\rangle$ is much larger than the other two cases where the SNR is larger and so is the error in the estimation of T and Ω.

5. Higher-order time–frequency sigma–delta schemes

In this section we will introduce higher-order time–frequency sigma–delta schemes to quantize the frame expansions of functions in $\mathcal{B}^{\varphi}$ for tight Weyl–Heisenberg frames. We will show that the approximation error is $\mathrm{O}(A^{-k})$ with a kth-order scheme when the frame bound is A. Let (φ, τ_0, ξ_0) be a tight Weyl–Heisenberg frame with frame bound A. Let f be in $\mathcal{B}^{\varphi}$; $c = (c_{n,m})$ with $c_{n,m} = \langle f, \varphi_{n,m}\rangle$ as before. Denote the real and imaginary parts of $c_{n,m}$ by $c_{n,m}^R$ and $c_{n,m}^I$, respectively. Let $(\Delta_1^{(k)}x)_{n,m} := \sum_{l=0}^{k}(-1)^l\binom{k}{l}x_{n-l,m}$ and $(\Delta_2^{(k)}x)_{n,m} := \sum_{l=0}^{k}(-1)^l\binom{k}{l}x_{n,m-l}$ for any sequence x. To define the kth-order time–frequency sigma–delta quantization scheme, consider the recursion relations where the superscript S is as described before:

$$\begin{aligned}\big(\Delta_1^{(k)}u^S\big)_{n,m} &= c_{n,m}^S - p_{n,m}^S,\\ p_{n,m}^S &= \mathrm{sign}\Big(\Theta\Big(\big(\Delta_1^{(0)}u^S\big)_{n-1,m}, \ldots, \big(\Delta_1^{(k-1)}u^S\big)_{n-1,m}, c_{n,m}^S\Big)\Big),\end{aligned} \tag{5.88}$$

$$\begin{aligned}\big(\Delta_2^{(k)}v^S\big)_{n,m} &= \bar{u}_{n,m}^S - r_{n,m}^S,\\ r_{n,m}^S &= \mathrm{sign}\Big(\Theta\Big(\big(\Delta_2(0)v^S\big)_{n,m-1}, \ldots, \big(\Delta_2^{(k-1)}u^S\big)_{n,m-1}, \bar{u}_{n,m}^S\Big)\Big),\end{aligned} \tag{5.89}$$

where $\bar{u}^S := u^S/C_{k,\Theta}$ and Θ is a function which guarantees that u^R, v^R, u^I, and v^I are uniformly bounded in l^∞ by $C_{k,\Theta}$. Note that the recursion relations (5.88) and (5.89) correspond to kth-order standard sigma–delta quantizers with $c_{n,m}^S$ and $\bar{u}_{n,m}^S$, respectively, as their input. Thus, since all these sequences are bounded in l^∞ by 1, such a Θ exists due to [3]. Note that

$$C_{k,\Theta}\big(\Delta_1^{(k)}\Delta_2^{(k)}v^R\big)_{n,m} = c_{n,m}^R - \big(p_{n,m}^R + C_{k,\Theta}\big(\Delta_1^{(k)}r^R\big)_{n,m}\big), \tag{5.90}$$

and similarly

$$C_{k,\Theta}\big(\Delta_1^{(k)}\Delta_2^{(k)}v^I\big)_{n,m} = c_{n,m}^I - \big(p_{n,m}^I + C_{k,\Theta}\big(\Delta_1^{(k)}r^I\big)_{n,m}\big). \tag{5.91}$$

We will now define the sequences q^R and q^I by $q^R_{n,m} = p^R_{n,m} + C_{k,\Theta}(\Delta_1^{(k)} r^R)_{n,m}$ and $q^I_{n,m} = p^I_{n,m} + C_{k,\Theta}(\Delta_1^{(k)} r^I)_{n,m}$. Finally, let us define T_{TF_k} by

$$T_{\mathrm{TF}_k}(c) := q, \tag{5.92}$$

where $q_{n,m} := q^R_{n,m} + i q^I_{n,m}$.

Theorem 9. *Let (φ, τ_0, ξ_0) be a tight Weyl–Heisenberg frame with frame bound A. Let f be in $\mathcal{B}^\varphi$ and define the sequence q by* (5.92)*, i.e., $q_{n,m}$ is obtained by quantizing the frame coefficients of f via a kth-order TF$\Sigma\Delta$ scheme. Fix a positive integer k and define*

$$\widetilde{F}_{A,k}(\tau, \xi) := \frac{1}{A} \sum_{n,m} q_{n,m} \langle \varphi_{n,m}, \varphi_{\tau,\xi} \rangle. \tag{5.93}$$

Suppose φ is chosen such that $\Phi(\tau, \xi) = \langle \varphi, \varphi_{\tau,\xi} \rangle$ satisfies

$$\frac{\partial^k}{\partial \tau^k} \frac{\partial^k}{\partial \xi^k} \bigl(\mathrm{e}^{i\tau\xi} \Phi(\tau, \xi)\bigr) \in L^1\bigl(\mathbb{R}^2\bigr). \tag{5.94}$$

Then

$$\bigl|V_\varphi f(\tau, \xi) - \widetilde{F}_{A,k}(\tau, \xi)\bigr| \leqslant \frac{1}{A^k} \sum_{l=0}^{k} C_{k,\varphi,l} |\tau|^l \tag{5.95}$$

with

$$C_{k,\varphi,l} = (2\pi)^{k-1} C_{k,\Theta} \|v\|_{l^\infty} \binom{k}{l} \bigl\| \partial_2^{(k-l)} \partial_1^k \Gamma \bigr\|_{L^1(\mathbb{R}^2)}, \tag{5.96}$$

where k is the order of the quantizer and $\Gamma(t, z) = \mathrm{e}^{itz} \Phi(t, z)$. We will call $\widetilde{F}_{A,k}$ the kth-order time–frequency sigma–delta approximation of $V_\varphi f$.

We need the following standard result to prove Theorem 9.

Lemma 2. *Let $\bar{\Delta}$ denote the forward difference operator, i.e., $(\bar{\Delta} x)_n := x_n - x_{n+1}$, as before. The following equality holds for any function $f \in C^k$:*

$$\bar{\Delta}^k f(x - n\alpha) = \alpha^{k-1} \int_0^{k\alpha} f^{(k)}\bigl(x - (n+k)\alpha + t\bigr) \rho_k\left(\frac{t}{\alpha}\right) \mathrm{d}t \tag{5.97}$$

for any α. In (5.97)*, ρ_k is the kth-order B-spline, $\rho_k = \chi_{[0,1]} * \cdots * \chi_{[0,1]}$ (k convolution factors).* (*Note that the support of ρ_k is on* $[0, k]$*, and $\sum_n \rho_k(x + n) = 1$ for all $y \in \mathbb{R}$.*)

Proof of Theorem 9. As in the proof of Theorem 1, we start by writing the error term

$$F(\tau, \xi) - \widetilde{F}_{A,k}(\tau, \xi) = \frac{C_{k,\Theta}}{A} \sum_{n,m} \bigl(\Delta_1^k \Delta_2^k v\bigr)_{n,m} V_\varphi \varphi_{n,m}(\tau, \xi), \tag{5.98}$$

$$= \frac{C_{k,\Theta}}{A} \sum_{n,m} v_{n,m} \bar{\Delta}_2^k \bar{\Delta}_1^k V_\varphi \varphi_{n,m}(\tau, \xi). \tag{5.99}$$

Now let us define $I_{n,m} = \bar{\Delta}_2^k \bar{\Delta}_1^k V_\varphi \varphi_{n,m}(\tau, \xi)$, which we can also write as

$$I_{n,m} = \mathrm{e}^{-i\tau\xi}\, \bar{\Delta}_2^k \bar{\Delta}_1^k \Omega_{\tau,\xi}(\tau - n\tau_0, m\xi_0) \tag{5.100}$$

with $\Omega_{\tau,\xi}(t, z) := \mathrm{e}^{iz\tau} \mathrm{e}^{it(\xi - z)} \Phi(t, \xi - z)$, as in the proof of Theorem 1. By Lemma 2 we can write (5.100) as

$$I_{n,m} = \mathrm{e}^{-i\tau\xi}\, \bar{\Delta}_2^k \tau_0^{k-1} \int_0^{k\tau_0} \partial_1^{(k)} \Omega_{\tau,\xi}\big(\tau - (n+k)\tau_0 + t, m\xi_0\big) \rho_k\left(\frac{t}{\tau_0}\right) \mathrm{d}t, \tag{5.101}$$

$$= \mathrm{e}^{-i\tau\xi}\, \tau_0^{k-1} \int_0^{k\tau_0} \big(\bar{\Delta}_2^k \partial_1^{(k)} \Omega_{\tau,\xi}\big(\tau - (n+k)\tau_0 + t, m\xi_0\big)\big) \rho_k\left(\frac{t}{\tau_0}\right) \mathrm{d}t, \tag{5.102}$$

$$= \mathrm{e}^{-i\tau\xi}\, \frac{(2\pi)^{k-1}}{A^{k-1}} \int_0^{k\tau_0} \int_0^{k\xi_0} \partial_2^{(k)} \partial_1^{(k)} \Omega_{\tau,\xi}\big(\tau - (n+k)\tau_0 + t, (m-k)\xi_0 + z\big) \rho_k\left(\frac{z}{\xi_0}\right) \rho_k\left(\frac{t}{\tau_0}\right) \mathrm{d}z\, \mathrm{d}t. \tag{5.103}$$

In the last equality we use the fact that $A = (2\pi)/(\tau_0\xi_0)$. Since the support of ρ_k is on $[0, k]$ we can replace the integration limits of both integrals in (5.103) by $-\infty$ and ∞. Thus after the appropriate change of variables in both integrals we get

$$I_{n,m} = \mathrm{e}^{-i\tau\xi}\, \frac{2\pi^{k-1}}{A^{k-1}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \partial_2^{(k)} \partial_1^{(k)} \Omega_{\tau,\xi}(p, s) \rho_k\left(\frac{p}{\tau_0} - \frac{p}{\tau} + n + k\right) \rho_k\left(\frac{s}{\xi_0} - m + k\right) \mathrm{d}p\, \mathrm{d}s. \tag{5.104}$$

Substituting (5.104) into (5.99) and taking the absolute value of the resulting expression, along with the fact that $\rho_k \geqslant 0$ and

$$\sum_{n,m} \rho_k\left(\frac{p}{\tau_0} - \frac{p}{\tau} + n + k\right) \rho_k\left(\frac{s}{\xi_0} - m + k\right) = 1, \tag{5.105}$$

yields:

$$\big|V_\varphi f(\tau, \xi) - \widetilde{F}_{A,k}(\tau, \xi)\big| \leqslant \frac{C_{k,\Theta} \|v\|_{l^\infty} (2\pi)^{(k-1)}}{A^k} \big\| \partial_2^{(k)} \partial_1^{(k)} \Omega_{\tau,\xi} \big\|_{L^1}. \tag{5.106}$$

Finally, using

$$\partial_2^{(k)} \partial_1^{(k)} \Omega_{\tau,\xi}(t, z) = \sum_{l=0}^{k} \binom{k}{l} (i\tau)^l \mathrm{e}^{iz\tau} \partial_2^{(k-l)} \partial_1^{(k)} \Gamma(t, \xi - z), \tag{5.107}$$

we get the result. □

Remark 12. The reasoning in Remark 1 still applies and thus Theorem 9 holds, at least approximately, if the frame (φ, τ_0, ξ_0) is almost tight.

Remark 13. A sufficient condition for $\Phi = V_\varphi \varphi$ to satisfy (5.94) is that the function φ is in the Schwartz space $\mathcal{S}(\mathbb{R})$.

Remark 14. We will again approximate f as a linear functional on some test function space. For a kth-order time–frequency sigma–delta quantization scheme an appropriate test function space is the modulation space $M^{1,1}_{m_k}$ with $m_k(\tau,\xi) := 1 + |\tau|^k$, i.e.,

$$M^{1,1}_{m_k} = \{g \in L^2(\mathbb{R}):\ (1 + |\tau^k|)V_\varphi g(\tau,\xi) \in L^1(\mathbb{R}^2)\}. \tag{5.108}$$

Let $g \in M^{1,1}_{m_k}$ and for $f \in \mathcal{B}^\varphi$, let $\widetilde{F}_{A,k}$ be defined as in (5.93). Then

$$\langle V_\varphi f - \widetilde{F}_{A,k}, V_\varphi g\rangle := \int \big(V_\varphi f(\tau,\xi) - \widetilde{F}_{A,k}(\tau,\xi)\big)V_\varphi g(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi \tag{5.109}$$

is finite; thus $\langle \widetilde{F}_{A,k}, V_\varphi g\rangle$ is well defined. We now define $\tilde{f}_{A,k}$ as a linear functional on $M^{1,1}_{m_k}$ such that

$$\langle \tilde{f}_{A,k}, g\rangle := \langle \widetilde{F}_{A,k}, V_\varphi g\rangle. \tag{5.110}$$

By Theorem 9 we can conclude

$$\big|\langle f, g\rangle - \langle \tilde{f}_{A,k}, g\rangle\big| \leqslant \frac{1}{A^k}\sum_{l=0}^{k} C_{\varphi,l}\big\|\tau^l V_\varphi g(\tau,\xi)\big\|_{L^1\mathbb{R}^2}, \tag{5.111}$$

where $C_{k,\varphi,l}$ is as in (5.96).

Remark 15. Let f_1 and f_2 be two functions in $\mathcal{B}^\varphi$, q^1, and q^2 the corresponding sequences produced by the kth-order time–frequency sigma–delta scheme, and let $\widetilde{F}^1_{A,k}$ and $\widetilde{F}^2_{A,k}$ be the kth-order time–frequency sigma–delta approximations of f_1 and f_2, respectively. Then, regardless of the order of the approximation, we have

$$\big\langle \widetilde{F}^1_{A,k} - \widetilde{F}^2_{A,k}, V_\varphi g\big\rangle = \frac{1}{A}\sum_{n,m}\big(q^1_{n,m} - q^2_{n,m}\big)\overline{\langle g, \varphi_{n,m}\rangle}. \tag{5.112}$$

Similarly, for any f in $\mathcal{B}^\varphi$, let $q = T_{\mathrm{TF}_k}(c)$ where c denotes the sequence of the frame coefficients of f; suppose $\widetilde{F}_{A,k}$ is the kth-order time–frequency sigma–delta approximation of f. Then we have

$$\langle F - \widetilde{F}_{A,k}, V_\varphi g\rangle = \frac{1}{A}\sum_{n,m}(c_{n,m} - q_{n,m})\overline{\langle g, \varphi_{n,m}\rangle}. \tag{5.113}$$

Remark 16. Theorems 3 and 5 are true regardless of the order k of the time–frequency sigma–delta scheme that is used to approximate a given function $f \in \mathcal{B}^\varphi$, as long as φ satisfies the conditions stated in Theorem 9 and the test functions are chosen appropriately. Theorems 4, 6, 7, and 8 need some modification to be true for the case where the quantizer is of kth-order. We state these modified versions below: Theorems 10, 11, 12, and 13 are the generalized versions of the aforementioned theorems, respectively. The proofs are similar to the first order case and will be omitted.

Theorem 10. *Let f_1, f_2 be in $\mathcal{B}^\varphi$, $F^j := V_\varphi f_j$ for $j = 1, 2$, $\widetilde{F}^j_{A,k}$ be the kth-order time–frequency sigma–delta approximation of F^j for some fixed positive integer k. Then, for $g \in M^{1,1}_{m_k}$,*

$$\big|\big\langle F^1 - F^2, V_\varphi g\big\rangle - \big\langle \widetilde{F}^1_{A,k} - \widetilde{F}^2_{A,k}, V_\varphi g\big\rangle\big| \leqslant \frac{4\pi}{A^k}\sum_{l=0}^{k} C_{k,\varphi,l}\big\|\tau^l V_\varphi g(\tau,\xi)\big\|_{L^1(\mathbb{R}^2)}, \tag{5.114}$$

where $C_{k,\varphi,l}$ is defined as in (5.96).

Theorem 11. *Let* $q = T_{\mathrm{TF}_k}(c)$ (*i.e., the quantization scheme is of order* k), *where* $c = (c_{n,m})_{(n,m)\in\mathbb{Z}^2}$ *with* $c_{n,m} = \langle f, \varphi_{n,m}\rangle$ *for some* f *in* $\mathcal{B}^\varphi$. *Let* N *be some fixed integer and define* $\widetilde{H}_A$ *as in* (3.60). *Then*

$$\left|V_\varphi T_{N\tau_0} f(\tau,\xi) - \widetilde{H}_A(\tau,\xi)\right| \leqslant \frac{1}{A^k}\sum_{l=0}^{k} \widetilde{C}_{k,\varphi,l}|\tau|^l \tag{5.115}$$

with

$$\widetilde{C}_{k,\varphi,l} = (2\pi)^{k-1} C_{k,\Theta}\|v\|_{l^\infty} \sum_{j=l}^{k} \binom{k}{j}\binom{j}{l}(N\tau_0)^{j-l}\left\|\partial_2^{(k-j)}\partial_1^{(k)}\Gamma\right\|. \tag{5.116}$$

Theorem 12. *Let* f *be in* $\mathcal{B}^\varphi$, $c = (\langle f, \varphi_{n,m}\rangle)$ *and* $q = (q_{n,m}) = T_{\mathrm{TF}_k}(c)$ *for some positive integer* k. *Fix an integer* M *and define* $\widetilde{H}_A$ *as in* (3.73). *Then*

$$\left|V_\varphi M_{M\xi_0} f(\tau,\xi) - \widetilde{H}_A(\tau,\xi)\right| \leqslant \frac{1}{A^k}\sum_{l=0}^{k} C_{k,\varphi,l}|\tau|^l, \tag{5.117}$$

where $C_{k,\varphi,l}$ *is as in* (5.96).

Theorem 13. *Let* f *be in* $\mathcal{B}^\varphi$, $c = (\langle f, \varphi_{n,m}\rangle)$ *and* $q = (q_{n,m}) = T_{\mathrm{TF}_k}(c)$. *For integers* N *and* M, *define* $\widetilde{H}_A^1$ *and* $\widetilde{H}_A^2$ *as in* (3.80) *and* (3.81), *respectively. Then*

$$\left|V_\varphi M_{M\xi_0} T_{N\tau_0} f(\tau,\xi) - \widetilde{H}_A^1(\tau,\xi)\right| \leqslant \frac{1}{A^k}\sum_{l=0}^{k} \widetilde{C}_{k,\varphi,l}|\tau|^l,$$

$$\left|V_\varphi T_{N\tau_0} M_{M\xi_0} f(\tau,\xi) - \widetilde{H}_A^2(\tau,\xi)\right| \leqslant \frac{1}{A^k}\sum_{l=0}^{k} \widetilde{C}_{k,\varphi,l}|\tau|^l,$$

where $\widetilde{C}_{k,\varphi,l}$ *is as in* (5.116).

6. Numerical experiment revisited

In this section, we will present the results of numerical experiments for the second-order TFΣΔ-I quantizer analogous to those discussed in Section 4 for the first-order quantizer. We choose $\varphi(t) = \pi^{1/4}\mathrm{e}^{-t^2/2}$. As we have discussed before, (φ, τ_0, ξ_0) constitutes a frame if τ_0 and ξ_0 is sufficiently small; moreover, the frame is almost tight with the frame bound $A \approx (2\pi)/(\tau_0\xi_0)$ if τ_0 and ξ_0 are sufficiently small and $\tau_0 \approx \xi_0$.

We will quantize the frame expansion of the function $f(t) = 0.5\mathrm{e}^{-(i0.1t^3+0.05t^2)}$, which is the same function we have used in Section 4. We have already computed the frame coefficients $\langle f, \varphi_{n,m}\rangle$ of f. Using the algorithm described in (5.88) and (5.89) with $k = 2$ and $\Theta(u, v, x) = u + 0.5v$ we obtain the quantized frame coefficients $q_{n,m}$ of f; these are shown in Fig. 6. Next, we fix the function G_{tot}, defined as in (4.86), as our test function and compute the inner product $\langle F - \widetilde{F}_{A,2}, G_{\mathrm{tot}}\rangle$ via (5.113) for various values of the frame bound A. Figure 7 shows the value of $\langle F - \widetilde{F}_{A,2}, G_{\mathrm{tot}}\rangle$ while A takes values between 25.13 and 1228.64. Similar to the first-order case, the decay of the approximation error is faster than

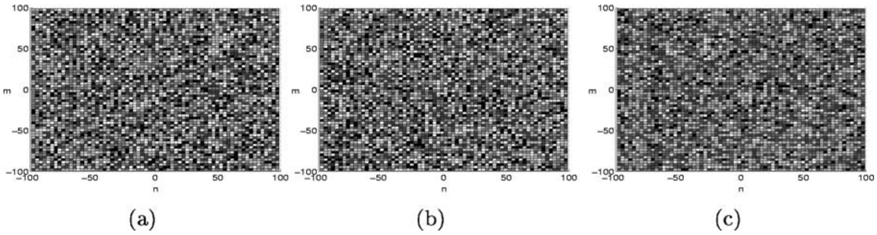


Fig. 6. The quantized frame coefficients $q_{n,m}$—obtained via the second-order scheme. Figure 6a shows the real part of the quantized coefficients; Fig. 6b shows the imaginary parts of the quantized coefficients—black corresponds to -10 and white corresponds to 10 in these figures. Figure 6c shows the absolute value of the quantized coefficients; in this figure black corresponds to 0 and white corresponds to $10\sqrt{2}$.

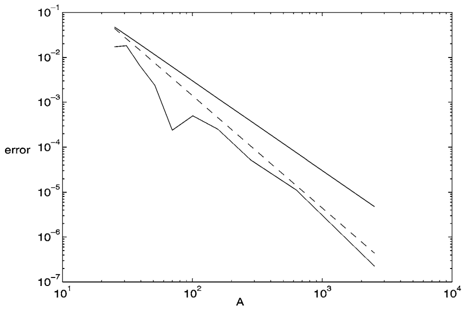
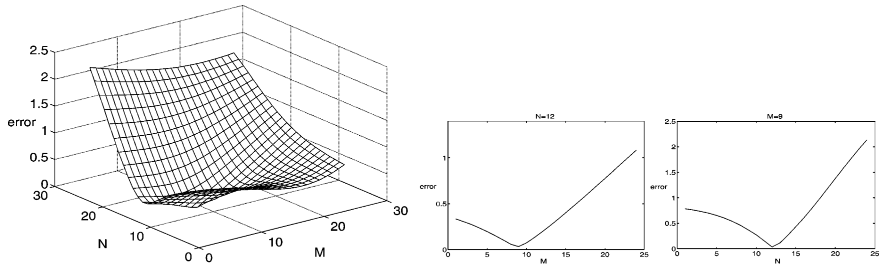


Fig. 7. The 'approximation error' $|\langle F - \widetilde{F}_{A,2}, G_{\text{tot}}\rangle|$ vs. the frame bound A for the second-order case. Both axes are logarithmic. The solid line seen in the figure is the graph $\{(A, 30A^{-2})\colon 25.13 < A < 1228.64\}$; the dashed line is the graph $\{(A, 150A^{-5/2})\colon 25.13 < A < 1228.64\}$.

the predicted rate, i.e., instead of being $O(A^{-2})$, the approximation error seems to be of order $A^{-5/2}$. This again matches the empirical error decay rate observed for the standard second-order sigma–delta quantizers.

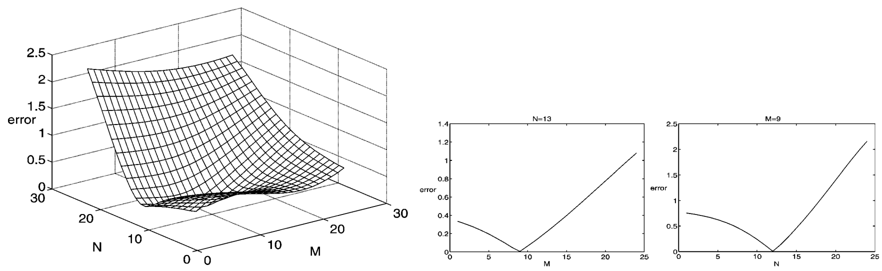
Next, we want to observe the translation invariance of the second-order quantizers. To this end, we repeat the experiment we did in Section 4: Fix the frame $(\varphi, 0.1, 0.1)$ and compute $q = T_{\text{TF}_2}(c)$, i.e., use a second order quantizer, where $c_{n,m} = \langle f, \varphi_{n,m}\rangle$. Now, as in Section 4, define $f_{T,\Omega}$ by $f_{T,\Omega} := M_{-\Omega}T_T f$. Let $c_{T,\Omega}$ be the sequence $(\langle f_{T,\Omega}, \varphi_{n,m}\rangle)$ and $q_{T,\Omega} := T_{\text{TF}_2}(c_{T,\Omega})$. Using q as a template, we will estimate T and Ω when we are only given the sequence $q_{T,\Omega}$. To accomplish this, we will compare $\widetilde{F}_{T,\Omega,A,2} := \sum (q_{T,\Omega})_{n,m}\langle\varphi_{n,m}, \varphi_{\tau,\xi}\rangle$ with $I_{N,M} := \sum(\gamma_N)^{m+M} q_{n+N,m+M}\langle\varphi_{n,m}, \varphi_{\tau,\xi}\rangle$ for various N and M by comparing the inner products $\langle\widetilde{F}_{T,\Omega,A,2} - I_{N,M}, G_{\text{tot}}\rangle$. We will calculate these inner products via (5.112). Since the frame constant A is large ($A \approx 628$ in this case), we expect according to Theorem 13 (although it is not guaranteed) to have $T \approx 0.1\overline{N}$ and $\Omega \approx 0.1\overline{M}$ where $(\overline{N}, \overline{M}) = \arg\inf_{(N,M)\in\mathbb{Z}^2}\langle\widetilde{F}_{T,\Omega,A} - I_{N,M}, G_{\text{tot}}\rangle$ if T and Ω are integer multiples of $\tau_0 = 0.1$ and $\xi_0 = 0.1$, respectively.

For $T = 1.2 = 12\tau_0$ and $\Omega = 0.9 = 9\tau_0$, we observe in Fig. 8 that the minimum is attained at $(N, M) = (12, 9)$, in other words our algorithm estimated the translation amounts T and Ω correctly. Next we test whether the algorithm can detect translation and modulation amounts that are *not* integer multiples of τ_0 and ξ_0 (of course with the resolution given by τ_0 and ξ_0). Figure 9 shows the result when



(a) (b) (c)

Fig. 8. The value $\langle \widetilde{F}_{T,\Omega,A,2} - I_{N,M}, G_{\mathrm{tot}}\rangle$ vs. N and M for $T = 1.2$ and $\Omega = 0.9$; the minimum is obtained at $N = 12$ and $M = 9$, which means that the algorithm predicts $T = 1.2$ and $\Omega = 0.9$, i.e., the correct values of T and Ω. Figure 8b shows $\langle \widetilde{F}_{T,\Omega,A,2} - I_{12,M}, G_{\mathrm{tot}}\rangle$ vs. M; Fig. 8c shows $\langle \widetilde{F}_{T,\Omega,A} - I_{N,9}, G_{\mathrm{tot}}\rangle$ vs. N.



(a) (b) (c)

Fig. 9. The value $\langle \widetilde{F}_{T,\Omega,A,2} - I_{N,M}, G_{\mathrm{tot}}\rangle$ vs. N and M for $T = 1.17$ and $\Omega = 0.93$; the minimum is obtained at $N = 12$ and $M = 9$, which means that the algorithm predicts $T = 1.2$ and $\Omega = 0.9$. Figure 9b shows $\langle \widetilde{F}_{T,\Omega,A} - I_{12,M}, G_{\mathrm{tot}}\rangle$ vs. M; Fig. 9c shows $\langle \widetilde{F}_{T,\Omega,A} - I_{N,9}, G_{\mathrm{tot}}\rangle$ vs. N.

$T = 1.17$ and $\Omega = 0.93$. One observes that the algorithm has estimated T and Ω as well as the resolution allows.

Finally, we add noise to our signal the way we described in Section 4, and again we use our algorithm to estimate the translation and modulation amounts T and Ω. We define $\widetilde{F}^{\nu}_{T,\Omega,A,2}$ is defined the same way we defined $\widetilde{F}^{\nu}_{T,\Omega,A}$ just above (4.87), only this time using the q produced by the second-order quantizer. In an experiment with $\mathrm{SNR} = 8.5$ dB, the algorithm estimated T and Ω as 1.2 and 0.7, respectively, where the true values of T and Ω are 1.1 and 0.9, respectively. When we decrease the SNR to 0 dB, the algorithm estimated T and Ω to be 1.4 and 0.6.

Acknowledgments

The author would like to thank Ingrid Daubechies for her helpful comments and advice. The author would also like to thank H.G. Feichtinger for all his comments and suggestions which improved the paper substantially. This work was partially funded by NSF KDI Grant DMS-9872890 and AFOSR Grant F49620-98-1-0044.

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