Hydraulic Fracture: multiscale processes and moving interfaces

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WORKSHOP ON ROCK MECHANICS AND LOGISTICS IN MINING

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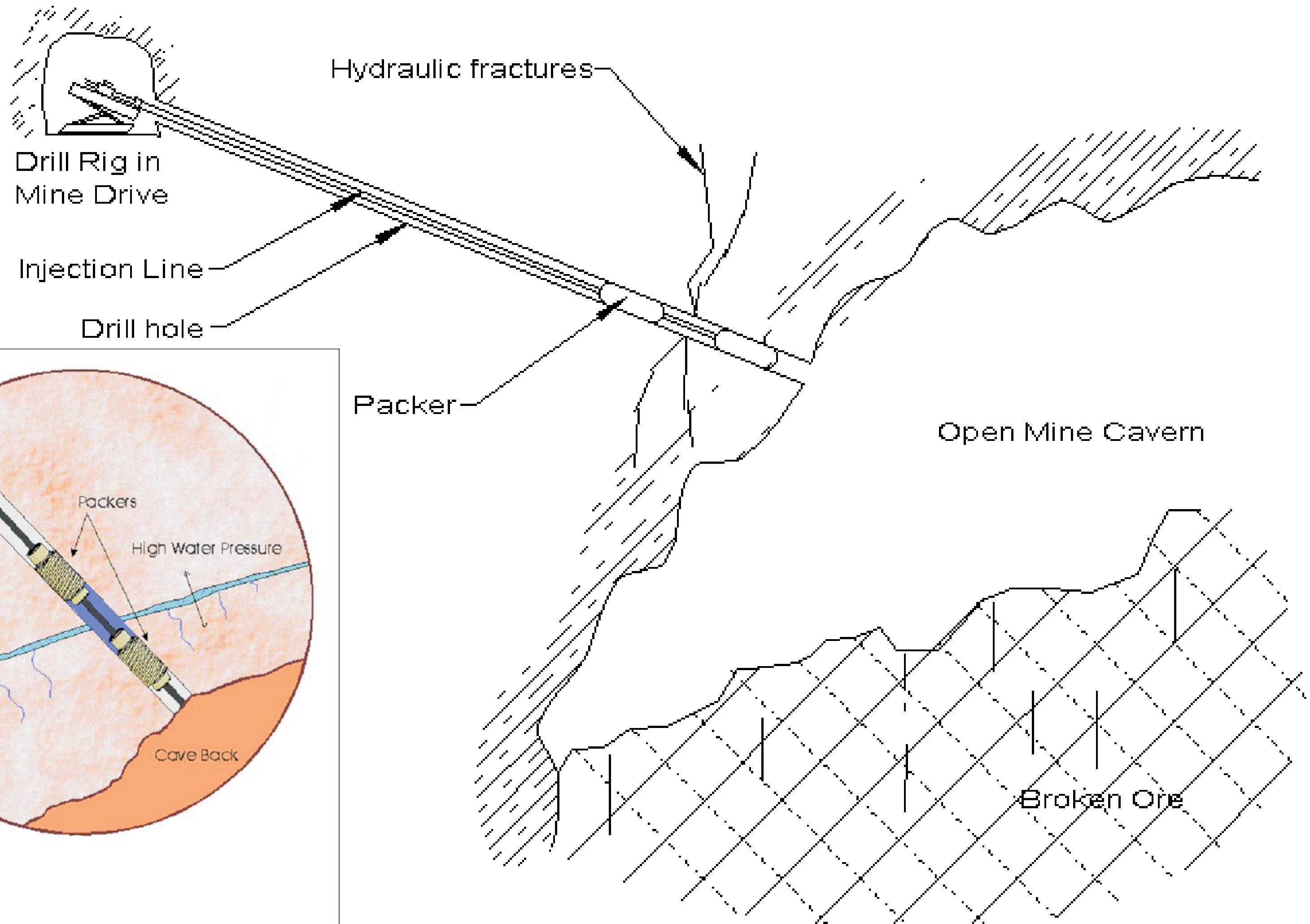
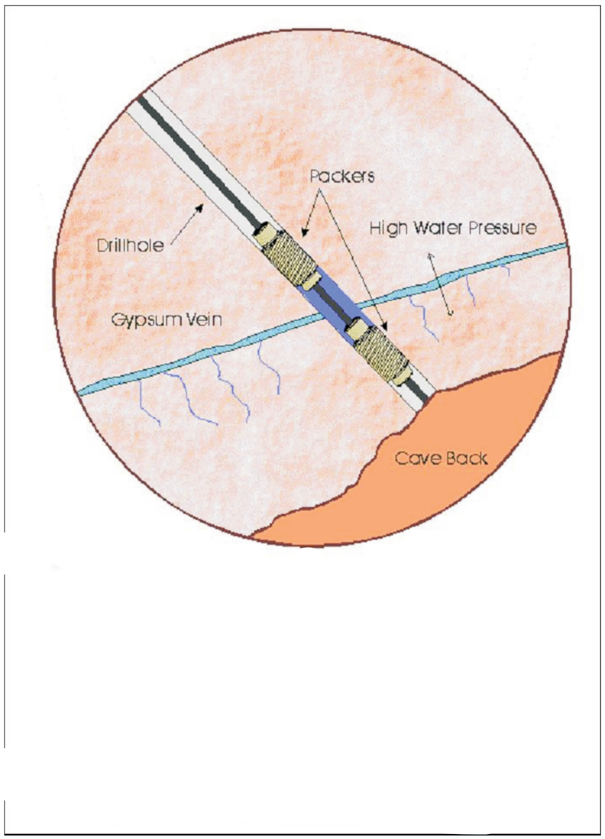
Outline

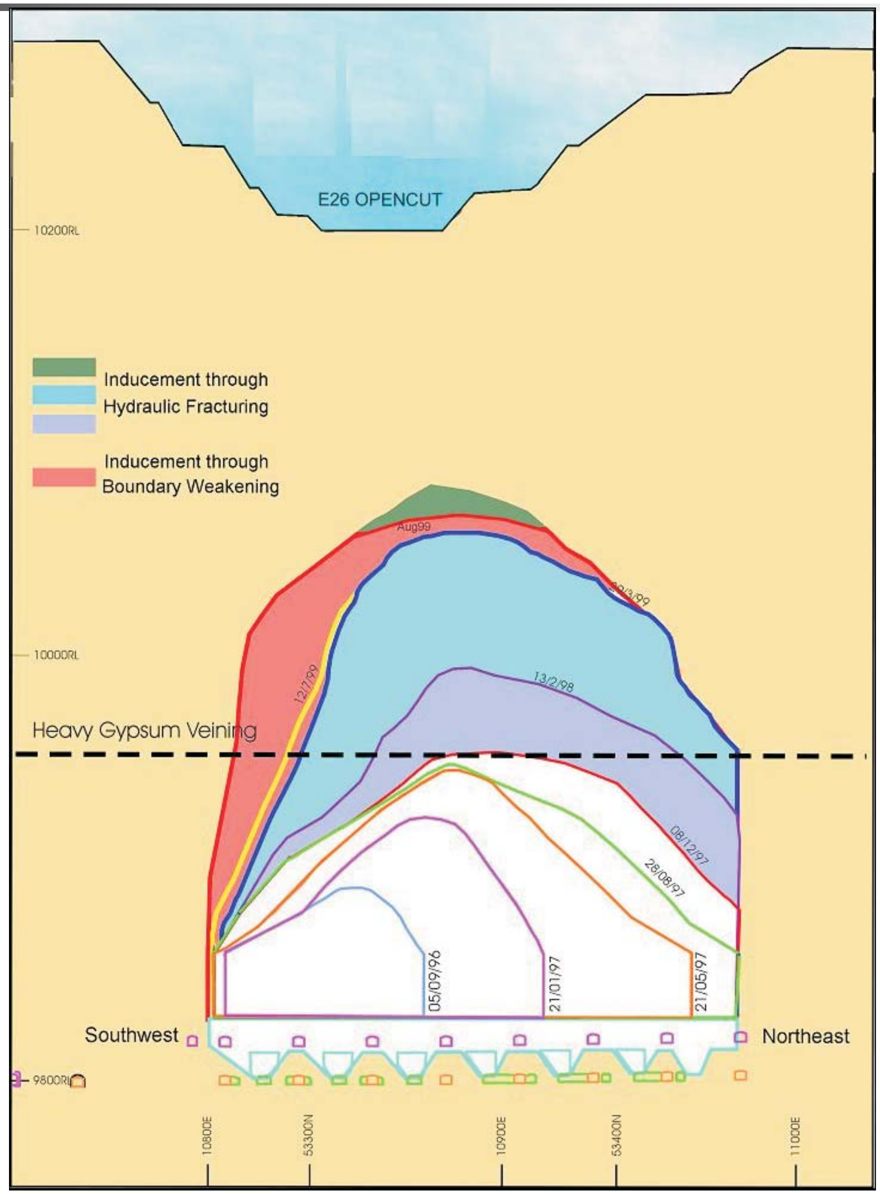
- What is a hydraulic fracture?
- Mathematical models of hydraulic fracture
- Scaling and special solutions for 1-2D models
- Numerical modeling for 2-3D problems
- Conclusions

HF Examples - block caving

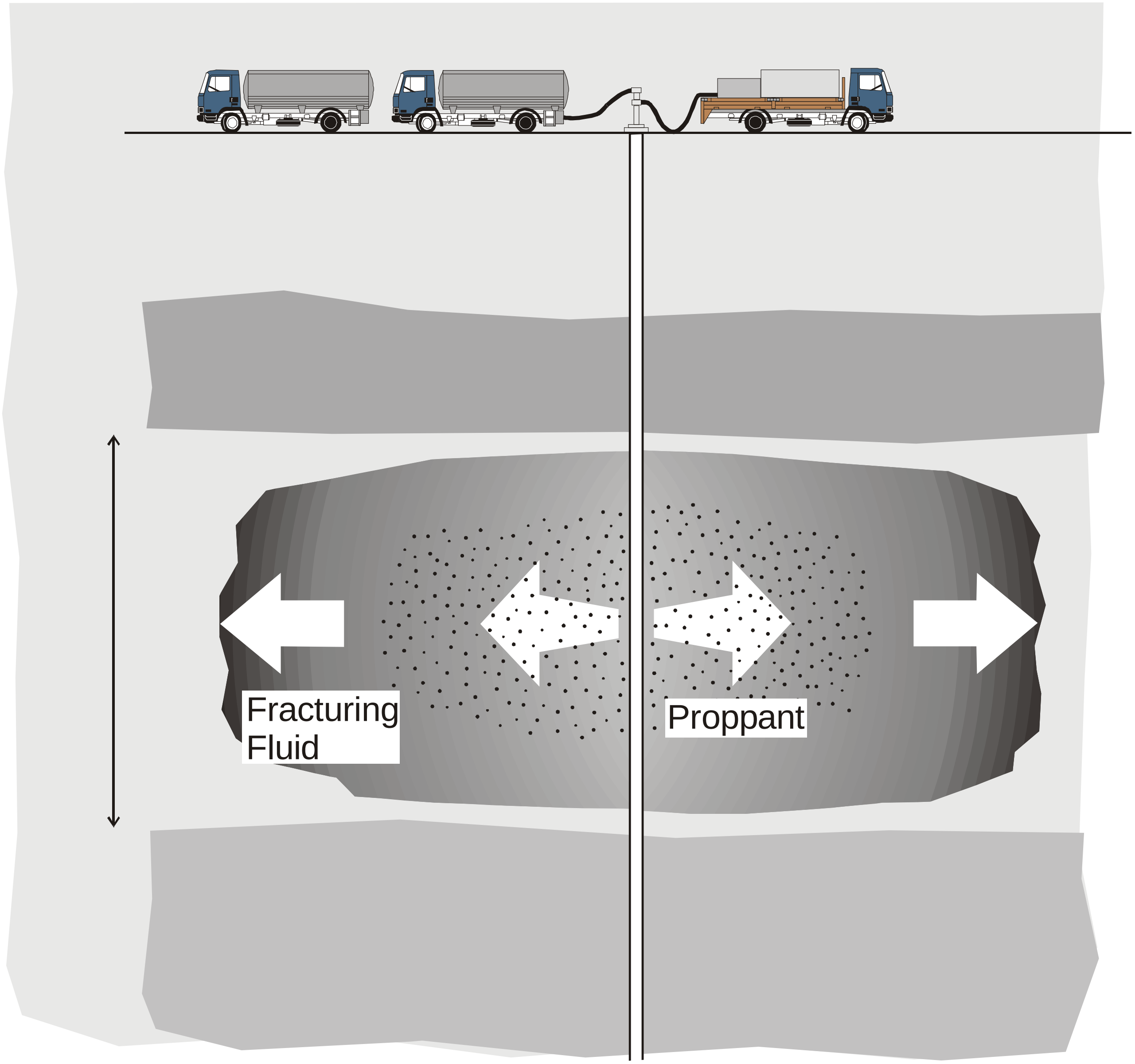




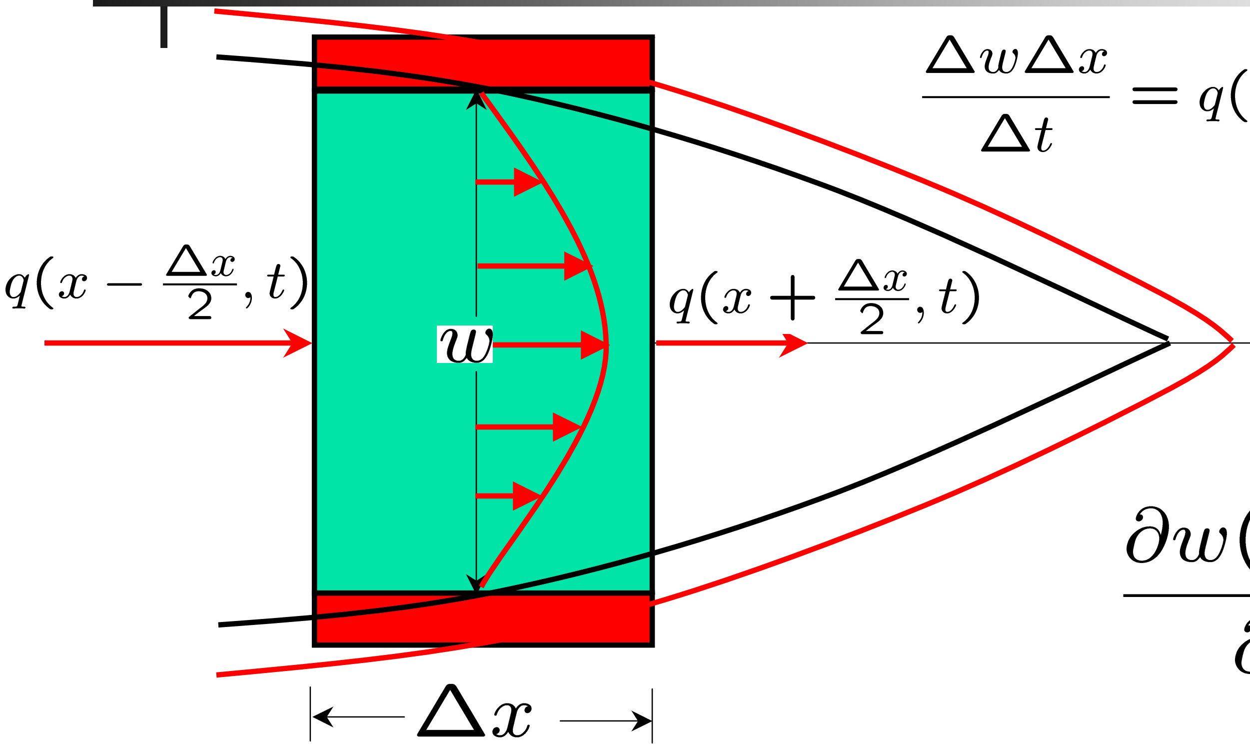
HF Example – caving (Jeffrey, CSIRO)



HF Examples – well stimulation



Model EQ 1: Conservation of mass



$$\frac{\Delta w \Delta x}{\Delta t} = q(x - \frac{\Delta x}{2}, t) - q(x + \frac{\Delta x}{2}, t)$$

$$\Downarrow \lim_{\Delta x,\ \Delta t \to 0}$$

$$\frac{\partial w(x,t)}{\partial t} = -\frac{\partial q(x,t)}{\partial x}$$

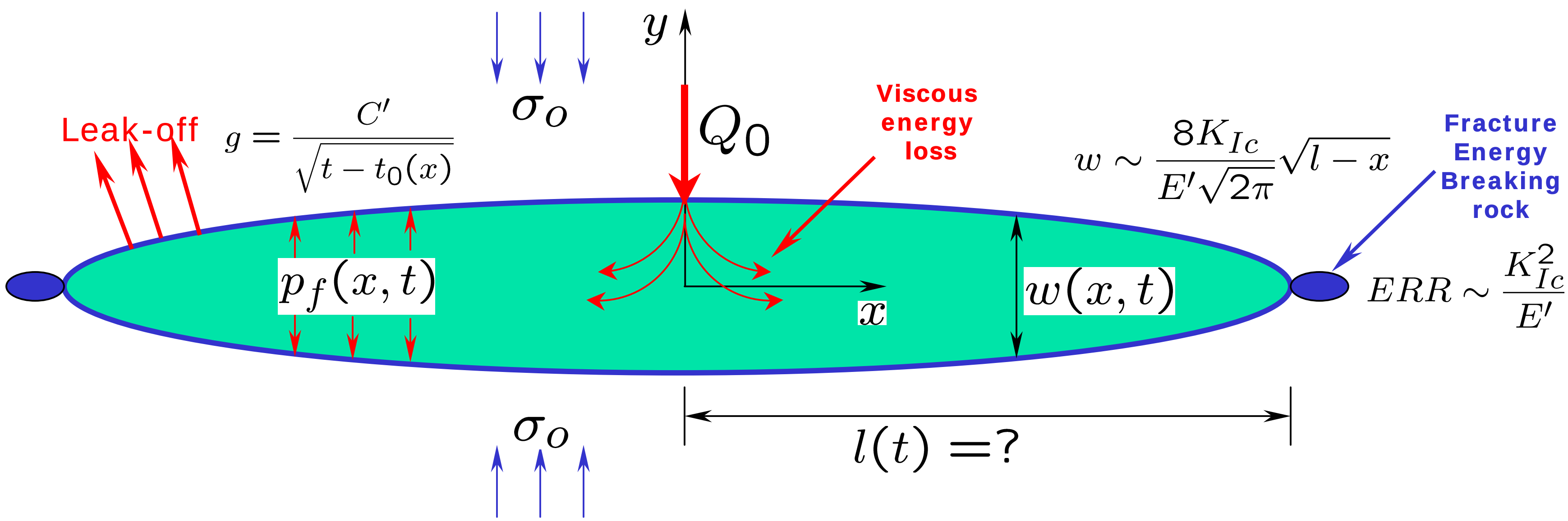
$$v_x = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left(\frac{w^2}{4} - y^2\right)$$

$$q(x,t) = -\frac{w^3}{12\mu}\frac{\partial p}{\partial x}$$

$$\boxed{\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(\frac{w^3}{\mu'}\frac{\partial p}{\partial x}\right)}$$

1-2D model and physical processes

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(\frac{w^3}{\mu'}\frac{\partial p_f}{\partial x}\right) + Q_0\delta(x) - g; \quad w^3\frac{\partial p_f}{\partial x}\Big]_{\pm l} = 0$$



$$p_f(x,t) - \sigma_o = -\frac{E'}{4\pi}\int_{-l(t)}^{l(t)} \frac{\partial w}{\partial \xi}\frac{d\xi}{\xi - x}; \quad w(\pm l, t) = 0$$

Scaling and dimensionless quantities

- Rescale: $\xi = x/l, \quad l = L\gamma, \quad w = \epsilon L \Omega, \quad p = \epsilon E' \Pi$
- Dimensionless quantities:

$$\mathcal{G}_v = \frac{Q_0 t}{\epsilon L^2}, \ \mathcal{G}_m = \frac{\mu'}{\epsilon^3 E' t}, \ \mathcal{G}_k = \frac{K'}{\epsilon E' L^{1/2}}, \ \mathcal{G}_c = \frac{C' t^{1/2}}{\epsilon L}$$

- Governing equations become:

$$\frac{t(\epsilon L)_t}{\epsilon L}\Omega + \dot{\Omega} t - \frac{t(L\gamma)_t}{L\gamma}\xi \frac{\partial \Omega}{\partial \xi} + \frac{\boxed{\mathcal{G}_c}}{\sqrt{1 - \frac{t_0(\xi l)}{t}}} = \frac{1}{\boxed{\mathcal{G}_m}\gamma^2}\frac{\partial}{\partial \xi}\left(\Omega^3 \frac{\partial \Pi}{\partial \xi}\right)$$

$$\Pi = -\frac{1}{2\pi\gamma}\int_0^1 \frac{\partial \Omega}{\partial \chi}\frac{d\chi}{\chi - \xi} \qquad \Omega \overset{\xi \to \pm 1}{=} \boxed{\mathcal{G}_k}\gamma^{1/2}\sqrt{1-\xi}$$

$$\boxed{\mathcal{G}_v} = 2\gamma \int_0^1 \Omega d\chi + 2\boxed{\mathcal{G}_c}\frac{1}{L}\int_0^1 l(\theta t)\theta^{-1/2}\int_0^1 \Gamma d\chi d\theta$$

Toughness dominated propagation

- Asymptotic behaviour of the Hilbert transform

$$\int_{-1}^{1} \frac{(1-\chi)^{\alpha-1}}{\chi-\xi} d\chi \overset{\xi \to 1_-}{\sim} \pi cot(\pi\alpha)(1-\xi)^{\alpha-1} + D$$

- Large toughness: $\mathcal{G}_k = 1; \ \mathcal{G}_m \ll 1; \ \mathcal{G}_c \ll 1$

$$\frac{1}{\gamma^2}\frac{\partial}{\partial \xi}\left(\Omega^3 \frac{\partial \Pi}{\partial \xi}\right) \approx 0 \quad \Rightarrow \quad \Omega^3 \frac{\partial \Pi}{\partial \xi} \approx c$$

$$\left.\begin{array}{c} \Omega \sim c_0 (1-\xi)^{\alpha} \\ \Pi \sim c_1 (1-\xi)^{\alpha-1} \end{array}\right\} \Rightarrow (1-\xi)^{3\alpha}(1-\xi)^{\alpha-2} = \bar{c}$$

$$\Pi \sim D; \quad \Omega \sim c_0\sqrt{1-\xi}$$

Viscosity dominated propagation

- Asymptotic behaviour of the Hilbert transform

$$\int_{-1}^{1} \frac{(1-\chi)^{\alpha-1}}{\chi-\xi} d\chi \overset{\xi\to 1_-}{\sim} \pi cot(\pi\alpha)(1-\xi)^{\alpha-1} + D$$

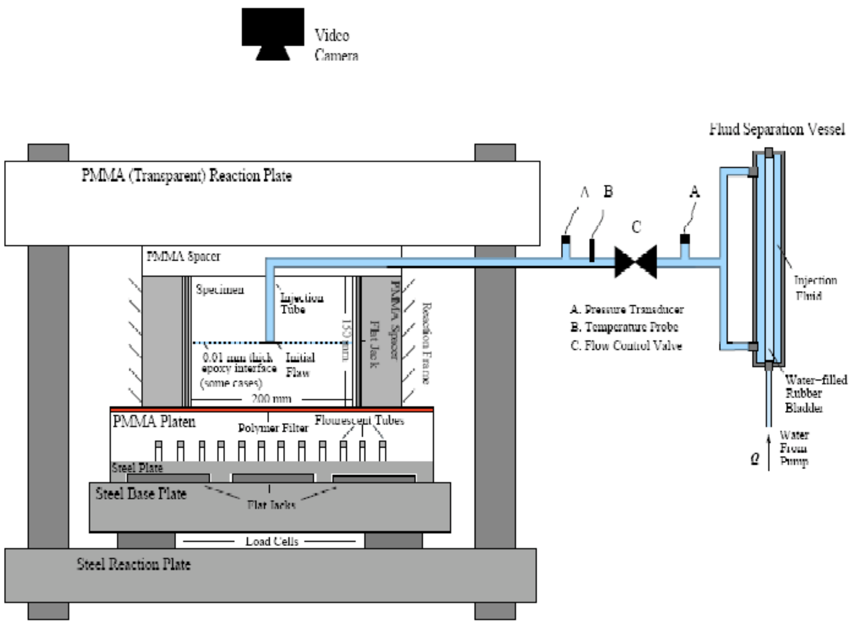
- Large viscosity: $\mathcal{G}_k \ll 1;\ \mathcal{G}_m = 1;\ \mathcal{G}_c \ll 1$

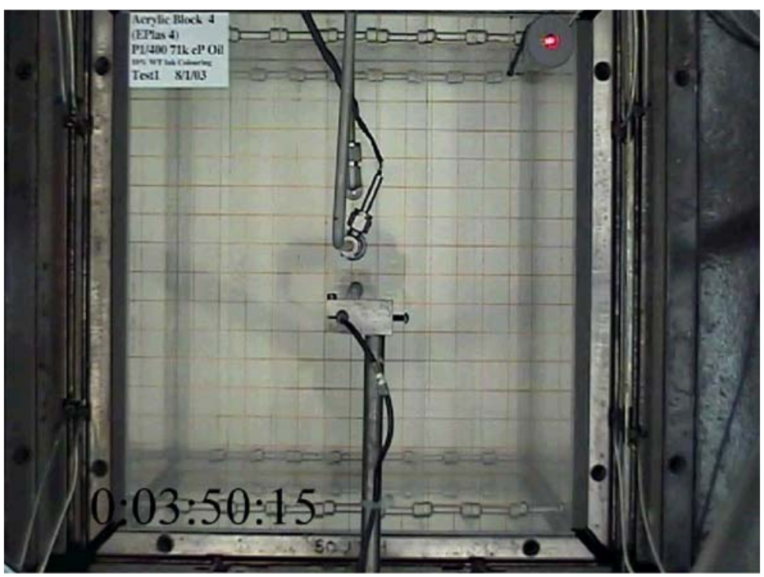
$$\frac{t(L\gamma)_t}{L\gamma}\xi\frac{\partial\Omega}{\partial\xi} \approx \frac{1}{\gamma^2\mathcal{G}_m}\frac{\partial}{\partial\xi}\left(\Omega^3\frac{\partial\Pi}{\partial\xi}\right) \overset{\xi\to 1}{\Rightarrow} \Omega^2\frac{\partial\Pi}{\partial\xi} \approx c$$

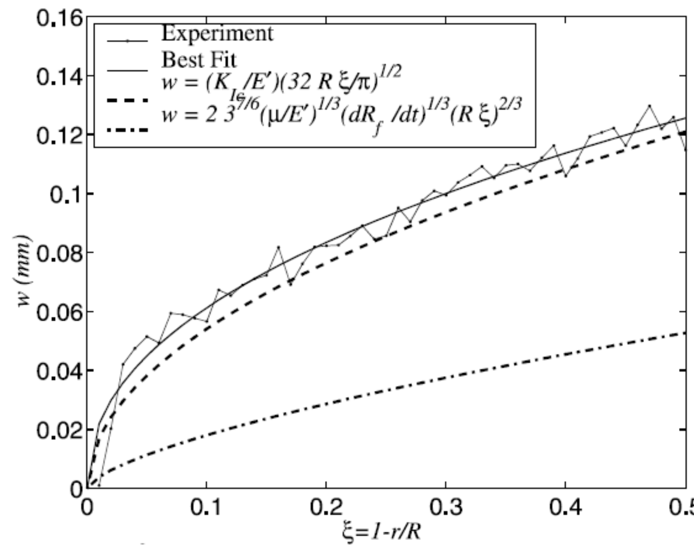
$$\left.\begin{array}{r} \Omega \sim c_2\,(1-\xi)^{\alpha} \\ \Pi \sim c_3\,(1-\xi)^{\alpha-1} \end{array}\right\} \Rightarrow (1-\xi)^{2\alpha}(1-\xi)^{\alpha-2} = \bar{c}$$

$$\Omega \sim c_2\,(1-\xi)^{2/3}; \quad \Pi \sim c_3\,(1-\xi)^{-1/3}$$

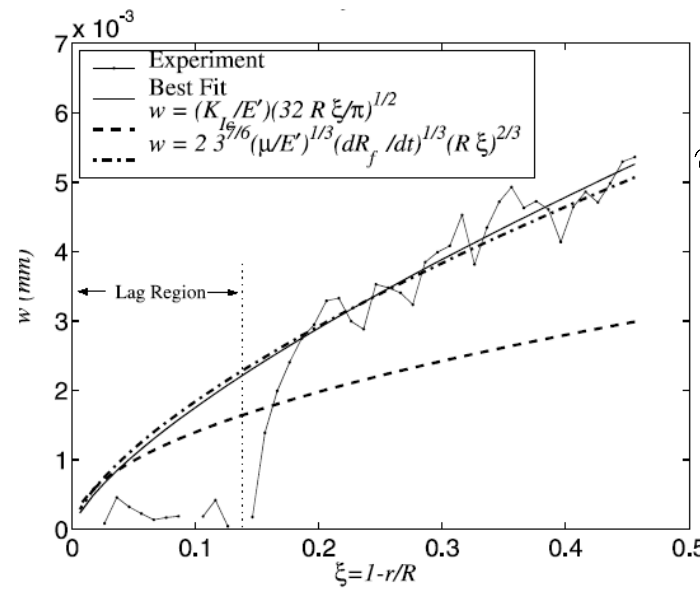
HF experiment (*Bunger & Jeffrey* CSIRO)





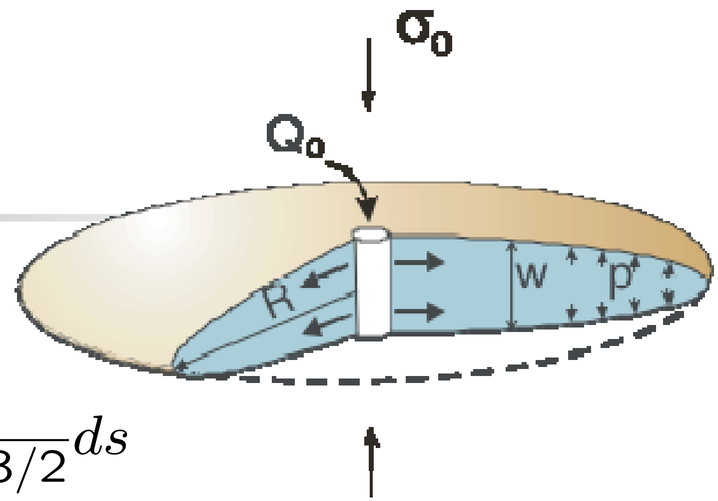


$$w \sim \frac{K'}{E'} s^{1/2}$$



$$w \sim c\,\frac{\mu' v}{E'} s^{3/2}$$

2-3D HF Equations



- Elasticity (non-locality)

$$p_f - \sigma_o = -\frac{E'}{8\pi}\int_{S(t)} \frac{w(x',y',t)}{\left[(x'-x)^2+(y'-y)^2\right]^{3/2}} ds$$

$$\Rightarrow \quad Cw = p_f - \sigma_o$$

- Lubrication (non-linearity)

$$\frac{\partial w}{\partial t} = \nabla\cdot\left(\frac{w^3}{\mu'}\nabla p_f\right) + Q(t)\delta(x,y)$$

$$\Rightarrow \quad \frac{\Delta w}{\Delta t} = A(w)p_f + S$$

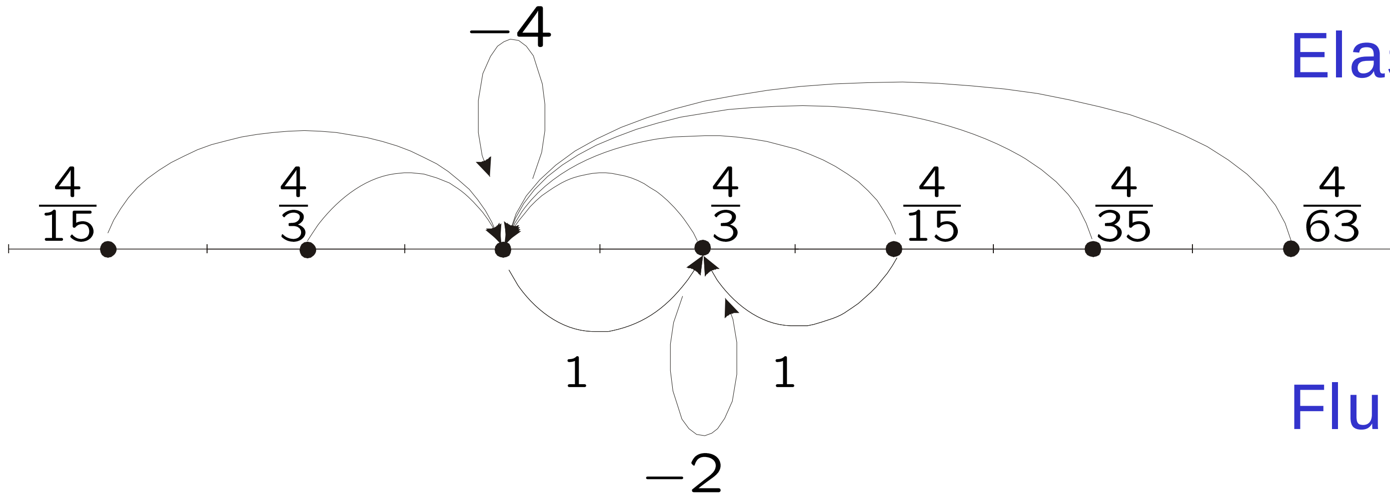
$$\frac{\partial w}{\partial t} = A(w)Cw$$

- Boundary conditions at moving front (free boundary)

$$\lim_{s\to 0}\frac{w}{s^{1/2}} = \frac{K'}{E'}, \qquad \lim_{s\to 0} w^3\frac{\partial p_f}{\partial s} = 0$$

Coupled equations – a model problem

Elasticity



Fluid Flow

n-2 n-1 n n+1 n+2

$$\lambda \int_{-l}^{l} \frac{w(s,t)}{(s-x)^2}\,ds = p \xrightarrow{\;C\;} \frac{\lambda}{\Delta x}\sum_{n=1}^{N} \frac{w_n}{(m-n)^2-\frac{1}{4}} = p_m$$

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(D(w)\frac{\partial p}{\partial x}\right) \xrightarrow{\;A\;} Ap_n = \frac{\bar{D}}{\Delta x^2}\left(p_{n+1} - 2p_n + p_{n-1}\right)$$

Conditioning of the Jacobian

- Evolution equation for w:

$$\left.\begin{aligned} p &= Cw \\ \frac{\partial w}{\partial t} &= A(w)p \end{aligned}\right\} \Rightarrow \frac{\partial w}{\partial t} = A(w)Cw$$

- Eigenvalues of AC:

$$\widehat{C}_k = \frac{\lambda}{\Delta x} \sum_{k=-\infty}^{\infty} \frac{e^{ikm\Delta x}}{m^2 - \frac{1}{4}} = \frac{\lambda}{\Delta x}\left[2\pi \sin\left(\frac{|k|\Delta x}{2}\right)\right]$$

$$\widehat{A}_k = \frac{\bar{D}}{\Delta x^2}(e^{ik\Delta x} - 2 + e^{-ik\Delta x}) = -\frac{4\bar{D}}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)$$

Explicit Scheme

$$\widehat{A}_k\widehat{C}_k = -\frac{8\pi|\lambda|\bar{D}}{\Delta x^3}\sin^3\left(\frac{|k|\Delta x}{2}\right) \Rightarrow \Delta t < \frac{\Delta x^3}{8\pi|\lambda|\bar{D}}$$

Use implicit time stepping

$$(I - \Delta t AC)w = f$$

General first order iterative method

- Consider solving: $ACw = f$

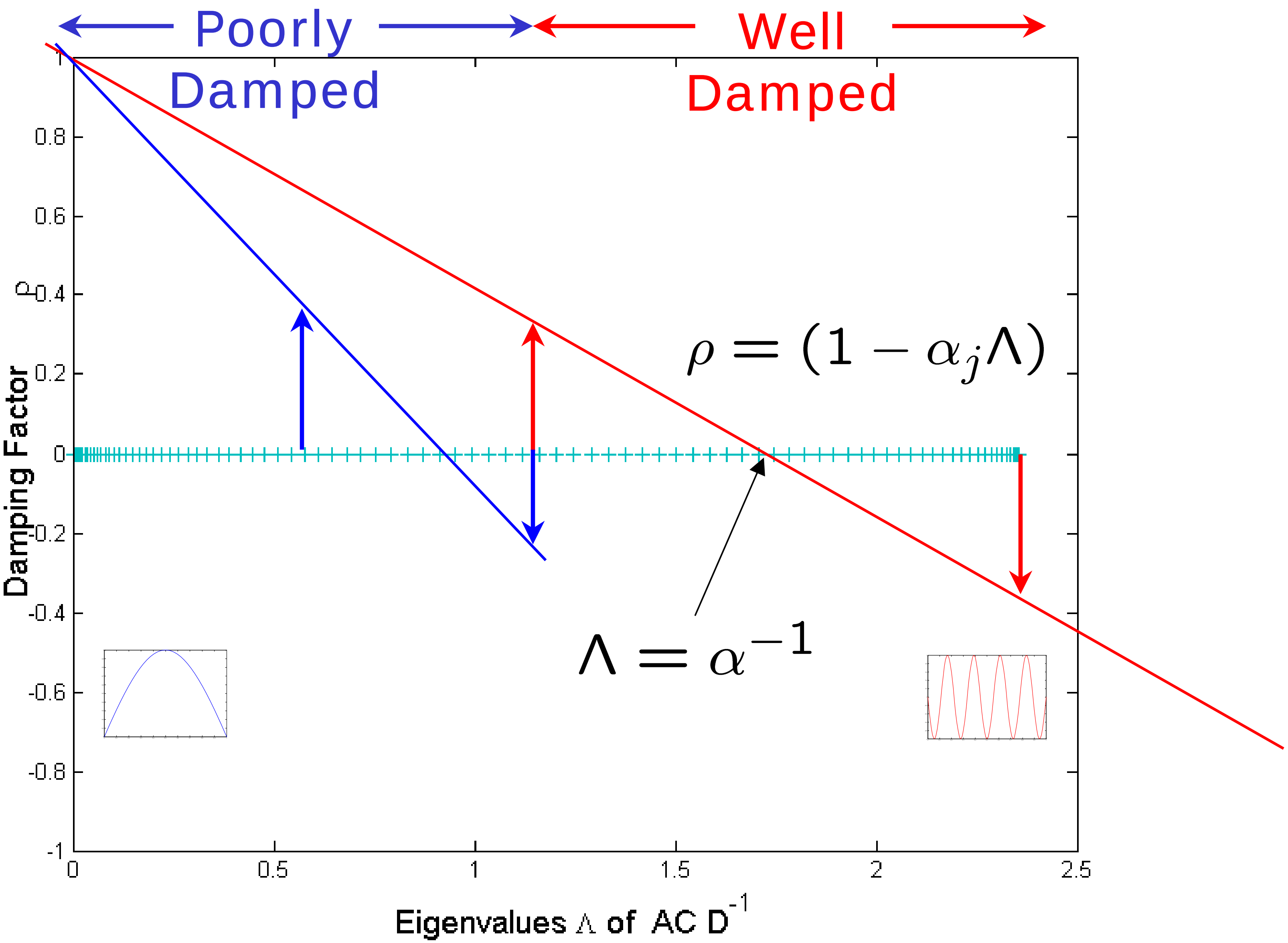
using the iterative method

$$w_{k+1} = w_k + \alpha B^{-1} \underbrace{(f - ACw_k)}_{r_k}$$

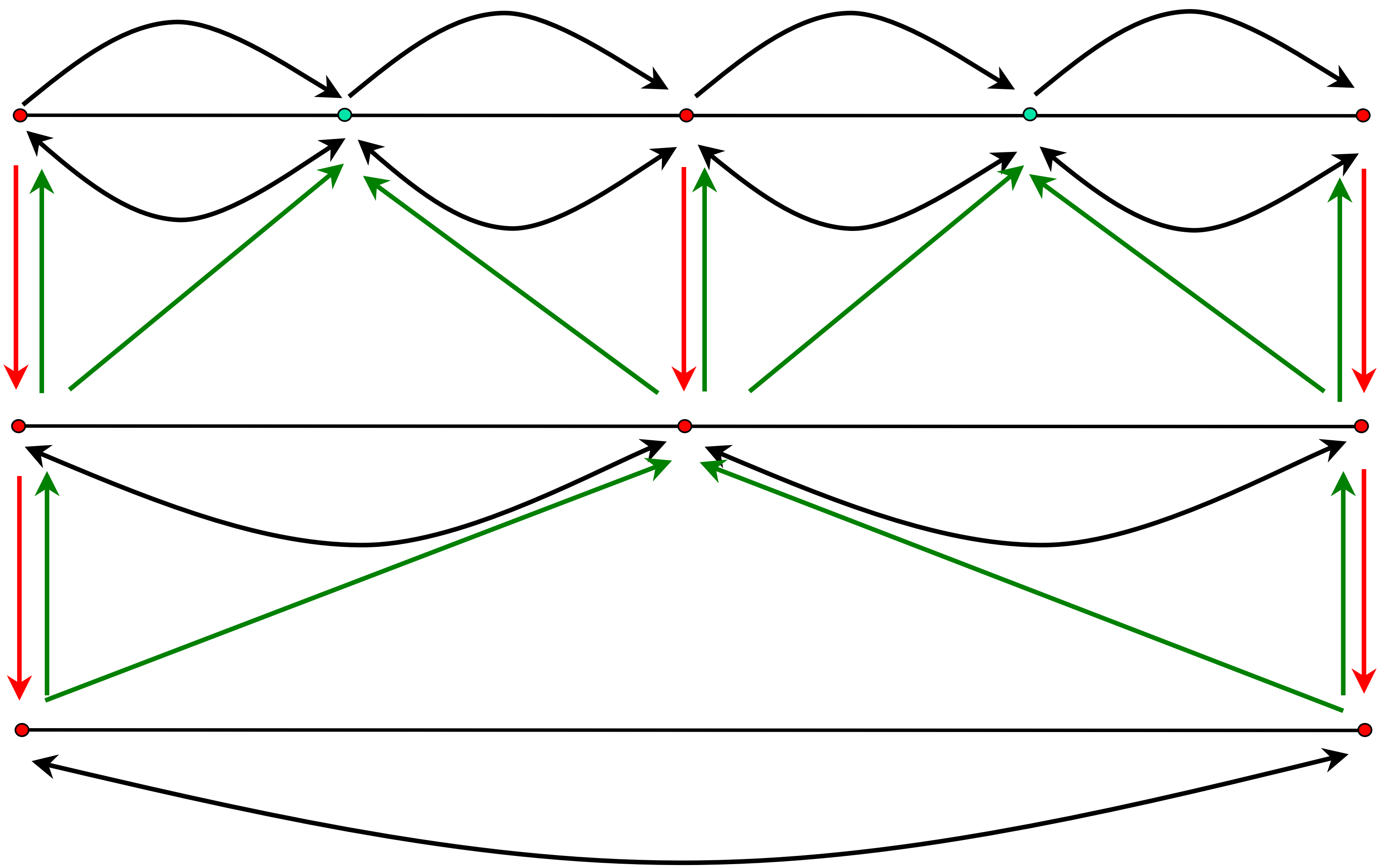
- Residual errors are damped according to

$$\begin{aligned} r_{k+1} &= f - ACw_{k+1} \\ &= f - AC(w_k + \alpha B^{-1} r_k) \\ &= \underbrace{(I - \alpha ACB^{-1})}_{\rho} r_k \end{aligned}$$

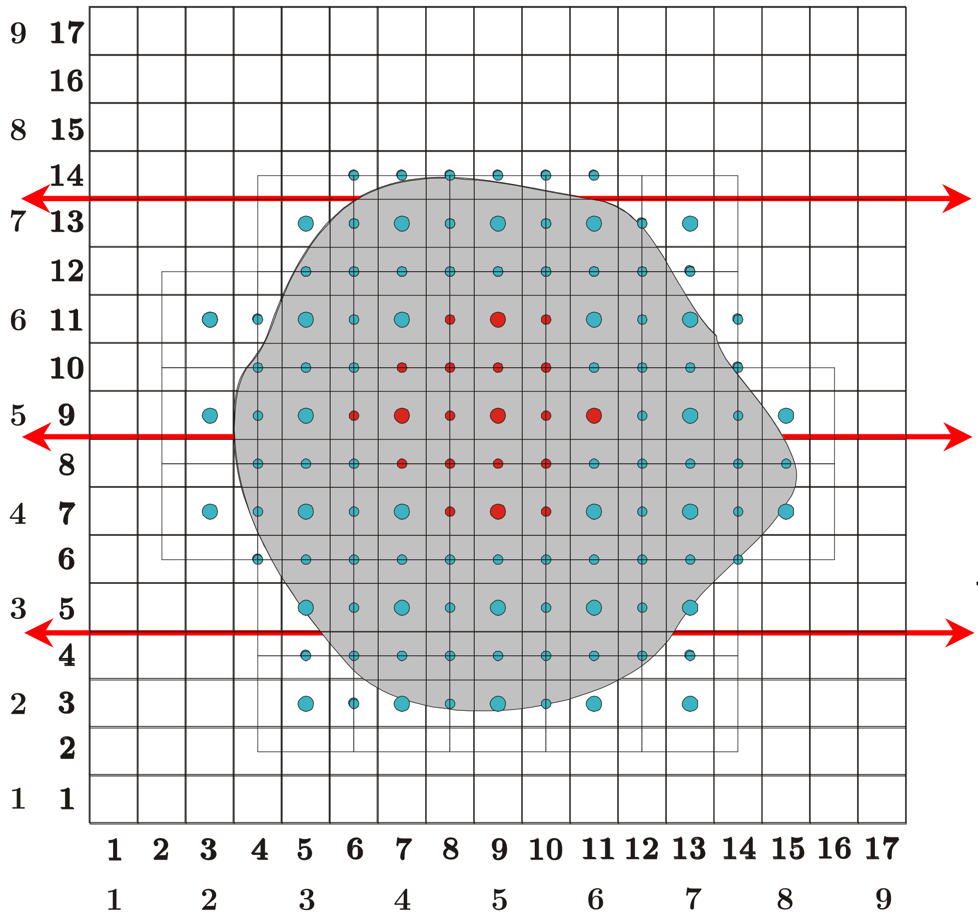
Damping factor



Multigrid Methods



MG approach for coupled HF Equations



$$Cw = p; \quad w \geq w_c$$

$$+$$

$$\frac{\partial w}{\partial t} = A(w)p + S$$

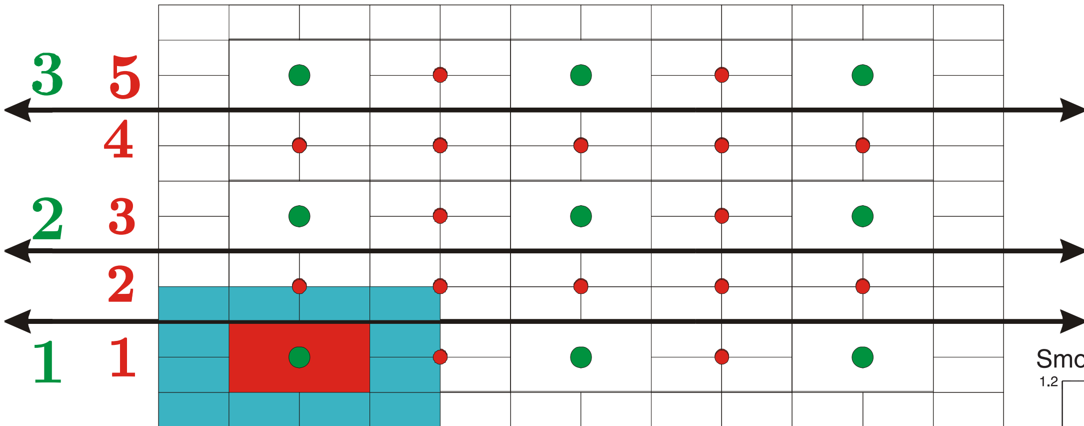
$$\Downarrow$$

$$\frac{\partial w}{\partial t} = A(w)Cw + S$$

$$(I - \Delta t A C)w = f$$

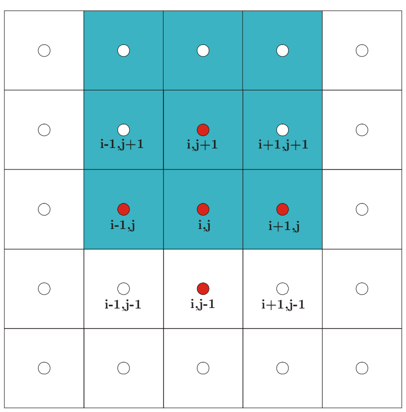
MG preconditioning of $\frac{\Delta w}{\Delta t} = A(w)Cw + S$

- C coarsening using dual mesh

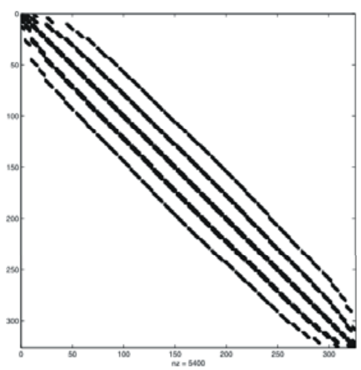


$$C^{2h}_{ij} = C^{h}_{ij} + \sum_{k \in N_j} C^{\frac{h}{2}}_{ik}$$

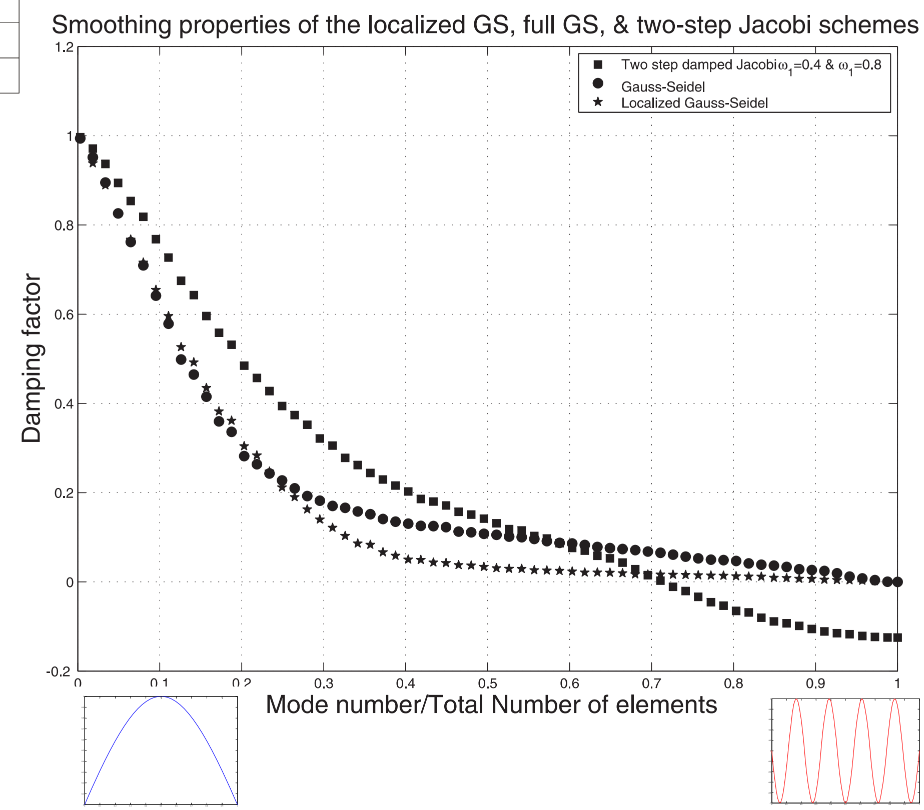
- Localized GS smoother



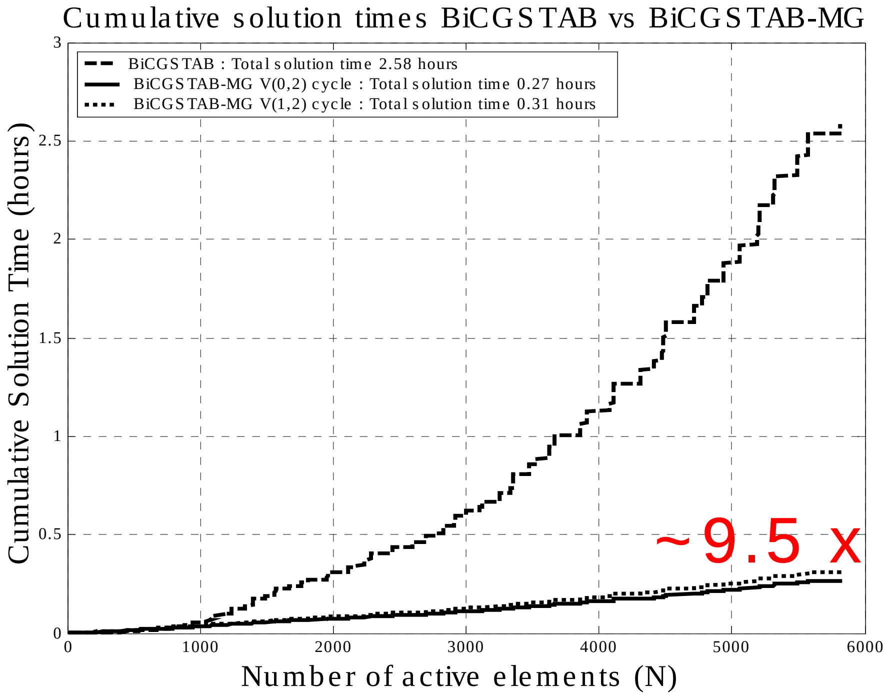
$\Rightarrow$

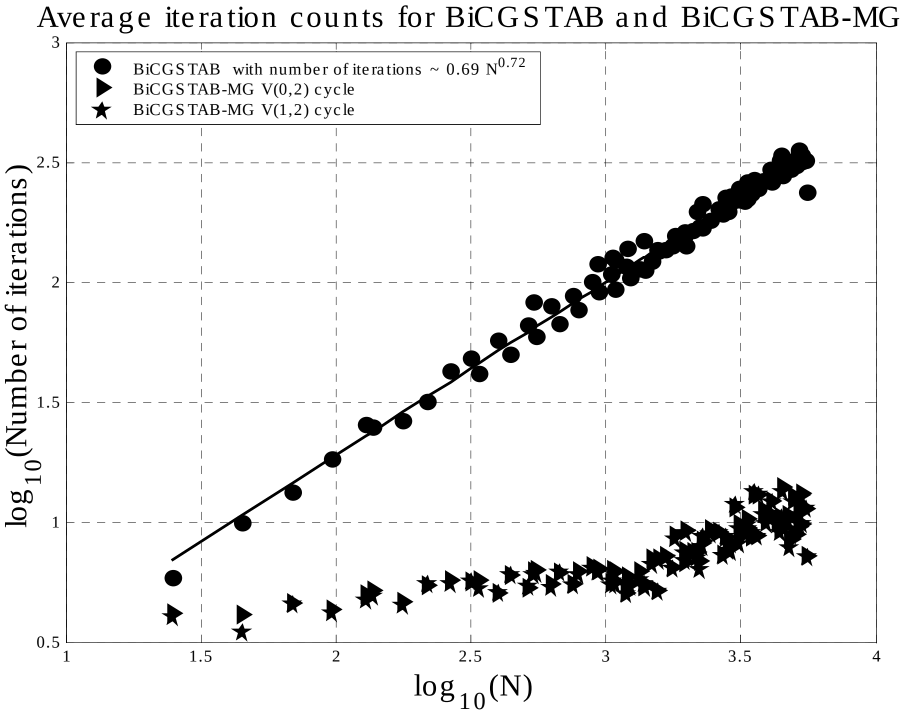


$$[AC]_{MN} \approx \sum_{K \in \mathcal{N}^5_M} A_{MK} C_{KN} \text{ when } N \in \mathcal{N}^9_K$$



Performance of MG Preconditioner



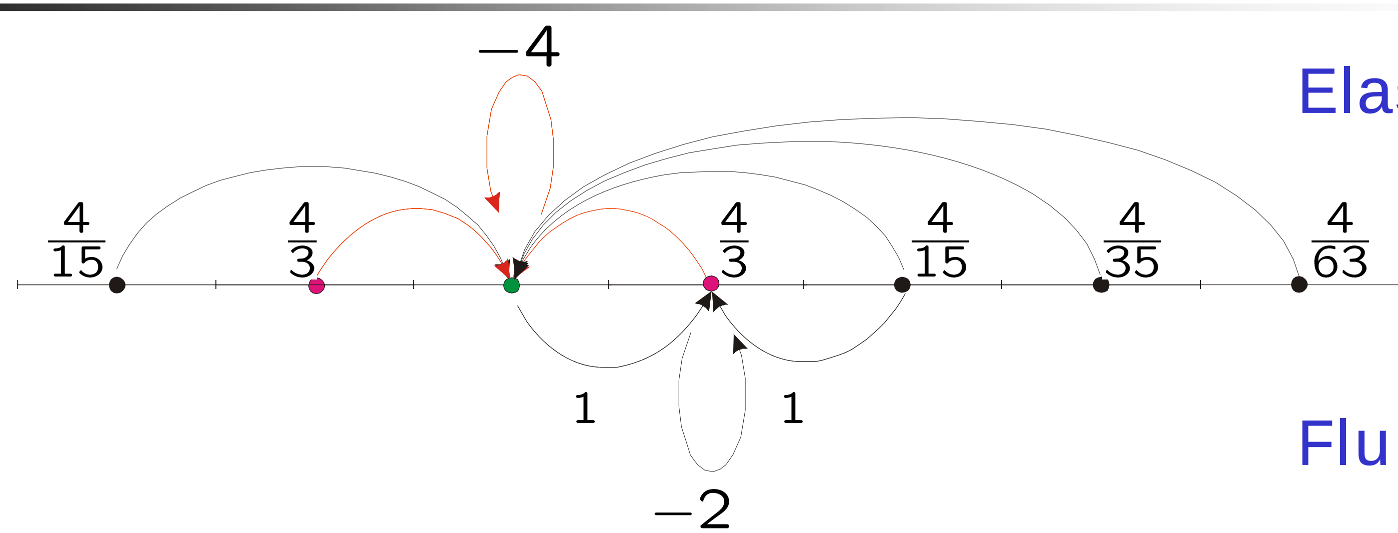


Incomplete LU factorization

```
function [A] = lu(A)
[n,n]=size(A);
for j=1:n
  for k=1:j-1
    for i=k+1:n
      A(i,j)=A(i,j)-A(i,k)*A(k,j);
    end
  end
  for i=j+1:n
    A(i,j)=A(i,j)/A(j,j);
  end
end
return
```

```
function [A] = basicILUc(A)
[n,n]=size(A);
for j=1:n
  for k=1:j-1
    if A(k,j) ~=0,
      for i=k+1:n
        if A(i,j) ~=0,
          A(i,j)=A(i,j)-A(i,k)*A(k,j);
        end
      end
    end
  end
  for i=j+1:n
    if A(i,j) ~=0,
      A(i,j)=A(i,j)/A(j,j);
    end
  end
end
return
```

Local Jacobian & Fourier Analysis



Elasticity

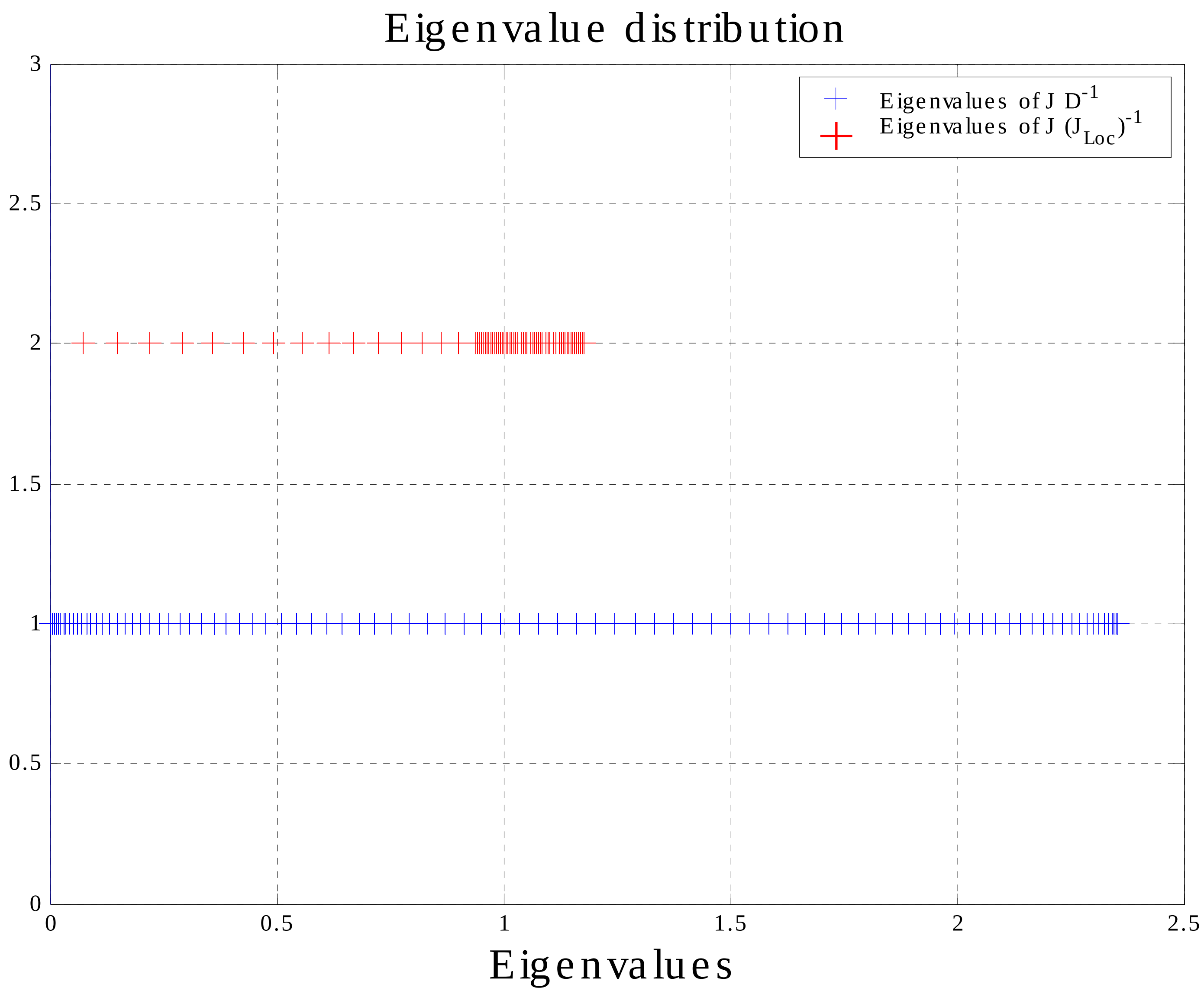
Fluid Flow

n-2 n-1 n n+1 n+2

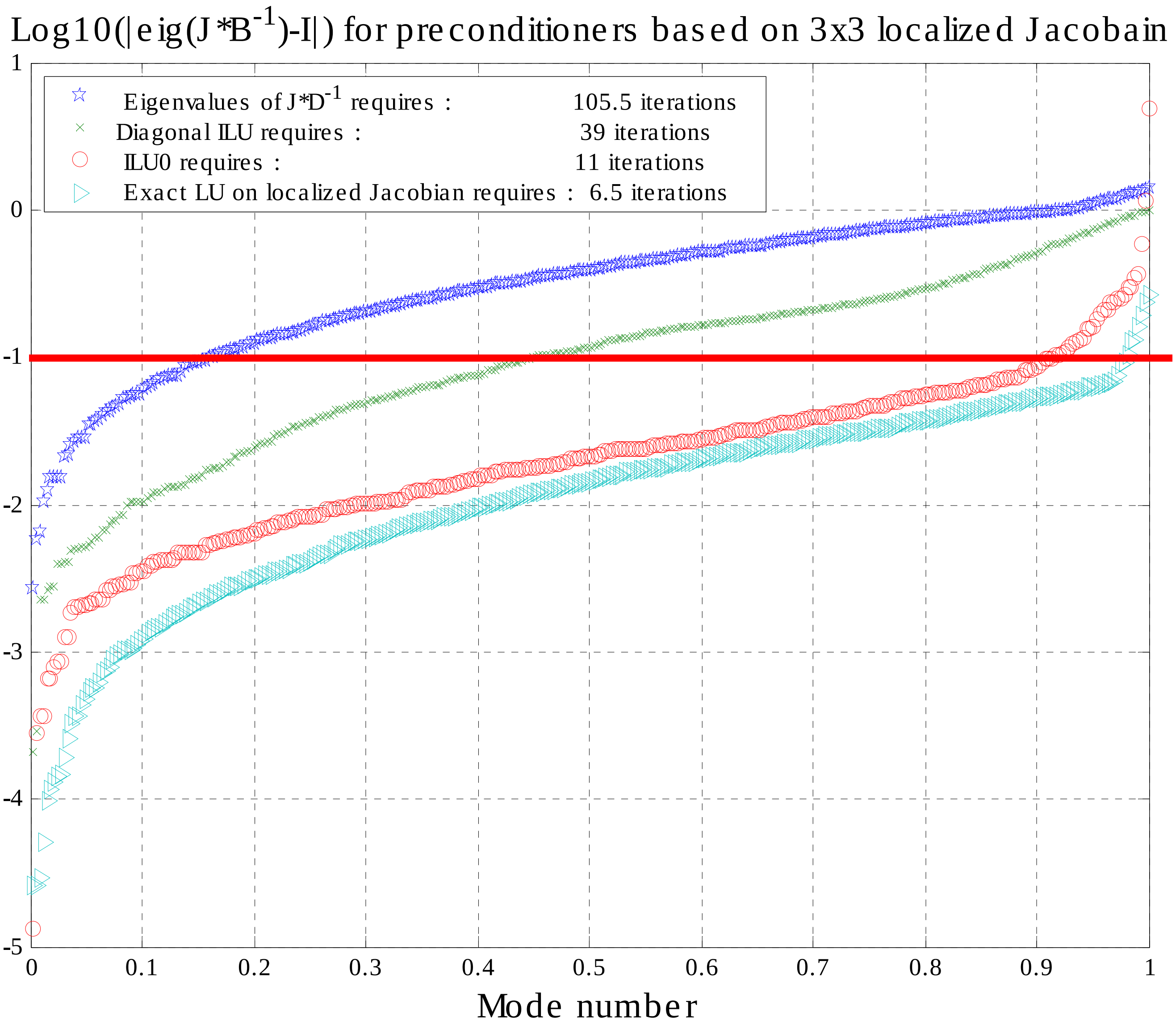
$$C_{Loc} = \frac{\lambda}{\Delta x}[\tfrac{4}{3} \ \ -4 \ \ \tfrac{4}{3}] \xrightarrow{*e^{ik\Delta x}} \hat{C}_{Loc,k} = \frac{4\lambda}{3\Delta x}(2cosk\Delta x - 3)$$

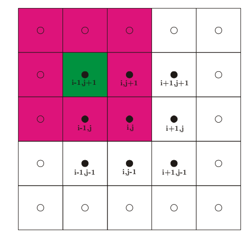
$$A = \frac{\bar{D}}{\Delta x^2}[1 \ \ -2 \ \ 1] \xrightarrow{*e^{ik\Delta x}} \hat{A}_k = -\frac{4\bar{D}}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)$$

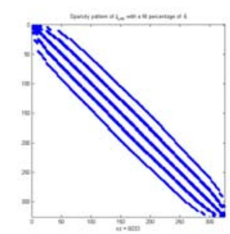
Spectrum of AC $(AC_{Loc})^{-1}$



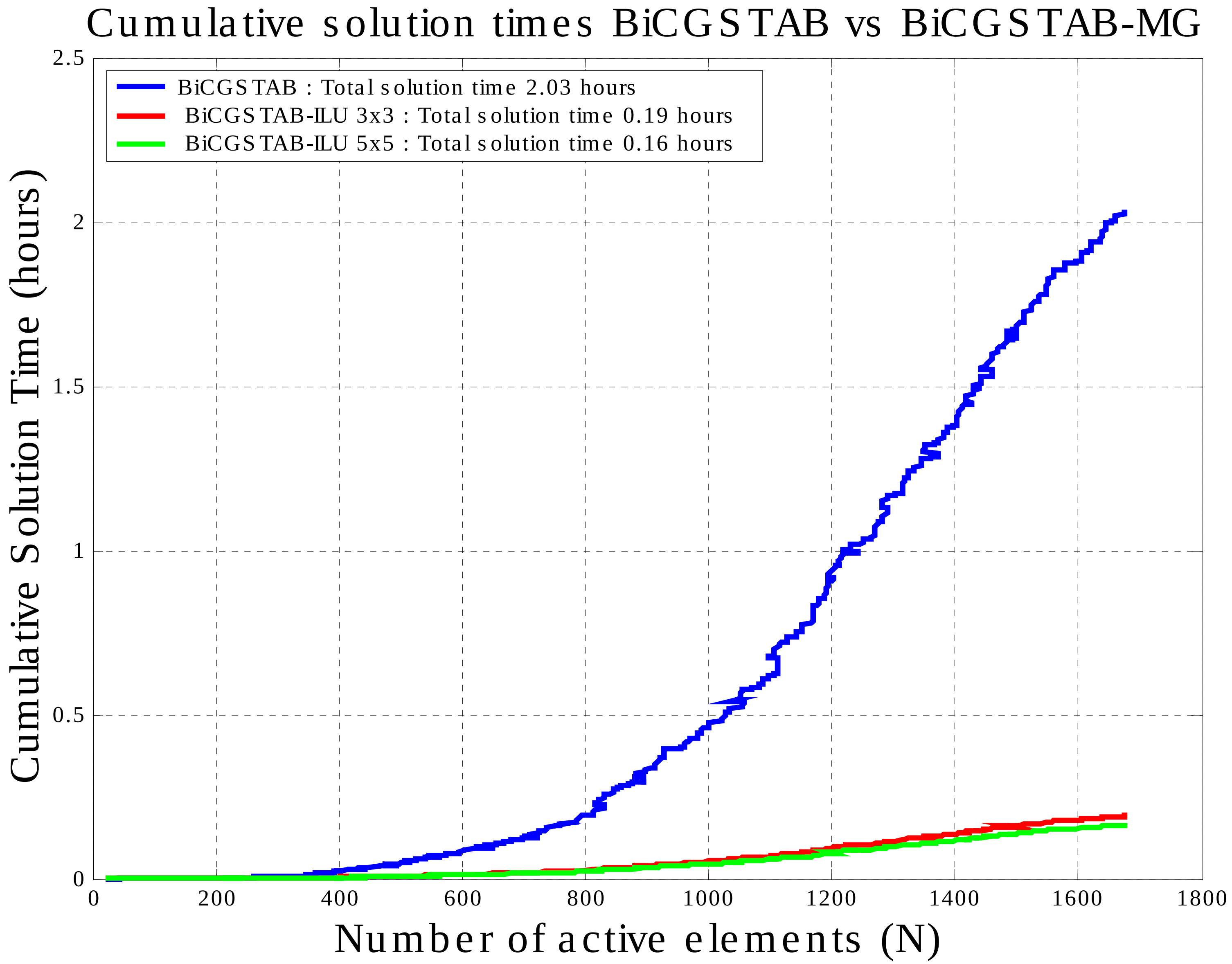
Eigenvalues of preconditioners



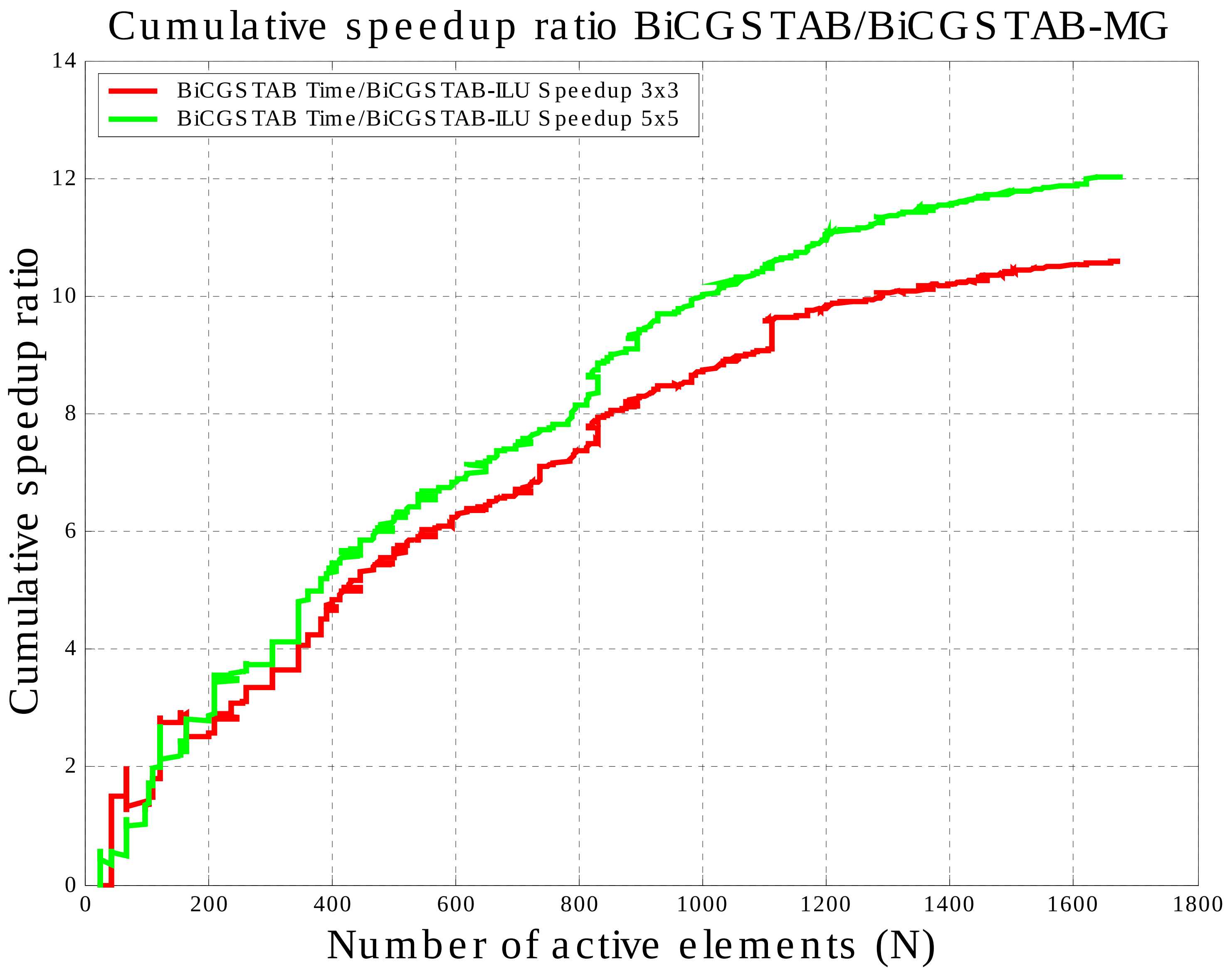




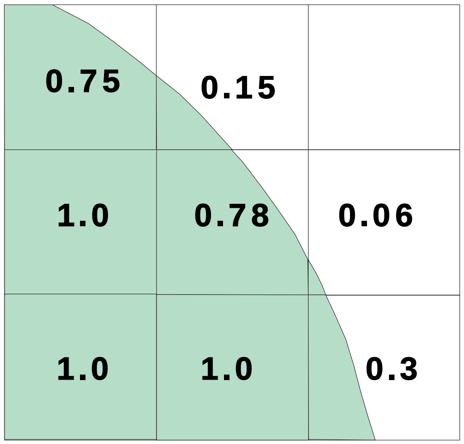
Numerical results for stress jump



Numerical results for stress jump



Front evolution via the VOF method



$$\frac{\partial \chi}{\partial t} + \mathbf{v} \cdot \nabla \chi = 0$$

$$\nabla \cdot \mathbf{v} = 0$$

$$A\frac{\partial}{\partial t}\left(\frac{1}{A}\int_A \chi dA\right) = -\int_A \nabla \cdot (\chi \mathbf{v})\, dA$$

$$\frac{\partial F}{\partial t} = -\frac{1}{A}\int_{\partial A} \chi v_n dl$$

Time stepping and front evolution

Time step loop: $t \leftarrow t + \Delta t$

VOF loop:

Coupled Solution

$$\left.\begin{aligned} \frac{\Delta w}{\Delta t} &= A(w)Cw + S \\ p &= Cw \end{aligned}\right] v = -\frac{w^2}{\mu'}\nabla p$$

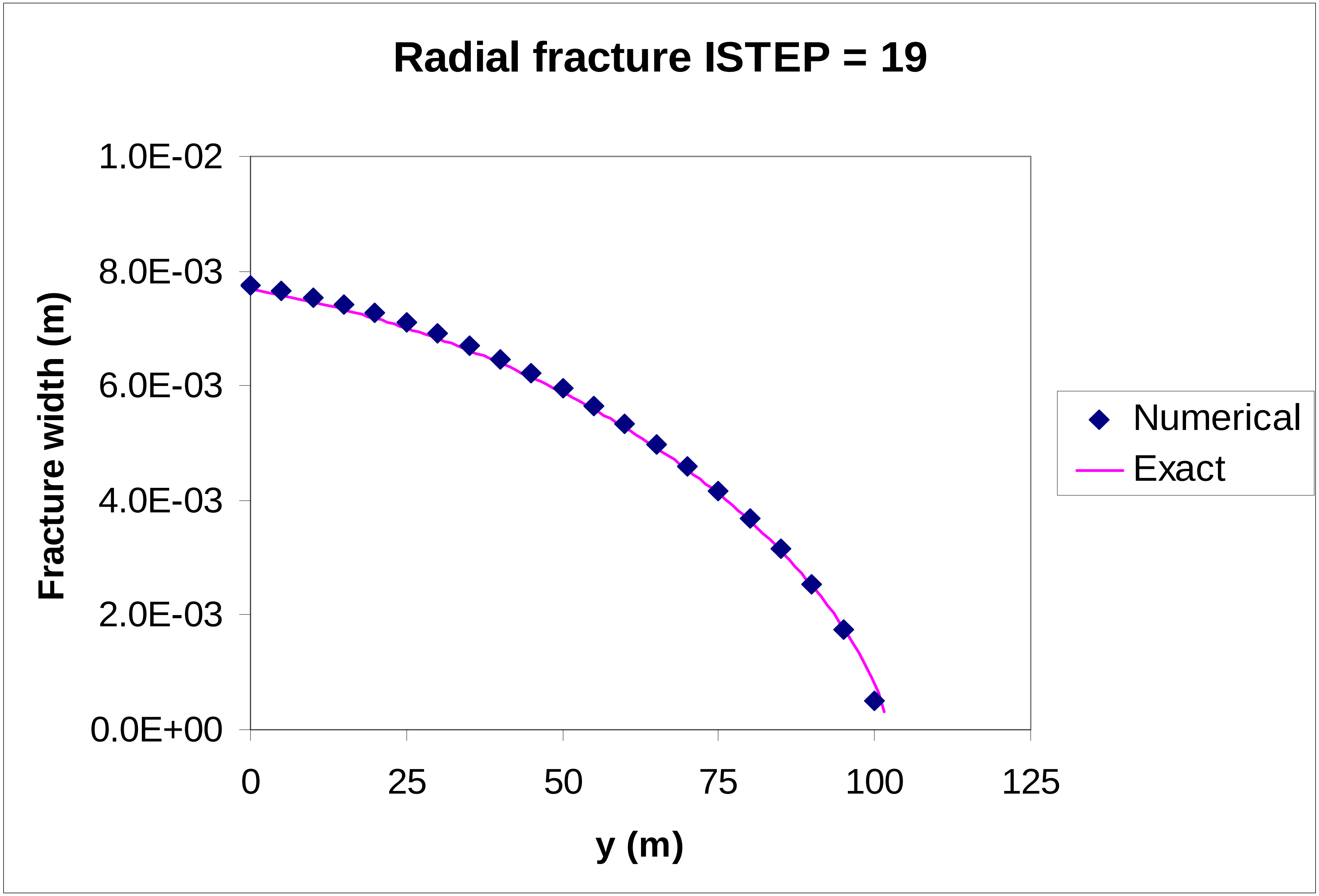
end

$$F_{k+1} = F_k - \frac{\Delta t}{A}\int_{\partial A} \chi v_n dl$$

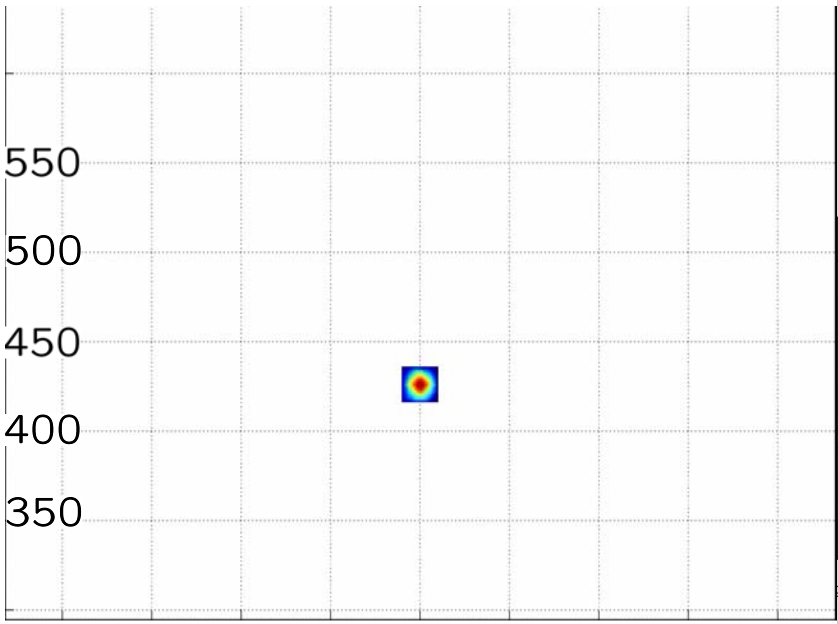
next VOF iteration

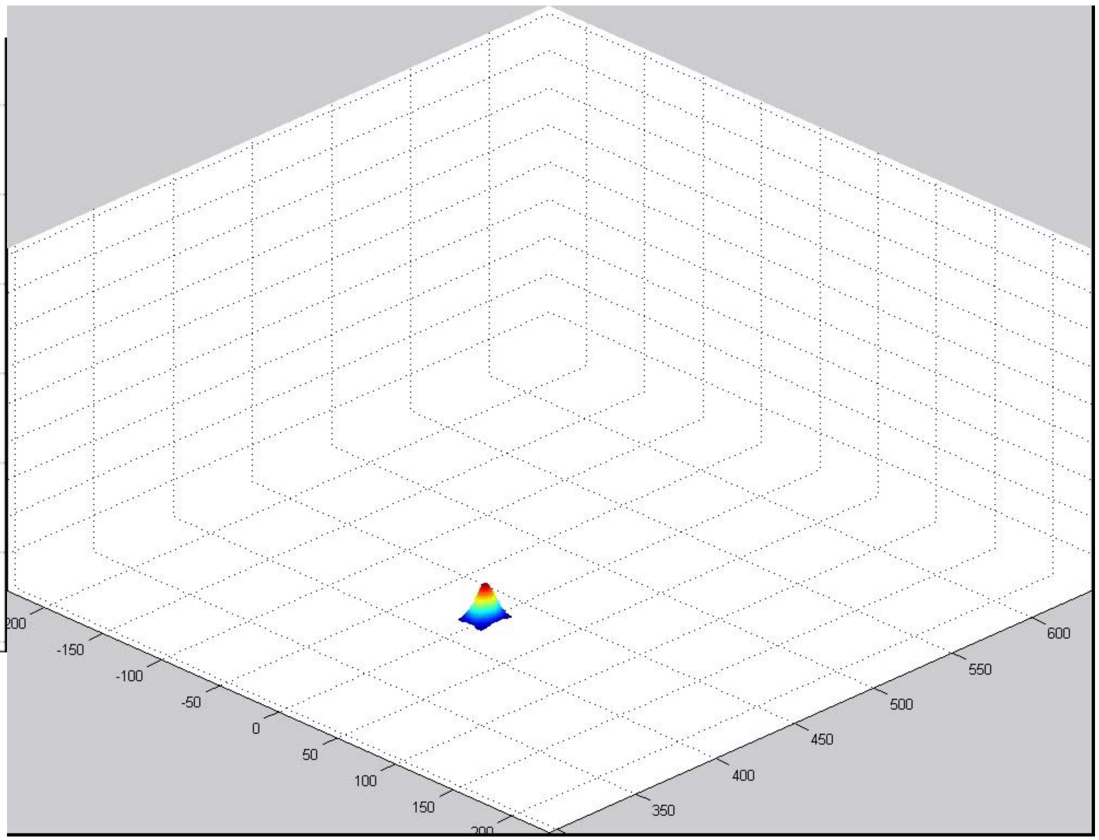
next time step

Radial Solution

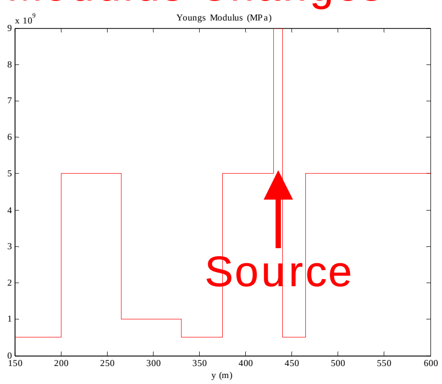


Fracture width for modulus contrast

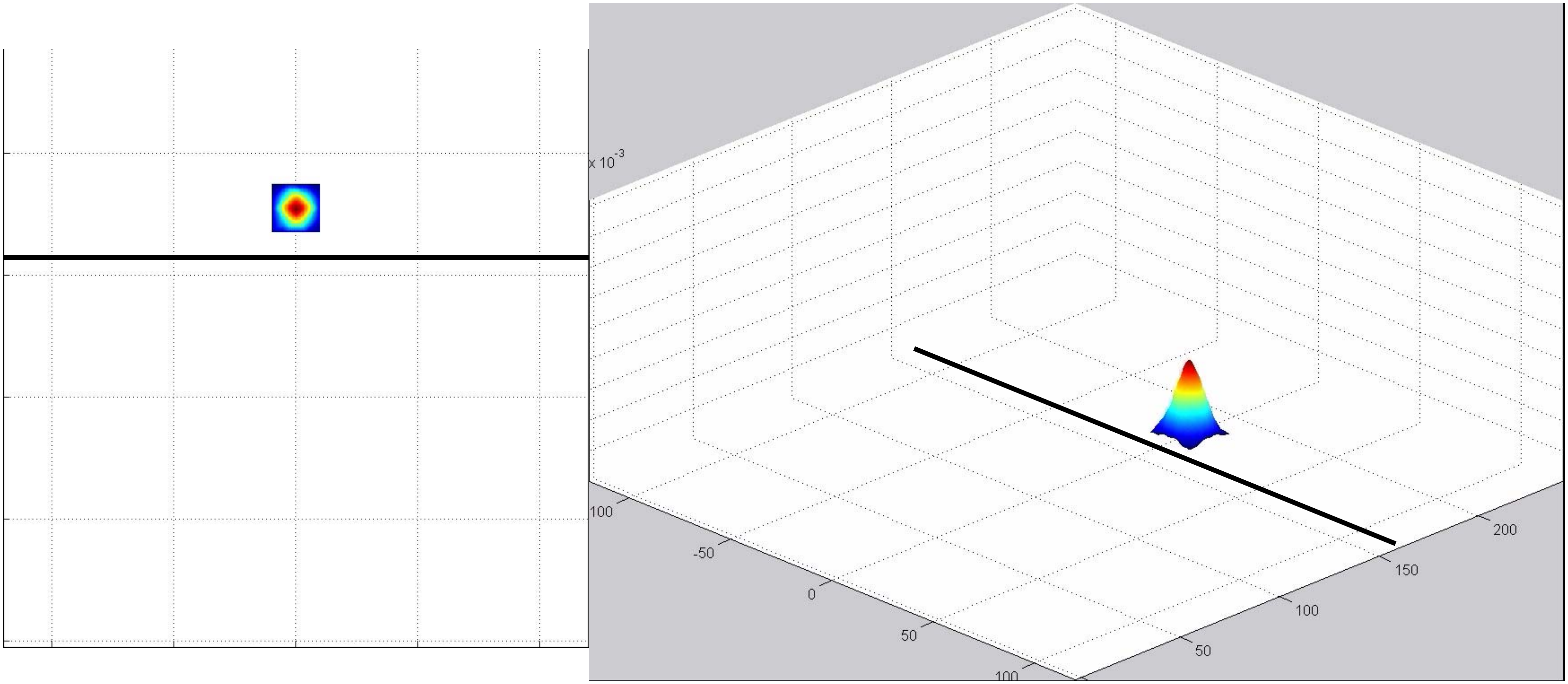




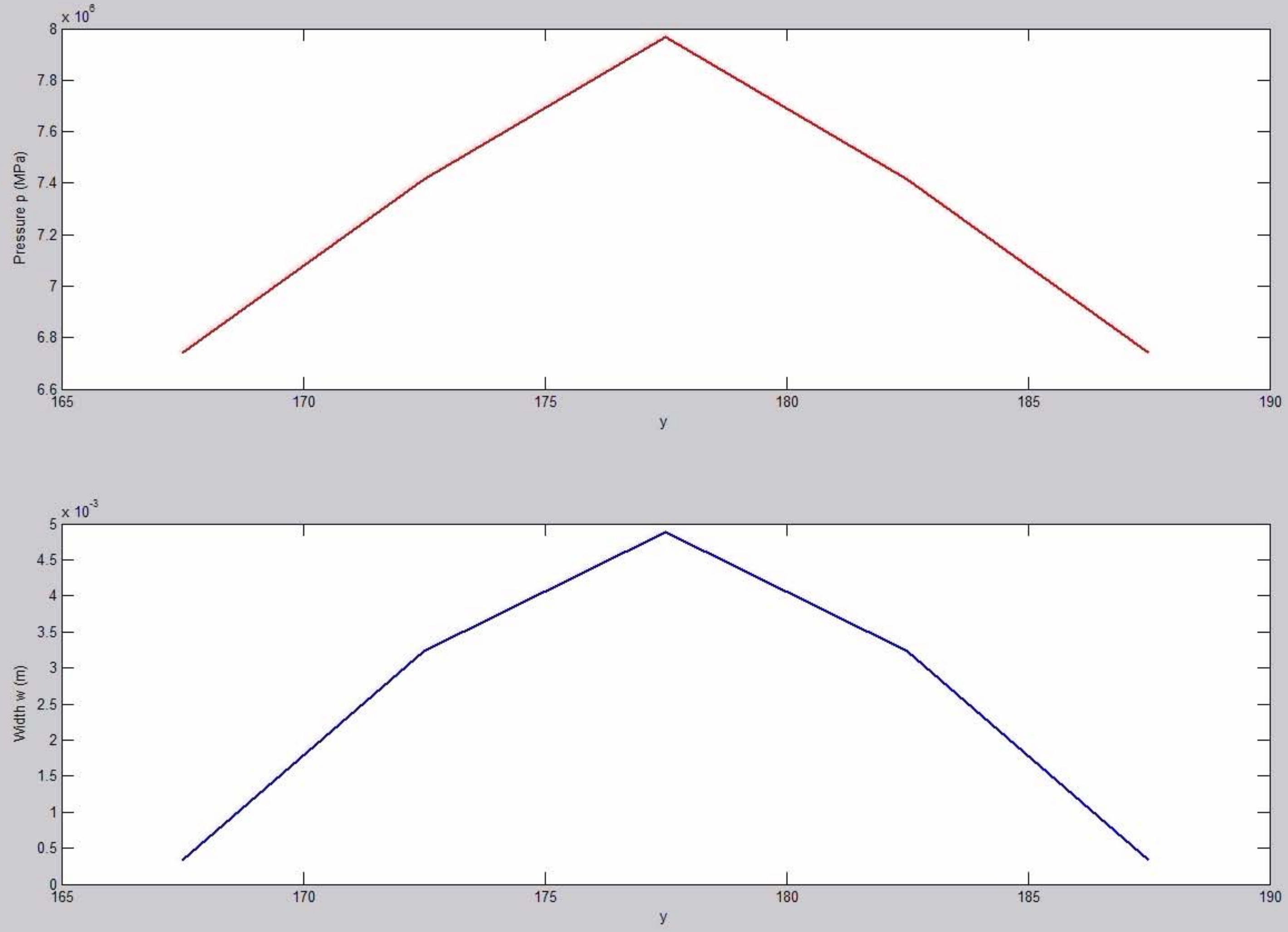
Modulus Changes



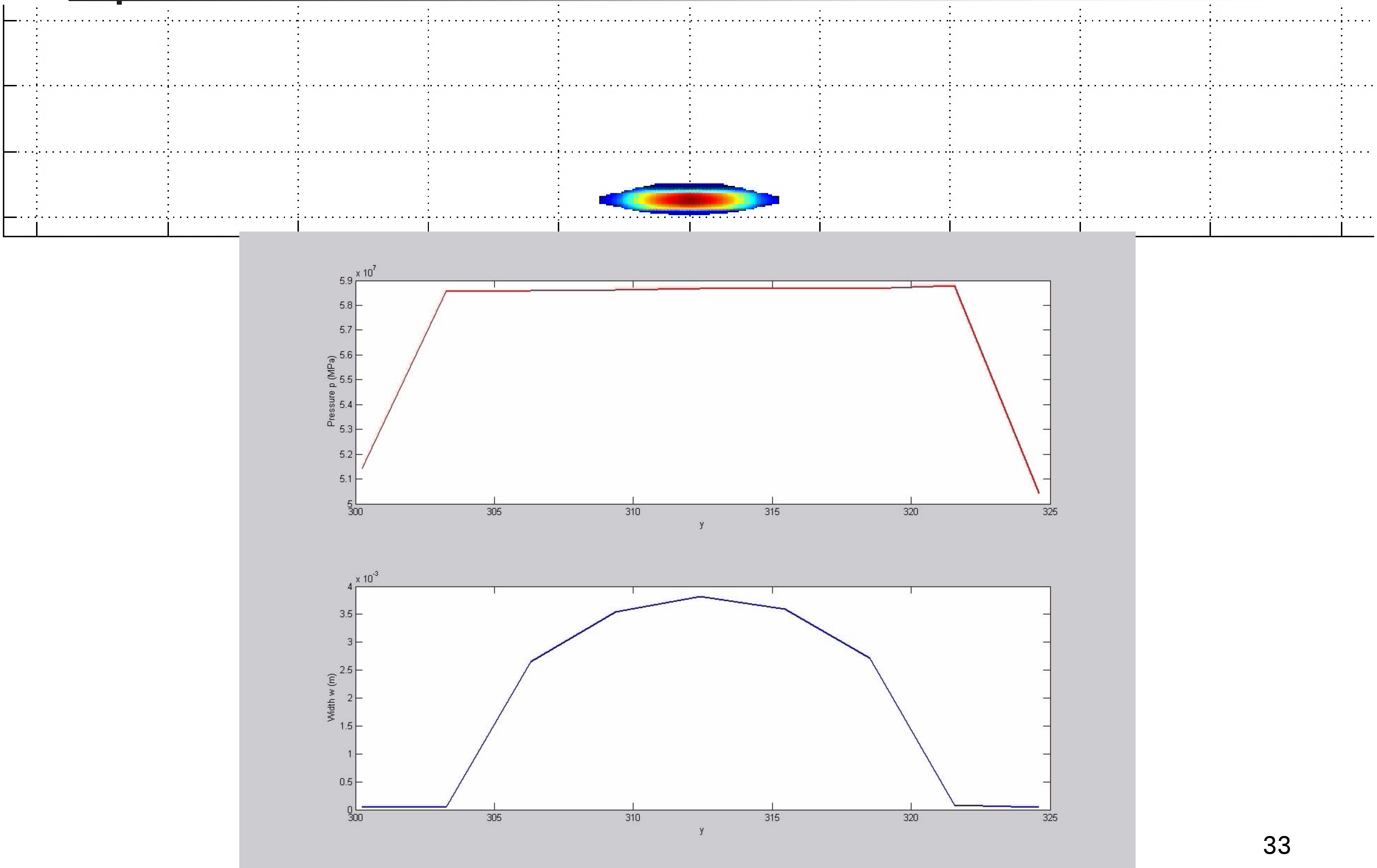
Fracture width for stress jump



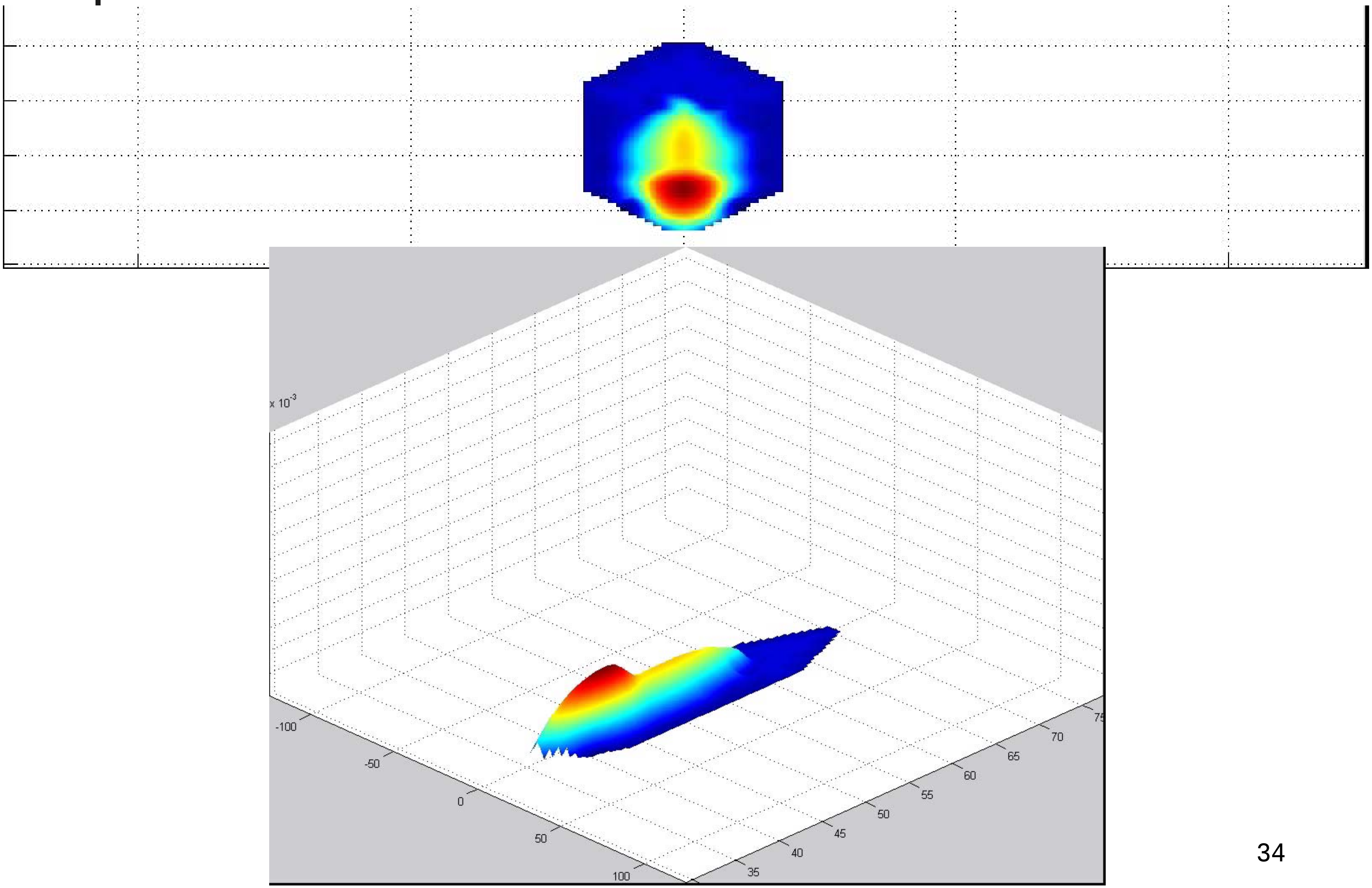
Pressure and width evolution



Channel fracture & pinch region



Hourglass HF with leakoff



Concluding remarks

- Examples of hydraulic fractures
- Scaling and physical processes in 1D models
  - Tip asymptotics: $\Omega \approx c_1 \sqrt{1-\xi}$ & $\Omega \sim c_3 (1-\xi)^{2/3}$
- Numerical models of 2-3D hydraulic fractures
  - The non-local, nonlinear & free boundary problem
  - An Eulerian approach and the coupled equations
  - A multigrid algorithm for the coupled problem
  - ILU Factorization for the Localized Jacobian
  - Front evolution via the VOF method
- Numerical results