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On the moving boundary conditions for a hydraulic fracture

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ABSTRACT

This paper re-examines the boundary conditions at the moving front of a hydraulic fracture when the fluid front has coalesced with the crack edge. This practically important particular case is treated as the zero fluid lag limit of the general case when the two fronts are distinct. The limiting process shows what becomes of the two boundary conditions on the fluid front, a pressure condition and a Stefan condition, when the lag vanishes. On the one hand, the pressure condition disappears as the net pressure (the difference between the fluid pressure and the magnitude of the far-field stress normal to the fracture) becomes singular. On the other hand, the Stefan condition, which equates the front velocity to the average fluid velocity, transforms into a zero flux boundary condition at the fornt. As a consequence, the velocity of the coalesced front does not appear explicitly in the boundary conditions. However, the front velocity can still be extracted from the near-tip aperture field by a nonlinear asymptotic analysis. The paper concludes with a description of an algorithm to propagate the combined front, which explicitly uses the known multiscale asymptotics of the fracture aperture.

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1. Introduction

Fluid-driven fractures represent a particular class of tensile fractures that propagate in solid media, typically under preexisting compressive stresses, as a result of internal pressurization by an injected viscous fluid. Hydraulic fractures are most commonly engineered for the stimulation of hydrocarbon-bearing rock strata to increase production of oil and gas wells (Economides & Nolte, 2000), but there are other industrial applications such as remediation projects in contaminated soils (Murdoch, 2002), waste disposal (Abou-Sayed et al., 1994), preconditioning and cave inducement in mining (Jeffrey & Mills, 2000). Furthermore, hydraulic fractures manifest at the geological scale as kilometer-long vertical dikes bringing magma from deep underground chambers to the earth's surface (Lister & Kerr, 1991; Rubin, 1995), or as subhorizontal fractures known as sills that divert magma from dikes (Pollard & Hozlhausen, 1979).

The design of hydraulic fracturing treatments relies, in part, on our ability to simulate the evolution of the fracture footprint and of the aperture field, as well as of the injection pressure, and to assess the dependence of these quantities on the fracturing fluid rheology, the injection rate, and the rock mechanical properties. However, simulating the propagation of a hydraulic fracture remains a formidable task, even under the ideal assumptions of an isotropic homogeneous linear elastic rock. The challenge stems on the one hand from solving the non-linear, history-dependent, and non-local equations

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governing the flow of a viscous fluid in a deformable permeable channel, and on the other hand from the moving boundary nature of the problem. In principle, there are two moving fronts – the crack edge and the fluid front lagging behind. But under stress conditions that are prevalent in hydrocarbon reservoirs, the lag between the two fronts is virtually non-existent. The two fronts must then be treated as having coalesced in numerical simulators, as a prohibitively dense discretization mesh would be required otherwise to capture the lag.

Paradoxically, it is more challenging to formulate a computational algorithm to propagate the front for the limiting case of zero lag. The complication arises because of the degeneracy of the Reynolds lubrication equation near the fracture tip, where the aperture tends to zero, and also because of a singularity in the leak-off velocity at the tip in permeable rock. As a result, the Stefan condition at the fluid front, which provides a condition on the fluid front velocity when the two fronts are distinct, degenerates into a zero flux boundary condition when the two fronts coalesce. With the front velocity not appearing explicitly in the conditions at the moving front, standard computational algorithms for solving moving boundary problems, such as the volume of fluid method (Voller, 2009) or the level set method (Osher & Sethian, 1988; Sethian, 1999) cannot be used as such.

While considerable effort has been invested in the modeling of hydraulic fracturing since the pioneering work of Khristianovic and Zheltov (1955) (see Adachi, Siebrits, Peirce, & Desroches (2007) and Bunger, Detournay, Garagash, & Peirce (2007) for an extensive list of references, with a particular focus on the Petroleum Industry), the realization that the global solution depends critically on the boundary conditions at the tip and on the details of the near-tip solution has only emerged in recent years. Indeed, when the first models of hydraulic fractures were being developed, the complexity of the problem linked to the existence of a moving boundary and to the degeneracy of the nonlinear equations near the tip was not fully recognized. In these early attempts, analytical solutions for plane strain and radial hydraulic fractures were built based on *ad hoc* assumptions (Abé, Mura, & Keer, 1976, 1979; Advani, Torok, Lee, & Choudhry, 1987; Geertsma & de Klerk, 1969; Nilson, 1986; Nilson & Griffiths, 1983), while numerical models inherited propagation algorithms from dry cracks based on linear elastic fracture mechanics (Advani, Lee, & Lee, 1990; Clifton, 1989; Clifton & Abou-Sayed, 1979; Shah, Carter, & Ingraffea, 1997; Sousa, Carter, & Ingraffea, 1993; Vandamme & Curran, 1989); these algorithms unwittingly forced a behavior in the tip region that was not always appropriate for the spatial resolution of the mesh (Lecampion et al., 2013). Furthermore, the non-linearity of the equations implies that it is possible to find multiple volume-conserving and equilibrated fracture width and fluid pressure fields associated with different fracture footprints at a given time. The role of the boundary conditions is to select the appropriate fracture width, fluid pressure, and fracture footprint combination.

Recent research efforts have led to a series of accurate benchmark solutions for simple hydraulic fracture geometries (with zero lag): plane strain (Adachi, 2001; Adachi & Detournay, 2002, 2008; Garagash, 2006a, 2006b; Garagash & Detournay, 2005; Hu & Garagash, 2010) and radial (Bunger, Detournay, & Garagash, 2005; Madyarova & Detournay, 2013; Savitski & Detournay, 2002). These solutions, which have also been partially verified by laboratory experiments (Bunger, 2005; Bunger et al., 2007), provide rigorous tests for numerical algorithms (Lecampion et al., 2013), and also are forcing a re-examination of the tip boundary conditions and of the importance of the solution in the vicinity of the fracture front.

Motivated by the recent resurgence of papers on the modeling of hydraulic fractures (Carrier & Granet, 2012; Chen, 2012; Damjanac, Detournay, Cundall, & Varun, 2013; Gordeliy & Detournay, 2011; Gordeliy & Peirce, 2013a, 2013b; Linkov, 2012; Mishuris, Wrobel, & Linkov, 2012; Mohammadnejad & Khoei, 2013; Hunsweck, Shen, & Lew, 2013; Zhou & Hou, 2013; Zhang & Jeffrey, 2012), we carefully re-examine here the conditions at the fluid and fracture fronts. Furthermore, we use tip asymptotic analysis to highlight the change in the required boundary conditions in the singular limit in which the fluid and the fracture fronts coalesce. The crack front velocity can then only be extracted from a non-linear asymptotic analysis of the solution in the tip, a challenging task in itself because of the multiscale nature of the tip solution. We conclude by describing an algorithm that exploits the tip asymptotics to both locate the free boundary and to determine the front velocity, and which is capable of achieving an accurate solution on a relatively coarse mesh.

2. Mathematical formulation

2.1. Problem definition and assumptions

We consider the propagation of a planar fracture, driven by the injection of a fluid in a rock medium, see Fig. 1. The hydraulic fracture is, in principle, characterized by the two distinct moving fronts that evolve with time *t*: one is the crack edge $C_c(t)$ and the other is the fluid front $C_f(t)$, which is contained inside $C_c(t)$. The contour $C_c(t)$ defines the crack footprint $\mathcal{A}_c(t)$, while $C_f(t)$ defines the fluid-filled fracture domain $\mathcal{A}_f(t) \subseteq \mathcal{A}_c(t)$. Under large far-field stress conditions, the lag between the crack edge and the fluid front becomes negligible (Garagash & Detournay, 2000), and the two fronts effectively coalesce to a single front denoted by C(t), which encompasses the crack domain $\mathcal{A}(t)$.

The fluid is viewed as being injected from a point source because the characteristic dimension of $C_f(t)$ and thus of $C_c(t)$ is much larger than the dimension of the source. The injection point serves as the origin for the vector **x** defining the position of any point in the fracture plane.

The main focus of this paper is on the nature of the boundary conditions at the moving front C(t) that results when the lag vanishes. Before addressing this question, we first formulate the complete set of equations for the case when $C_c(t)$ is distinct from $C_f(t)$. A complete formulation of the problem requires that the governing equations, the boundary conditions on $C_c(t)$



Fig. 1. Planar hydraulic fracture with two distinct moving fronts: the crack edge $C_c(t)$ and the fluid front $C_f(t)$.

and on $C_{f}(t)$, and the initial conditions be specified. The set of equations and conditions represent a closed system, from which it is possible to determine the evolution of the footprints of both the fracture and of its fluid-filled part, as well as the evolution of the aperture and pressure fields.

To simplify the problem and make it more tractable, we start by introducing four sets of assumptions, pertaining to the rock, the fluid, the in-situ stress, and the lag region, respectively. First, the rock is assumed to be homogeneous, linearly elastic, brittle, and impermeable and of infinite extent (although the assumption of impermeability will be relaxed at some point in the discussion); thus only two parameters are needed to characterize the rock: the so-called plane strain modulus E' (equal to $E/(1 - v^2)$ where E is Young's modulus and v is Poisson's ratio) and the toughness K_{lc} . Second, the fracturing fluid is assumed to be incompressible and Newtonian with viscosity μ and density ρ_f . Third, the orientation of the minimum insitu compressive principal stress $\sigma_0(\mathbf{x})$ is assumed to remain constant. The assumptions on the far-field stress and on the rock medium ensure that the fluid-driven fracture propagates in Mode I (pure tension). In other words, the mode I stress intensity factor K_I equals the rock toughness K_{lc} and $K_{ll} = K_{lll} = 0$ at every point of the crack edge $C_c(t)$. Finally, the lag region between the crack edge and the fluid front is assumed to be filled with vapors from the fracturing fluid, at a pressure that is negligible compared to the magnitude of the far-field stress.

2.2. Governing equations, initial and boundary conditions

Two fundamental equations govern the fracture aperture *w* and the fluid pressure p_f : a non-local elasticity equation relating the net pressure $p = p_f - \sigma_0$ to the aperture *w* and the Reynolds lubrication equation. For the cases considered here, i.e., a domain of infinite extent, and a linear elastic homogeneous material, the elasticity equation can be expressed as a hypersingular integral equation given by Crouch and Starfield (1983), Hills, Kelly, Dai, and Korsunsky (1996)

$$p(\mathbf{x},t) = p_{\rm f}(\mathbf{x},t) - \sigma_0(\mathbf{x}) = -\frac{E}{8\pi(1-\nu^2)} \int_{A_c(t)} \frac{w(\mathbf{x}',t) dA_c(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}$$
(1)

The Reynolds equation is the nonlinear PDE (Batchelor, 1967)

$$\frac{\partial \boldsymbol{w}}{\partial t} = \frac{1}{12\mu} \boldsymbol{\nabla} \cdot \left[\boldsymbol{w}^3 \left(\boldsymbol{\nabla} \boldsymbol{p}_{\rm f} - \boldsymbol{\rho}_f \boldsymbol{g} \right) \right] + \boldsymbol{Q}(t) \delta(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathcal{A}_f(t)$$
⁽²⁾

obtained by combining Poiseuille's law for the fluid flux **q**

$$\mathbf{q} = -\frac{w^3}{12\mu} \Big(\nabla p_{\rm f} - \rho_f \mathbf{g} \Big) \tag{3}$$

with the continuity equation

$$\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{q} = Q(t)\delta(\mathbf{x}) \tag{4}$$

The boundary condition at the injection point has been incorporated directly in the continuity equation and in the Reynolds equation, via the singular term $Q(t)\delta(\mathbf{x})$. In the above, the symbol \mathbf{g} stands for the acceleration due to gravity.

If the fluid front $C_f(t)$ is distinct from the crack front $C_c(t)$, there are two boundary conditions on each front. On the crack front $C_c(t) : w = 0$ and $K_I = K_{Ic}$; and on the fluid front $C_f(t) : p_f = 0$ and the Stefan condition

$$\boldsymbol{V}_f = \boldsymbol{q}/\boldsymbol{w} \text{ on } C_f \tag{5}$$

This latter condition simply expresses that the fluid front velocity V_f is equal to the average fluid velocity v at the fluid front – itself equal to the flux q divided by the crack aperture w at the fluid front. (On the fluid front, the flux q is orthogonal to $C_f(t)$).

At time t = 0, the solution is assumed to be known, i.e, the two contours $C_c(0)$ and $C_f(0)$, and the fracture aperture $w(\mathbf{x}, 0)$. Alternatively the early time solution could be given by the so-called "O-solution," a similarity solution for a radial fracture characterized by power law growth of the two fronts, with $C_f(t)$ growing faster than $C_c(t)$ (Bunger & Detournay, 2007).

Eqs. (1) and (2), together with two boundary conditions at each of the two fronts, and given the initial conditions $\{C_c(0), C_f(0), and w(\mathbf{x}, 0)\}$ with $\mathbf{x} \in A_c(0)$ (or alternatively a similarity solution at early time) constitute a closed system for tracking the evolution of the hydraulic fracture.

3. Particular case of zero lag

Under certain conditions, which can be broadly defined to correspond to a "large" far-field stress σ_0 , the lag between the fluid front and the crack edge becomes negligible. This particular case actually turns out to be typical of most hydraulic fracturing treatments. It needs to be addressed carefully because of the degeneracy of the fluid equations at the tip.

In the presence of a "large" far-field stress, the two distinct fronts effectively coalesce so that a solution with one combined front rapidly emerges over a time scale that is very small compared to the other time scales characterizing the fracture evolution. For example, there is an extremely rapid transition from the O-solution to an intermediate asymptotic solution – the so-called "M-solution", for a radial fracture with no lag, which is obtained by setting $K_{lc} = 0$ (Savitski & Detournay, 2002). The transition between the O- to the M-solution takes place over the time scale $E^2 \mu / \sigma_0^3$ (Bunger & Detournay, 2007), which is typically of order of seconds. The similarity M-solution is also characterized by a power law evolution of the fracture radius; it is effectively the early time solution for planar hydraulic fractures that are completely filled by the injected fluid.

The combined front is now characterized by the conditions $\mathbf{q} = 0$ and w = 0, besides the propagation criterion $K_I = K_{lc}$. To justify this we reason as follows: the Stefan condition $\mathbf{V}_f = \mathbf{q}/w$ and the condition $p = -\sigma_0$ (equivalent to $p_f = 0$) on $C_f(t)$, reduces to $\mathbf{q} = 0$ when $C_c = C_f = C$. Indeed, the Stefan condition degenerates: since w = 0 at the front and since \mathbf{V}_f is finite the only possibility is that $\mathbf{q} = 0$. In fact, the front velocity $\mathbf{V}_f = \mathbf{V}_c = \mathbf{V}$ has to be extracted from an asymptotic analysis of the nonlinear system of equations consisting of the elasticity and lubrication equations, and the conditions $\mathbf{q} = 0$, w = 0, and $K_I = K_{Ic}$, as will be shown in Section 4. For impermeable rock, $\mathbf{V} = \mathbf{q}/w$ on C(t), i.e. the front velocity is equal to the average fluid velocity \mathbf{q}/w , but this is not the case when there is leak-off.

It is useful to understand how the two conditions on $C_f(t)$, $\mathbf{q}(\mathbf{x}_f) = \mathbf{V}_f(\mathbf{x}_f)w(\mathbf{x}_f)$ and $p(\mathbf{x}_f) = -\sigma_0$, with $\mathbf{x}_f \in C_f$ reduce to $\mathbf{q}(\mathbf{x}_c) = 0$ when $C_c = C_f = C$, noting that $\mathbf{x}_c \in C_c$. Consider a situation where the two fronts are close, and let $\mathbf{x}_f(\mathbf{x}_c)$ be the point on the fluid front that is closest to \mathbf{x}_c . In other words, $\mathbf{x}_f^* = \mathbf{x}_c - \lambda \mathbf{n}_c$, where λ is the local distance between the two fronts and \mathbf{n}_c is the outward unit normal to the front; both λ and \mathbf{n}_c are evidently functions of \mathbf{x}_c . We now imagine that σ_0 increases and becomes large compared to the reference stress $\sigma_* = 3\pi\mu V E'^2/8K_{lc}^2$ (Garagash & Detournay, 2000), where $V = |\mathbf{V}_c| \simeq |\mathbf{V}_f|$. With increasing σ_0 , the lag λ decreases and vanishes as $\sigma_0 \to \infty$. In fact, for $\sigma_0/\sigma_* \gtrsim 10$, $\lambda \simeq 3.210^{-2}\lambda_* \exp(-\sigma_0/\sigma_*)$ with $\lambda_* = K_{lc}^6/\mu^2 V^2 E'^4$. When $\mathbf{x}_f^*(\mathbf{x}_c) \to \mathbf{x}_c$, $w(\mathbf{x}_f^*) \to 0$ while $\mathbf{V}_f(\mathbf{x}_*^*) \to \mathbf{V}_c(\mathbf{x}_c) = \mathbf{V}(\mathbf{x}_c)$, thus indeed $\mathbf{q}(\mathbf{x}_f^*) \to 0$, and $p(\mathbf{x}_f^*) \to -\infty$. In other words, when the lag λ vanishes, the boundary condition on the pressure disappears and the Stefan condition on the fluid front velocity transforms into a condition on the flux since the two front velocities have now become equal. The three conditions on C(t) do not contain the velocity \mathbf{V} explicitly, which is needed to evolve the front. However, as shown in Section 4, the front velocity can be extracted from a nonlinear asymptotic analysis.

These boundary conditions at the crack front, in the presence of a vanishing lag, have been the source of considerable confusion. The main points that we want to stress here are that the three conditions on C(t), w = 0, q = 0, and $K_I = K_{Ic}$ are independent, although they appear to be related. Indeed, according to linear elastic fracture mechanics, the crack aperture in the immediate vicinity of the crack edge is given by

$$w \sim^{s \to 0} \left(\frac{32}{\pi}\right)^{1/2} \frac{K_{lc}}{E} s^{1/2}$$
 (6)

where *s* is a moving coordinate having an origin on the crack edge, which measures the signed distance from the front, with positive value corresponding to a point inside the crack. Evidently, (6) implies that w = 0 on the crack edge, s = 0. Also, as discussed above, the Stefan condition $V_f = q/w$ transforms into q = 0 when the fluid front coincides with the crack edge. Nonetheless, each boundary condition carries different information. Indeed,

- Given the aperture field $w(\mathbf{x}, t)$, and the domain contour C(t), the condition $\mathbf{q} = 0$, which amounts to a Neumann boundary condition, is sufficient to solve for the fluid pressure $p_t(\mathbf{x}, t)$ or the net pressure $p(\mathbf{x}, t)$, up to a constant.
- Given the fluid pressure $p_f(\mathbf{x}, t)$ (up to a constant), c(t), and the crack volume (equal to the injected volume of fluid $\mathcal{V}_f(t) = \int_0^t Q(\tau) d\tau + \mathcal{V}_f(0)$), the condition w = 0 is sufficient to solve for the aperture $w(\mathbf{x}, t)$, noting that the undetermined constant in the fluid pressure is determined by the solvability condition $\mathcal{V}_c(t) = \mathcal{V}_f(t)$, typical of Neumann problems.
- The propagation criterion $K_I = K_{Ic}$ provides the supplementary information to determine the position of the crack edge C(t).

4. Asymptotic analysis and front velocity in the absence of a lag

4.1. Preamble

The velocity of the front C(t), in the absence of a lag, can be extracted from an asymptotic analysis of the elasticity and lubrication equations *together* with the three conditions w = 0, q = 0, and $K_I = K_{Ic}$ on C(t). The cases $\lambda > 0$, for which a classical Stefan condition applies, are thus radically different from the case $\lambda = 0$, in which the front velocity can only be determined in terms of quantities involving a distinguished limit at the front.

Our proof builds on the reverse problem, i.e., the solution w(x, t) and p(x, t) in the vicinity of C(t) is completely determined by the elasticity and lubrication equations and the three conditions on C(t), provided that the front velocity is known. First we note that the 1D nature of the solution in the vicinity of the front, i.e., the spatial variation of the fields w and p near C(t), takes place in the direction normal to the front (see Peirce & Detournay (2008) for details). Furthermore, we show next from considerations involving the asymptotic forms of the continuity and the elasticity equations that the near-tip pressure, aperture, and flux fields depend on time, via the dependence of the tip velocity on time.

4.2. Continuity

4.2.1. Impermeable case

Near the moving front C(t), the continuity equation (4) can be rewritten as

$$\frac{Dw}{Dt} + V\frac{\partial w}{\partial s} - \frac{\partial q}{\partial s} = 0$$
⁽⁷⁾

where the operator D/Dt represents the rate of change at a fixed *s*. (Although not embodied in the notation, (7) deals with the asymptotic form of the fields $w(\mathbf{x}, t)$ and $q(\mathbf{x}, t)$, which only depends on the signed distance *s* from the front and on time *t*). However, the convective term $V\partial w/\partial s$ dominates the time derivative Dw/Dt close to the front (Adachi & Detournay, 2008; Garagash, Detournay, & Adachi, 2011), since $w \sim s^{\alpha}$ with $1/2 \leq \alpha < 1$ as $s \to 0$ ($\alpha = 1/2$ if $K_{lc} > 0$, and $\alpha = 2/3$ if $K_{lc} = 0$). Hence the continuity equation reduces to

$$V\frac{dw}{ds} - \frac{dq}{ds} = 0 \tag{8}$$

where the hat indicates that the dependence of the field on time, is only via the dependence of the tip velocity on time, i.e., $\hat{w}(s; V)$ and $\hat{q}(s; V)$. Integrating the continuity equation (8) with $\hat{w}(0) = \hat{q}(0) = 0$ yields (Desroches et al., 1994)

$$\hat{q} = V\hat{w}, \quad s \ll \ell \tag{9}$$

In *impermeable* rocks the average fluid velocity $\hat{v} = \hat{q}/\hat{w}$ is thus uniform in the vicinity of the front *c* and equal to the instantaneous tip velocity *V*.

Eq. (9) *apparently* resembles the Stefan condition (5) on $C_f(t)$. The resemblance is superficial, however. The Stefan condition applies strictly at \mathbf{x}_f ; i.e., it does not necessarily imply that behind the front $C_f(t)$ the average fluid velocity v is equal to the front velocity V_f over any significant distance. Indeed, (9) relies on neglecting $D\hat{w}/Dt$ on account of the singularity in the aperture gradient at the tip, which results from the fact that either $\alpha = 1/2$ (an asymptotic result obtained by invoking elasticity) or $\alpha = 2/3$ (obtained by invoking both elasticity and lubrication). In other words, (9) is an asymptotic condition that applies in a region behind the crack tip while (5) is an interface condition. The region of validity of the asymptotic condition is actually quite large (approximately one order of magnitude smaller than the fracture dimension for plane strain and radial geometries (Adachi & Detournay, 2002; Garagash & Detournay, 2005; Savitski & Detournay, 2002)) on account of the singular behavior of the solution at the tip. Finally, the Stefan condition $q(\mathbf{x}_f) = V_f(\mathbf{x}_f)w(\mathbf{x}_f)$ applies whether or not there is leak-off, while (9) is valid only in the absence of leak-off, as will be shown in Section 4.2.

4.2.2. Permeable case

In the case of one dimensional Carter leak-off (Carter et al., 1957), the crack front velocity *V* is no longer equal to the average fluid velocity $\hat{\nu}(0)$. Indeed, the leak-off specific discharge $f(\mathbf{x}, t)$ is given by

$$f(\mathbf{x},t) = \frac{2C_l}{\sqrt{t - t_o(\mathbf{x})}} \tag{10}$$

where t_0 is the time of first exposure, i.e., the time at which the crack front was at \mathbf{x} , C_l is the Carter leak-off coefficient, and the factor 2 accounts for the two walls of the fracture. Near the front, $t - \hat{t}_0(s) = s/V$; hence

$$\hat{f}(s) = 2C_l \sqrt{\frac{V}{s}} \tag{11}$$

Although there are issues applying Carter's law in the near tip region, the basic structure of the law, i.e. the inverse square root dependence on the elapsed time $t - \hat{t}_0(s)$, remains in the case where a more general diffusive process is considered (Detournay & Garagash, 2003; Kovalyshen & Detournay, 2013). In other words, (10) or (11) apply, provided that the leak-off coefficient C_l is re-interpreted. With leak-off, the continuity equation (8) becomes

$$V\frac{d\hat{w}}{ds} - \frac{d\hat{q}}{ds} + \hat{f} = 0 \tag{12}$$

which, after integration, yields

$$\hat{q} = V\hat{w} + 4C_l\sqrt{Vs} \quad s \ll \ell \tag{13}$$

In the toughness-dominated regime, the average fluid velocity $\hat{v} = \hat{q}/\hat{w}$ is finite and uniform in the region of dominance of the square root asymptote given in (6), but the front velocity is no longer equal to the average fluid velocity in that region, but rather given by

$$V = \left(\sqrt{V_* + \hat{\nu}(0)} - \sqrt{V_*}\right)^2 \tag{14}$$

with

$$V_* = \frac{\pi}{8} \left(\frac{C_l E'}{K_{lc}} \right)^2 \tag{15}$$

On the other hand, in the viscosity-dominated regime and provided that the dimensionless leak-off coefficient $\chi = \sqrt{V_*/V} \gtrsim 100$, an intermediate asymptote emerges, which is characterized by $\hat{w} \sim s^{5/8}$ (Garagash et al., 2011; Lenoach, 1995). In the region of dominance of this leak-off viscosity asymptote, the average fluid velocity varies as $s^{-1/8}$ (and thus does not have a finite limit at s = 0).

4.3. Multiscale tip asymptotics

Besides (13), the asymptotic fields are governed by elasticity and Poiseuille's law. Again, these two equations take a particular form near the crack front. The elasticity equation (1) for a field point close to the (smooth) crack front C(t) degenerates to Peirce and Detournay (2008)

$$\hat{p} = \frac{E'}{4\pi} \int_0^\infty \frac{\mathrm{d}\hat{w}}{\mathrm{d}z} \frac{\mathrm{d}z}{\mathrm{s}-z},\tag{16}$$

while Poiseuille's law (3) reduces to

$$\hat{q} = \frac{\hat{w}^3}{12\mu} \frac{\mathrm{d}\hat{p}}{\mathrm{d}s}.$$
(17)

Note that any smooth spatial variation of the far-field stress σ_0 in the elasticity equation (σ_0 enters the equation through the definition of the net pressure) as well as the gravity term in Poiseuille's law can be ignored when viewed at the tip scale.

The system of Eqs. (6), (13), (16), and (17), which embed the boundary conditions for \hat{w} and \hat{q} at s = 0 as well as the propagation criterion, constitute a closed set that can be solved for $\hat{p}(s)$, $\hat{q}(s)$, and $\hat{w}(s)$ on $[0, \infty)$ (Garagash et al., 2011). The tip asymptotic fields are actually governed by the equations for a semi-infinite fluid-driven fracture steadily propagating at a

constant velocity, corresponding to the current propagation speed of the finite fracture. The tip solution is thus autonomous. The general solution for the aperture can be expressed as $\hat{\Omega}(\hat{\xi}) = \hat{w}(\hat{s}/\hat{\ell}_{mk})/\hat{w}_{mk}$ where $\hat{\ell}_{mk}$ and \hat{w}_{mk} are reference length scales for the distance from the front and for the aperture, respectively

$$\hat{\ell}_{mk} = \frac{\eta_{\ell} K_{lc}^6}{E'^4 \mu^2 V^2}, \quad \hat{w}_{mk} = \frac{\eta_w K_{lc}^4}{E'^3 \mu V},$$

with $\eta_{\ell} = 2^{11}/3^2 \pi^3 \simeq 7.339$ and $\eta_w = 2^8/3\pi^2 \simeq 8.646$. Furthermore, the solution $\hat{\Omega}(\hat{\xi})$ only depends on the number $\chi = \sqrt{V_*/V}$. It is characterized by a near-field (toughness) asymptote $\hat{\Omega}(\hat{\xi})^{\frac{1}{\xi} \to 0} \hat{\xi}^{1/2}$, a far-field (viscosity) asymptote $\hat{\Omega}(\hat{\xi})^{\frac{1}{\xi} \to 0} \hat{\xi}^{1/2}$, a far-field (viscosity) asymptote $\hat{\Omega}(\hat{\xi})^{\frac{1}{\xi} \to 0} \hat{\xi}^{1/2}$, a far-field (viscosity) asymptote $\hat{\Omega}(\hat{\xi})^{\frac{1}{\xi} \to 0} \hat{g}^{1/2}$, with $\beta_m = 2^{1/3} \cdot 3^{5/6}$, and also an intermediate (leak-off/viscosity) asymptote $\hat{\Omega}(\hat{\xi}) \sim \hat{\xi}^{5/8}$ if $\chi \gtrsim 100$. In the absence of leak-off ($\chi = 0$), the toughness asymptote $\hat{w}/\hat{w}_{mk} \sim (s/\hat{\ell}_{mk})^{1/2}$ applies for $s \lesssim 10^{-6} \hat{\ell}_{mk}$ and the viscous dissipation asymptote $\hat{w}/\hat{w}_{mk} \sim \beta_m (s/\hat{\ell}_{mk})^{2/3}$ for $s \gtrsim \hat{\ell}_{mk}$. This particular result shows that if $\hat{\ell}_{mk}$ is much smaller than the fracture dimension (pragmatically less than 10%), then the fracture propagates in the viscosity regime, since at the scale of the fracture, the relevant tip asymptote is the viscosity asymptote, which shields the toughness asymptote. In this regime, the energy dissipation associated with the breaking of the rock is negligible compared to the viscous dissipation. Hydraulic fracturing treatments for the stimulation of oil and gas are dominantly in the viscosity regime (Bunger et al., 2007; Detournay, Peirce, & Bunger, 2007).

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The above results indicate that the near-tip aperture field only depends on the front velocity, given known material parameters. Next, we describe a technique to update the front position and simultaneously calculate the front velocity, from the aperture computed at points in the vicinity of the front.

5. Computational considerations with the tip asymptotes

Evidently the calculation of the two fields $w(\mathbf{x}, t)$ and $p_f(\mathbf{x}, t)$ and of the contour C(t) are coupled; their solutions at any given time *t* requires iteration between the position of the front and the solution of a non-linear system of algebraic equations, obtained by combining discretized forms of the elasticity equation (1) and of the Reynolds equation (2). The challenge of solving this particular problem is further compounded by the fact that the non-linear coupling between the Reynolds and the elasticity is specially intense in the tip region, where there is rapid variation of the crack aperture. The issues associated with this coupling, in particular the emergence of different asymptotes depending on the local crack velocity, have been discussed in a series of papers (Garagash, 2009; Lecampion et al., 2013; Peirce & Detournay, 2008).

Recently, a novel Implicit Level Set Algorithm (ILSA) has been developed (Peirce & Detournay, 2008), which uses the limiting behavior of the fracture width, obtained from asymptotic analysis, in order to locate the unknown free boundary without requiring explicit knowledge of the front velocity field. This scheme is able to select the appropriate asymptote based on local conditions along the perimeter of the fracture footprint and to embed this asymptote and incorporate information at a much finer length scale in a weak sense by matched tip volumes. In a recent comparative study of a number of different numerical formulations (Lecampion et al., 2013), the ILSA scheme was shown to provide extremely accurate results while requiring relatively few numerical resources.

To describe the details of the algorithm, assume that the three-dimensional elastic equilibrium equation (1) are discretized using constant width rectangular elements that are collocated at element centers. Further let the Reynolds lubrication equation (2) be discretized using a finite volume method also defined with respect to quantities sampled at the centers of the rectangular elements. At the periphery of the fracture, which may not conform to the structured rectangular mesh, the boundary is represented using a concept of partially filled tip elements that are used to define average fracture widths, which are also sampled at element centers. The distinguishing feature of this algorithm is its ability to locate the fracture free boundary using the asymptotic behavior of the hydraulic fracture width that is applicable at a particular point on the fracture perimeter. The free boundary is located by the following iterative process; given an initial guess for the fracture boundary C. determine the corresponding trial fracture width w and fluid pressure field $p_{\rm f}$; in the ribbon of elements that are completely filled with fluid and which share at least one side with a partially filled tip element, use these trial width values to estimate the distance to the free boundary by inverting the applicable tip asymptotic expansion; use these estimates of the distance to the free boundary as initial conditions for the eikonal equation $|\nabla T(\mathbf{x})| = 1$, whose level set curve $T(\mathbf{x}) = 0$ is the free boundary C that we seek. The fracture boundary is then moved to the curve $T(\mathbf{x}) = 0$ and the iterative process is repeated until convergence is achieved. The algorithm is able to use the multi-scale hydraulic fracture tip asymptotic solution (Lecampion et al., 2013; Peirce, 2014) and can thus automatically capture the different types of propagation regimes with a relatively coarse mesh.

The approach discussed above should be contrasted with the method recently advocated by Linkov and coworkers (Linkov, 2012; Mishuris et al., 2012) to determine the front velocity *V*. Their approach is based on equating the front velocity *V* to the average fluid velocity v = q/w in the vicinity of the front (in impermeable media). This "speed equation," as referred to by these authors, is in fact the integrated continuity equation (9). Issues with this approach will be discussed in detail elsewhere. We only want to point out here that calculation of the front velocity from (9) requires understanding how to accurately compute both *w* and *q* in the vicinity of the front. While *w* can be computed with good accuracy near the crack tip, the numerical evaluation of *q* is problematic as it involves applying divided differences to a singular pressure field. Furthermore, if there is leak-off and if the fracture propagates in the viscosity-dominated regime, the region under the umbrella of the square root asymptote (where the fluid velocity is uniform, although not equal to the front velocity) is much smaller than the spatial resolution of a "reasonable" mesh (say O(10) elements per fracture dimension) (Lecampion et al., 2013). While it is, in principle, still possible to extract the front velocity *V* from the fluid velocity field, it would be very challenging to determine the spatial variation of fluid velocity within the constraints of a reasonable mesh discretization. In contrast, the implicit procedure outlined in Section 5 does not rely on knowing an accurate front velocity but yields it as a quantity that can be determined *a posteriori* from the front position.

6. Conclusions

In this paper we have re-examined the nature of the boundary conditions at the moving front of a fluid driven fracture, in the practically important case when the fluid front and the crack edge have virtually coalesced. By treating the zero fluid lag case as the limit of the general case when the two fronts are distinct, we have analyzed the transformation of the two boundary conditions at the fluid front when the fluid lag vanishes; namely the degeneracy of the Stefan condition into a zero flux condition and the disappearance of the pressure condition. The front velocity, needed to evolve the fracture footprint with time in numerical simulations, does not therefore appear explicitly in the boundary conditions for the zero lag case as is usual in problems with a moving boundary, where a Stefan condition provides an explicit expression for the velocity.

Although the front velocity is equal to the average fluid velocity in the vicinity of the tip when the rock is impermeable, a large variation of the fluid velocity in the tip region is expected when there is leakoff, in view of the multiscale nature of the solution near the tip – a consequence of the combination of nonlinear, non-local, and history dependence of the goverving equations. In addition, considering the challenge of accurately computing the fluid velocity near the front, in view of the pressure singularity, any algorithmic scheme to extract the front velocity from the near tip fluid velocity would require an extremely fine grid resolution. However, the front velocity can be determined from a nonlinear asymptotic analysis of the fracture aperture field. Such an approach, combined with an implicit level set algorithm is able to accurately propagate the fracture footprint on a coarse mesh.

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