# Lecture 20: Heat conduction with time dependent boundary conditions using Eigenfunction Expansions

(*Compiled 19 December 2017*)

The ultimate goal of this lecture is to demonstrate a method to solve heat conduction problems in which there are time dependent boundary conditions. The idea is to construct the simplest possible function, $w(x,t)$ say, that satisfies the inhomogeneous, time-dependent boundary conditions. The solution $u(x,t)$ that we seek is then decomposed into a sum of $w(x,t)$ and another function $v(x,t)$, which satisfies the homogeneous boundary conditions. When these two functions are substituted into the heat equation, it is found that $v(x,t)$ must satisfy the heat equation subject to a source that can be time dependent. As in Lecture 19, this forced heat conduction equation is solved by the method of eigenfunction expansions.

**Key Concepts: Time-dependent Boundary conditions, distributed sources/sinks, Method of Eigenfunction Expansions**.

## 20 Heat Conduction Problems with Time Dependent Boundary Conditions

### 20.1 Inhomogeneous Derivative Boundary Conditions using Eigenfunction Expansions

**Example 20.1** *Let us revisit the problem with inhomogeneous derivative BC (Example 18.2) - but we will now use Eigenfunction Expansions.*

$$u_t = \alpha^2 u_{xx} \qquad 0 < x < L, \quad t > 0 \tag{20.1}$$

$$\text{BC: } u_x(0,t) = A \quad u_x(L,t) = B \tag{20.2}$$

$$\text{IC: } u(x,0) = g(x) \tag{20.3}$$

First look for a function of the form $h(x) = ax^2 + bx$ that satisfies the inhomogeneous BC:

$$\begin{aligned} h(x) &= ax^2 + bx, \quad h_x(x) = 2ax + b \\ h_x(0) &= b = A \quad h_x(L) = 2aL + A = B \Rightarrow a = (B-A)/2L \\ h(x) &= \left(\frac{B-A}{2L}\right) x^2 + Ax. \end{aligned}$$

Now let

$$u(x,t) = h(x) + v(x,t).$$

Substitute into the PDE:

$$u_t = \big(h(\cancel{x}) + v(x,t)\big)_t = \alpha^2 u_{xx} = \alpha^2 \big(h(x) + v(x,t)\big)_{xx} = \alpha^2 \cdot 2a + \alpha^2 v_{xx}.$$

Therefore

$$v_t = \alpha^2 v_{xx} + 2a\alpha^2 \tag{20.4}$$

$$A = u_x(0,t) = h_x(0) + v_x(0,t) = A + V_x(0,t) \Rightarrow v_x(0,t) = 0 \tag{20.5}$$

$$B = u_x(L,t) = h_x(L) + V_x(L,t) = B + V_x(L,t) \Rightarrow v_x(L,t) = 0 \tag{20.6}$$

$$g(x) = u(x,0) = h(x) + v(x,0) \Rightarrow v(x,0) = g(x) - h(x). \tag{20.7}$$

We now use an Eigenfunction Expansion to solve the BVP (20.4)-(20.7). Because of the homogeneous Neumann BC we assume an expansion of the form

$$\begin{aligned} v(x,t) &= \hat{v}_0(t)/2 + \sum_{n=1}^{\infty} \hat{v}_n(t)\cos\left(\frac{n\pi x}{L}\right) \\ v_t &= \dot{\hat{v}}_0(t)/2 + \sum_{n=1}^{\infty} \dot{\hat{v}}_n(t)\cos\left(\frac{n\pi x}{L}\right) \\ v_x &= \sum_{n=1}^{\infty} \hat{v}_n(t)\left\{-\left(\frac{n\pi}{L}\right)\right\}\sin\left(\frac{n\pi x}{L}\right), v_{xx} = \sum_{n=1}^{\infty} \hat{v}_n(t)\left\{-\left(\frac{n\pi}{L}\right)^2\right\}\cos\left(\frac{n\pi x}{L}\right). \end{aligned}$$

We also expand the inhomogeneous term in (20.4) in terms of the Eigenfunctions:

$$2a\alpha^2 = a_0/2 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \quad a_0 = 4a\alpha^2, a_n = 0 \quad n \geq 1.$$

Therefore

$$0 = v_t - \alpha^2 v_{xx} - 2a\alpha^2 = \dot{\hat{v}}_0(t)/2 - 2a\alpha^2 + \sum_{n=1}^{\infty}\left\{\dot{\hat{v}}_n + \alpha^2\left(\frac{n\pi}{L}\right)^2\hat{v}_n\right\}\cos\left(\frac{n\pi x}{L}\right).$$

Therefore

$$\begin{aligned} \dot{\hat{v}}_0(t) &= 4a\alpha^2 \Rightarrow \hat{v}_0(t) = 4a\alpha^2 t + c_0 \\ \dot{\hat{v}}_n(t) &= -\alpha^2\left(\frac{n\pi}{L}\right)^2\hat{v}_n \Rightarrow \hat{v}_n(t) = \hat{v}_n(0)\mathrm{e}^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}. \end{aligned}$$

Therefore

$$v(x,t) = \frac{4a\alpha^2 t + c_0}{2} + \sum_{n=1}^{\infty}\hat{v}_n(0)\mathrm{e}^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\cos\left(\frac{n\pi x}{L}\right).$$

$$\begin{aligned} g(x) - h(x) = g(x) - \left\{\left(\frac{B-A}{2L}\right)x^2 + Ax\right\} &= v(x,0) = \frac{c_0}{2} + \sum_{n=1}^{\infty}\hat{v}_n(0)\cos\left(\frac{n\pi x}{L}\right) \\ c_0 &= \frac{2}{L}\int_0^1\left[g(x) - \left\{\left(\frac{B-A}{2L}\right)x^2 + Ax\right\}\right]\,dx \\ \hat{v}_n(0) &= \frac{2}{L}\int_0^1\left[g(x) - \left\{\left(\frac{B-A}{2L}\right)x^2 + Ax\right\}\right]\cos\left(\frac{n\pi x}{L}\right)\,dx. \end{aligned}$$

Thus

$$u(x,t) = \left(\frac{B-A}{2L}\right)x^2 + Ax + 2a\alpha^2 t + \frac{c_0}{2} + \sum_{n=1}^{\infty}\hat{v}_n(0)\mathrm{e}^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\cos\left(\frac{n\pi x}{L}\right)$$

which is identical to the solution obtained in Example 18.2.

## 20.2 Time-dependent Boundary Conditions using Eigenfunction Expansions

**Example 20.2** *Time Dependent Boundary Conditions - general case:*

$$\begin{aligned} u_t &= \alpha^2 u_{xx}, \qquad 0 < x < L \\ \textit{BC: } u(0,t) &= \phi_0(t) \quad u(L,t) = \phi_1(t) \\ \textit{IC: } u(x,0) &= f(x). \end{aligned} \tag{20.8}$$

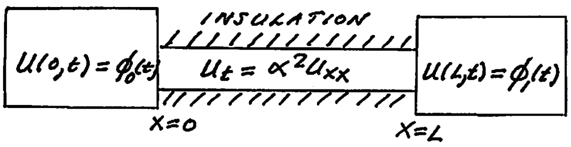


FIGURE 1. Bar subject to a time-dependent Dirichlet BC

Let $\boxed{w(x,t) = \phi_0(t) + x\left(\dfrac{\phi_1(t) - \phi_0(t)}{L}\right)}$ $\Rightarrow$ $w(0,t) = \phi_0(t)$; $w(L,t) = \phi_1(t)$. Now let $u(x,t) = w(x,t) + v(x,t)$. Then

$$\begin{aligned} w_t + v_t &= \alpha^2(\cancel{w_{xx}} + v_{xx}) \\ v_t &= \alpha^2 v_{xx} - w_t \qquad w_t = \dot{\phi}_0 + \frac{x}{L}(\dot{\phi}_1 - \dot{\phi}_0) \\ \text{BC: } u(0,t) &= \phi_0(t) = w(0,t) + v(0,t) = \phi_0(t) + v(0,t) \Rightarrow v(0,t) = 0 \\ u(L,t) &= \phi_1(t) = w(L,t) + v(L,t) = \phi_1(t) + v(L,t) \Rightarrow v(L,t) = 0 \\ \text{IC: } u(x,0) &= f(x) = w(x,0) + v(x,0) \Rightarrow v(x,0) = f(x) - w(x,0). \end{aligned} \tag{20.9}$$

Thus we need to solve the following BVP for $v(x,t)$:

$$\begin{aligned} v_t &= \alpha^2 v_{xx} - w_t \\ \text{BC: } v(0,t) &= 0 \quad v(L,t) = 0 \\ \text{IC: } v(x,0) &= f(x) - w(x,0). \end{aligned} \tag{20.10}$$

Now $v(x,t)$ can be found using an eigenfunction expansion. The eigenfunctions and eigenvalues associated with the

Dirichlet B C are

$$
\begin{aligned}
\lambda_n &= \left(\frac{n\pi}{L}\right) \qquad n = 1, 2, \ldots \qquad X_n(x) = \sin(\lambda_n x) \\
\text{let } S(x,t) &= -w_t \\
&= -(\dot{\phi}_1 - \dot{\phi}_0)\left(\frac{x}{L}\right) - \dot{\phi}_0 \\
&= \sum_{n=1}^{\infty} \hat{S}_n(t)\sin(\lambda_n x) \\
\text{and } v(x,t) &= \sum_{n=1}^{\infty} \hat{v}_n(t)\sin(\lambda_n x) \\
\text{then } v_t &= \sum_{n=1}^{\infty} \dot{\hat{v}}_n(t)\sin(\lambda_n x) \text{ and } v_{xx} = \sum_{n=1}^{\infty} \hat{v}_n(t)\left\{-\lambda_n^2\right\}\sin(\lambda_n x) \\
\text{thus } 0 &= v_t - \alpha^2 v_{xx} - S(x,t)
\end{aligned}
$$

Therefore

$$
0 = \sum_{n=1}^{\infty} \left\{\dot{\hat{v}}_n + \alpha^2\lambda_n^2\hat{v}_n - \hat{S}_n(t)\right\}\sin(\lambda_n x) \tag{20.11}
$$

Since the eigenfunctions are linearly independent it follows that $\{\ \} = 0$ in (20.11) or

$$
\frac{d\hat{v}_n}{dt} + \alpha^2\lambda_n^2\hat{v}_n = \hat{S}_n(t) \tag{20.12}
$$

but (20.12) is just a first order linear ODE with an integrating factor

$$
F(t) = e^{\alpha^2\lambda_n^2 t}
$$

Thus

$$
\frac{d}{dt}\left(e^{\alpha^2\lambda_n^2 t}\hat{v}_n(t)\right) = e^{\alpha^2\lambda_n^2 t}\hat{S}_n(t)
$$

Integrating we obtain

$$
e^{\alpha^2\lambda_n^2 t}\hat{v}_n(t) = \int_0^t e^{\alpha^2\lambda_n^2 \tau}\hat{S}_n(\tau)\,d\tau + c_n
$$

or

$$
\hat{v}_n(t) = \int_0^t e^{-\alpha^2\lambda_n^2(t-\tau)}\hat{S}_n(\tau)\,d\tau + e^{-\alpha\lambda_n^2 t}c_n
$$

Thus

$$
v(x,t) = \sum_{n=1}^{\infty}\left\{\int_0^t e^{-\alpha^2\lambda_n^2(t-\tau)}\hat{S}_n(\tau)\,d\tau + e^{-\alpha^2\lambda_n^2 t}c_n\right\}\sin(\lambda_n x)
$$

All we need to do to complete the solution of this problem is to determine the coefficients $c_n$. These we obtain from the initial condition as follows

$$
g(x) - \left[\{\phi_1(0) - \phi_0(0)\}\left(\frac{x}{L}\right) + \phi_0(0)\right] = \sum_{n=1}^{\infty} c_n \sin(\lambda_n x)
$$

But this is just a Fourier sine series in which

$$c_n = \frac{2}{L}\int_0^L \left(g(x) - \left[\{\phi_1(0) - \phi_0(0)\}\left(\frac{x}{L}\right) - \phi_0(0)\right]\right)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

Finally

$$u(x,t) = (\phi_1(t) - \phi_0(t))\left(\frac{x}{L}\right) + \phi_0(t) + \sum_{n=1}^{\infty}\left\{\int_0^t e^{-\alpha^2\lambda_n^2(t-\tau)}\hat{S}_n(\tau)\,d\tau + e^{-\alpha^2\lambda_n^2 t}c_n\right\}\sin\lambda_n x.$$

*Specific case:* Let $\phi_0(t) = At$, $\phi_1(t) = 0$, and $f(x) = 0$.
In this case

$$w(x,t) = At + \frac{x}{L}(0 - At) = At\left(1 - \frac{x}{L}\right). \tag{20.13}$$

$$\begin{aligned} u_t &= \alpha^2 u_{xx} \qquad 0 < x < L \\ \text{BC: } u(0,t) &= At \quad u(L,t) = 0 \\ \text{IC: } u(x,t) &= 0. \end{aligned} \tag{20.14}$$

Let $u(x,t) = w(x,t) + v(x,t)$ where $w(x,t) = At\left(1 - \frac{x}{L}\right)$. Then

$$\begin{aligned} v_t &= \alpha^2 v_{xx} - A\left(1 - \frac{x}{L}\right) \\ v(0,t) &= 0 = v(L,t) \\ v(x,0) &= 0. \end{aligned} \tag{20.15}$$

Let

$$\begin{aligned} s(x,t) = -A\left(1 - \frac{x}{L}\right) &= \sum_{n=1}^{\infty}\hat{s}_n(t)\sin\left(\frac{n\pi x}{L}\right) \\ \hat{s}_n &= \frac{2}{L}\int_0^L A\left(\frac{x}{L} - 1\right)\sin\left(\frac{n\pi x}{L}\right)\,dx \\ &= -\frac{2A}{n\pi}. \end{aligned} \tag{20.16}$$

Now let

$$\begin{aligned} v(x,t) &= \sum_{n=1}^{\infty}\hat{v}_n(t)\sin\left(\frac{n\pi x}{L}\right) \\ v_t &= \sum_{n=1}^{\infty}\dot{\hat{v}}_n(t)\sin\left(\frac{n\pi x}{L}\right), \quad v_{xx} = -\sum_{n=1}^{\infty}\hat{v}_n(t)\left(\frac{n\pi}{L}\right)^2\sin\left(\frac{n\pi x}{L}\right). \end{aligned} \tag{20.17}$$

Therefore

$$0 = v_t - \alpha^2 v_{xx} - s(x,t) = \sum_{n=1}^{\infty}\left\{\dot{\hat{v}}_n(t) + \alpha^2\left(\frac{n\pi}{L}\right)^2\hat{v}_n + \frac{2A}{n\pi}\right\}\sin\left(\frac{n\pi x}{L}\right). \tag{20.18}$$

Therefore

$$\dot{\hat{v}}_n(t) + \alpha^2\left(\frac{n\pi}{L}\right)^2 \hat{v}_n(t) = -\frac{2A}{n\pi} \tag{20.19}$$

$$\left(\mathrm{e}^{+\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\hat{v}_n(t)\right) = -\frac{2A}{n\pi}\mathrm{e}^{\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \tag{20.20}$$

$$\mathrm{e}^{\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\hat{v}_n(t) = -\frac{2AL^2}{\alpha^2(n\pi)^3}\mathrm{e}^{\alpha^2\left(\frac{n\pi}{L}\right)^2 t} + B_n \tag{20.21}$$

$$\hat{v}_n(t) = -\frac{2AL^2}{\alpha^2(n\pi)^3} + B_n\mathrm{e}^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \tag{20.22}$$

$$0 = \hat{v}_n(0) = -\frac{2AL^2}{\alpha^2(n\pi)^3} + B_n. \tag{20.23}$$

Therefore

$$\hat{v}_n(t) = \frac{2AL^2}{\alpha^2(n\pi)^3}\left(\mathrm{e}^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} - 1\right). \tag{20.24}$$

Therefore

$$u(x,t) = At\left(1 - \frac{x}{L}\right) + \frac{2AL^2}{\pi^3\alpha^2}\sum_{n=1}^{\infty}\frac{(\mathrm{e}^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} - 1)}{n^3}\sin\left(\frac{n\pi x}{L}\right). \tag{20.25}$$

### 20.2.1 *Summary of guesses for $w(x,t)$ to remove different inhomogeneous boundary conditions*

Consider the following heat equation subject to a loss represented by $-\gamma u$ and a source $S(x,t)$:

$$u_t = \alpha^2 u_{xx} - \gamma u + S(x,t)$$

*Mixed BC I*

$$u(0,t) = \phi_0(t), \quad u_x(L,t) = \phi_1(t), \quad w = \phi_0 + \phi_1 x$$

*Mixed BC II*

$$u_x(0,t) = \phi_0(t), \quad u(L,t) = \phi_1(t), \quad w = (\phi_1 - \phi_0 L) + \phi_0 x$$

*Dirichlet BC*

$$u(0,t) = \phi_0(t), \quad u(L,t) = \phi_1(t), \quad w = \phi_0 + (\phi_1 - \phi_0)\frac{x}{L}$$

*Neumann BC*

$$u_x(0,t) = \phi_0(t), \quad u_x(L,t) = \phi_1(t), \quad w = \phi_0 x + (\phi_1 - \phi_0)\frac{x^2}{2L}$$

Let $u(x,t) = w(x,t) + v(x,t)$

$$w_t + v_t = \alpha^2(w_{xx} + v_{xx}) - \gamma(w+v) + S(x,t)$$
$$v_t = \alpha^2 v_{xx} - \gamma v + \left\{\alpha^2 w_{xx} - \gamma w - w_t\right\} + S(x,t)$$