

Lecture 20: Heat conduction with time dependent boundary conditions using Eigenfunction Expansions

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The ultimate goal of this lecture is to demonstrate a method to solve heat conduction problems in which there are time dependent boundary conditions. The idea is to construct the simplest possible function, $w(x, t)$ say, that satisfies the inhomogeneous, time-dependent boundary conditions. The solution $u(x, t)$ that we seek is then decomposed into a sum of $w(x, t)$ and another function $v(x, t)$, which satisfies the homogeneous boundary conditions. When these two functions are substituted into the heat equation, it is found that $v(x, t)$ must satisfy the heat equation subject to a source that can be time dependent. As in Lecture 19, this forced heat conduction equation is solved by the method of eigenfunction expansions.

Key Concepts: Time-dependent Boundary conditions, distributed sources/sinks, Method of Eigenfunction Expansions.

20 Heat Conduction Problems with Time Dependent Boundary Conditions

20.1 Inhomogeneous Derivative Boundary Conditions using Eigenfunction Expansions

Example 20.1 Let us revisit the problem with inhomogeneous derivative BC (Example 18.2) - but we will now use Eigenfunction Expansions.

$$u_t = \alpha^2 u_{xx} \quad 0 < x < L, \quad t > 0 \quad (20.1)$$

$$BC: u_x(0, t) = A \quad u_x(L, t) = B \quad (20.2)$$

$$IC: u(x, 0) = g(x) \quad (20.3)$$

First look for a function of the form $h(x) = ax^2 + bx$ that satisfies the inhomogeneous BC:

$$h(x) = ax^2 + bx, \quad h_x(x) = 2ax + b$$

$$h_x(0) = b = A \quad h_x(L) = 2aL + A = B \Rightarrow a = (B - A)/2L$$

$$h(x) = \left(\frac{B - A}{2L} \right) x^2 + Ax.$$

Now let

$$u(x, t) = h(x) + v(x, t).$$

Substitute into the PDE:

$$u_t = (h(x) + v(x, t))_t = \alpha^2 u_{xx} = \alpha^2 (h(x) + v(x, t))_{xx} = \alpha^2 \cdot 2a + \alpha^2 v_{xx}.$$

Therefore

$$v_t = \alpha^2 v_{xx} + 2a\alpha^2 \quad (20.4)$$

$$A = u_x(0, t) = h_x(0) + v_x(0, t) = A + V_x(0, t) \Rightarrow v_x(0, t) = 0 \quad (20.5)$$

$$B = u_x(L, t) = h_x(L) + V_x(L, t) = B + V_x(L, t) \Rightarrow v_x(L, t) = 0 \quad (20.6)$$

$$g(x) = u(x, 0) = h(x) + v(x, 0) \Rightarrow v(x, 0) = g(x) - h(x). \quad (20.7)$$

We now use an Eigenfunction Expansion to solve the BVP (20.4)-(20.7). Because of the homogeneous Neumann BC we assume an expansion of the form

$$\begin{aligned} v(x, t) &= \hat{v}_0(t)/2 + \sum_{n=1}^{\infty} \hat{v}_n(t) \cos\left(\frac{n\pi x}{L}\right) \\ v_t &= \dot{\hat{v}}_0(t)/2 + \sum_{n=1}^{\infty} \dot{\hat{v}}_n(t) \cos\left(\frac{n\pi x}{L}\right) \\ v_x &= \sum_{n=1}^{\infty} \hat{v}_n(t) \left\{ -\left(\frac{n\pi}{L}\right) \right\} \sin\left(\frac{n\pi x}{L}\right), v_{xx} = \sum_{n=1}^{\infty} \hat{v}_n(t) \left\{ -\left(\frac{n\pi}{L}\right)^2 \right\} \cos\left(\frac{n\pi x}{L}\right). \end{aligned}$$

We also expand the inhomogeneous term in (20.4) in terms of the Eigenfunctions:

$$2a\alpha^2 = \hat{a}_0/2 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \quad a_0 = 4a\alpha^2, a_n = 0 \quad n \geq 1.$$

Therefore

$$0 = v_t - \alpha^2 v_{xx} - 2a\alpha^2 = \dot{\hat{v}}_0(t)/2 - 2a\alpha^2 + \sum_{n=1}^{\infty} \left\{ \dot{\hat{v}}_n + \alpha^2 \left(\frac{n\pi}{L}\right)^2 \hat{v}_n \right\} \cos\left(\frac{n\pi x}{L}\right).$$

Therefore

$$\begin{aligned} \dot{\hat{v}}_0(t) &= 4a\alpha^2 \Rightarrow \hat{v}_0(t) = 4a\alpha^2 t + c_0 \\ \dot{\hat{v}}_n(t) &= -\alpha^2 \left(\frac{n\pi}{L}\right)^2 \hat{v}_n \Rightarrow \hat{v}_n(t) = \hat{v}_n(0) e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t}. \end{aligned}$$

Therefore

$$v(x, t) = \frac{4a\alpha^2 t + c_0}{2} + \sum_{n=1}^{\infty} \hat{v}_n(0) e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right).$$

$$g(x) - h(x) = g(x) - \left\{ \left(\frac{B-A}{2L}\right) x^2 + Ax \right\} = v(x, 0) = \frac{c_0}{2} + \sum_{n=1}^{\infty} \hat{v}_n(0) \cos\left(\frac{n\pi x}{L}\right)$$

$$c_0 = \frac{2}{L} \int_0^1 \left[g(x) - \left\{ \left(\frac{B-A}{2L}\right) x^2 + Ax \right\} \right] dx$$

$$\hat{v}_n(0) = \frac{2}{L} \int_0^1 \left[g(x) - \left\{ \left(\frac{B-A}{2L}\right) x^2 + Ax \right\} \right] \cos\left(\frac{n\pi x}{L}\right) dx.$$

Thus

$$u(x, t) = \left(\frac{B-A}{2L}\right) x^2 + Ax + 2a\alpha^2 t + \frac{c_0}{2} + \sum_{n=1}^{\infty} \hat{v}_n(0) e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right)$$

which is identical to the solution obtained in Example 18.2.

20.2 Time-dependent Boundary Conditions using Eigenfunction Expansions

Example 20.2 Time Dependent Boundary Conditions - general case:

$$\begin{aligned}
 u_t &= \alpha^2 u_{xx}, & 0 < x < L \\
 BC: u(0, t) &= \phi_0(t) & u(L, t) &= \phi_1(t) \\
 IC: u(x, 0) &= f(x). & &
 \end{aligned} \tag{20.8}$$

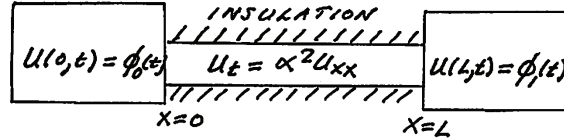


FIGURE 1. Bar subject to a time-dependent Dirichlet BC

Let $w(x, t) = \phi_0(t) + x \left(\frac{\phi_1(t) - \phi_0(t)}{L} \right) \Rightarrow w(0, t) = \phi_0(t); w(L, t) = \phi_1(t)$. Now let $u(x, t) = w(x, t) + v(x, t)$.
Then

$$\begin{aligned}
 w_t + v_t &= \alpha^2 (w_{xx} + v_{xx}) \\
 v_t &= \alpha^2 v_{xx} - w_t & w_t &= \dot{\phi}_0 + \frac{x}{L} (\dot{\phi}_1 - \dot{\phi}_0) \\
 BC: u(0, t) &= \phi_0(t) = w(0, t) + v(0, t) = \phi_0(t) + v(0, t) \Rightarrow v(0, t) = 0 \\
 u(L, t) &= \phi_1(t) = w(L, t) + v(L, t) = \phi_1(t) + v(L, t) \Rightarrow v(L, t) = 0 \\
 IC: u(x, 0) &= f(x) = w(x, 0) + v(x, 0) \Rightarrow v(x, 0) = f(x) - w(x, 0).
 \end{aligned} \tag{20.9}$$

Thus we need to solve the following BVP for $v(x, t)$:

$$\begin{aligned}
 v_t &= \alpha^2 v_{xx} - w_t \\
 BC: v(0, t) &= 0 & v(L, t) &= 0 \\
 IC: v(x, 0) &= f(x) - w(x, 0).
 \end{aligned} \tag{20.10}$$

Now $v(x, t)$ can be found using an eigenfunction expansion. The eigenfunctions and eigenvalues associated with the

Dirichlet B C are

$$\begin{aligned} \lambda_n &= \left(\frac{n\pi}{L}\right) \quad n = 1, 2, \dots \quad X_n(x) = \sin(\lambda_n x) \\ \text{let } S(x, t) &= -w_t \\ &= -(\dot{\phi}_1 - \dot{\phi}_0) \left(\frac{x}{L}\right) - \dot{\phi}_0 \\ &= \sum_{n=1}^{\infty} \hat{S}_n(t) \sin(\lambda_n x) \\ \text{and } v(x, t) &= \sum_{n=1}^{\infty} \hat{v}_n(t) \sin(\lambda_n x) \\ \text{then } v_t &= \sum_{n=1}^{\infty} \dot{\hat{v}}_n(t) \sin(\lambda_n x) \quad \text{and } v_{xx} = \sum_{n=1}^{\infty} \hat{v}_n(t) \{-\lambda_n^2\} \sin(\lambda_n x) \\ \text{thus } 0 &= v_t - \alpha^2 v_{xx} - S(x, t) \end{aligned}$$

Therefore

$$0 = \sum_{n=1}^{\infty} \left\{ \dot{\hat{v}}_n + \alpha^2 \lambda_n^2 \hat{v}_n - \hat{S}_n(t) \right\} \sin(\lambda_n x) \quad (20.11)$$

Since the eigenfunctions are linearly independent it follows that $\{ \} = 0$ in (20.11) or

$$\frac{d\hat{v}_n}{dt} + \alpha^2 \lambda_n^2 \hat{v}_n = \hat{S}_n(t) \quad (20.12)$$

but (20.12) is just a first order linear ODE with an integrating factor

$$F(t) = e^{\alpha^2 \lambda_n^2 t}$$

Thus

$$\frac{d}{dt} \left(e^{\alpha^2 \lambda_n^2 t} \hat{v}_n(t) \right) = e^{\alpha^2 \lambda_n^2 t} \hat{S}_n(t)$$

Integrating we obtain

$$e^{\alpha^2 \lambda_n^2 t} \hat{v}_n(t) = \int_0^t e^{\alpha^2 \lambda_n^2 \tau} \hat{S}_n(\tau) d\tau + c_n$$

or

$$\hat{v}_n(t) = \int_0^t e^{-\alpha^2 \lambda_n^2 (t-\tau)} \hat{S}_n(\tau) d\tau + e^{-\alpha^2 \lambda_n^2 t} c_n$$

Thus

$$v(x, t) = \sum_{n=1}^{\infty} \left\{ \int_0^t e^{-\alpha^2 \lambda_n^2 (t-\tau)} \hat{S}_n(\tau) d\tau + e^{-\alpha^2 \lambda_n^2 t} c_n \right\} \sin(\lambda_n x)$$

All we need to do to complete the solution of this problem is to determine the coefficients c_n . These we obtain from the initial condition as follows

$$g(x) - \left[\{\phi_1(0) - \phi_0(0)\} \left(\frac{x}{L}\right) + \phi_0(0) \right] = \sum_{n=1}^{\infty} c_n \sin(\lambda_n x)$$

But this is just a Fourier sine series in which

$$c_n = \frac{2}{L} \int_0^L \left(g(x) - \left[\{\phi_1(0) - \phi_0(0)\} \left(\frac{x}{L} \right) - \phi_0(0) \right] \right) \sin \left(\frac{n\pi x}{L} \right) dx$$

Finally

$$u(x, t) = (\phi_1(t) - \phi_0(t)) \left(\frac{x}{L} \right) + \phi_0(t) + \sum_{n=1}^{\infty} \left\{ \int_0^t e^{-\alpha^2 \lambda_n^2 (t-\tau)} \hat{S}_n(\tau) d\tau + e^{-\alpha^2 \lambda_n^2 t} c_n \right\} \sin \lambda_n x.$$

Specific case: Let $\phi_0(t) = At$, $\phi_1(t) = 0$, and $f(x) = 0$.

In this case

$$w(x, t) = At + \frac{x}{L}(0 - At) = At \left(1 - \frac{x}{L} \right). \quad (20.13)$$

$$u_t = \alpha^2 u_{xx} \quad 0 < x < L$$

$$\text{BC: } u(0, t) = At \quad u(L, t) = 0 \quad (20.14)$$

$$\text{IC: } u(x, t) = 0.$$

Let $u(x, t) = w(x, t) + v(x, t)$ where $w(x, t) = At \left(1 - \frac{x}{L} \right)$. Then

$$v_t = \alpha^2 v_{xx} - A \left(1 - \frac{x}{L} \right)$$

$$v(0, t) = 0 = v(L, t) \quad (20.15)$$

$$v(x, 0) = 0.$$

Let

$$s(x, t) = -A \left(1 - \frac{x}{L} \right) = \sum_{n=1}^{\infty} \hat{s}_n(t) \sin \left(\frac{n\pi x}{L} \right)$$

$$\begin{aligned} \hat{s}_n &= \frac{2}{L} \int_0^L A \left(\frac{x}{L} - 1 \right) \sin \left(\frac{n\pi x}{L} \right) dx \\ &= -\frac{2A}{n\pi}. \end{aligned} \quad (20.16)$$

Now let

$$v(x, t) = \sum_{n=1}^{\infty} \hat{v}_n(t) \sin \left(\frac{n\pi x}{L} \right) \quad (20.17)$$

$$v_t = \sum_{n=1}^{\infty} \dot{\hat{v}}_n(t) \sin \left(\frac{n\pi x}{L} \right), \quad v_{xx} = - \sum_{n=1}^{\infty} \hat{v}_n(t) \left(\frac{n\pi}{L} \right)^2 \sin \left(\frac{n\pi x}{L} \right).$$

Therefore

$$0 = v_t - \alpha^2 v_{xx} - s(x, t) = \sum_{n=1}^{\infty} \left\{ \dot{\hat{v}}_n(t) + \alpha^2 \left(\frac{n\pi}{L} \right)^2 \hat{v}_n + \frac{2A}{n\pi} \right\} \sin \left(\frac{n\pi x}{L} \right). \quad (20.18)$$

Therefore

$$\dot{\hat{v}}_n(t) + \alpha^2 \left(\frac{n\pi}{L}\right)^2 \hat{v}_n(t) = -\frac{2A}{n\pi} \quad (20.19)$$

$$\left(e^{+\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \hat{v}_n(t)\right) = -\frac{2A}{n\pi} e^{\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \quad (20.20)$$

$$e^{\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \hat{v}_n(t) = -\frac{2AL^2}{\alpha^2 (n\pi)^3} e^{\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} + B_n \quad (20.21)$$

$$\hat{v}_n(t) = -\frac{2AL^2}{\alpha^2 (n\pi)^3} + B_n e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \quad (20.22)$$

$$0 = \hat{v}_n(0) = -\frac{2AL^2}{\alpha^2 (n\pi)^3} + B_n. \quad (20.23)$$

Therefore

$$\hat{v}_n(t) = \frac{2AL^2}{\alpha^2 (n\pi)^3} \left(e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} - 1\right). \quad (20.24)$$

Therefore

$$u(x, t) = At \left(1 - \frac{x}{L}\right) + \frac{2AL^2}{\pi^3 \alpha^2} \sum_{n=1}^{\infty} \frac{(e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} - 1)}{n^3} \sin\left(\frac{n\pi x}{L}\right). \quad (20.25)$$

20.2.1 Summary of guesses for $w(x, t)$ to remove different inhomogeneous boundary conditions

Consider the following heat equation subject to a loss represented by $-\gamma u$ and a source $S(x, t)$:

$$u_t = \alpha^2 u_{xx} - \gamma u + S(x, t)$$

Mixed BC I

$$u(0, t) = \phi_0(t), \quad u_x(L, t) = \phi_1(t), \quad w = \phi_0 + \phi_1 x$$

Mixed BC II

$$u_x(0, t) = \phi_0(t), \quad u(L, t) = \phi_1(t), \quad w = (\phi_1 - \phi_0 L) + \phi_0 x$$

Dirichlet BC

$$u(0, t) = \phi_0(t), \quad u(L, t) = \phi_1(t), \quad w = \phi_0 + (\phi_1 - \phi_0) \frac{x}{L}$$

Neumann BC

$$u_x(0, t) = \phi_0(t), \quad u_x(L, t) = \phi_1(t), \quad w = \phi_0 x + (\phi_1 - \phi_0) \frac{x^2}{2L}$$

Let $u(x, t) = w(x, t) + v(x, t)$

$$w_t + v_t = \alpha^2 (w_{xx} + v_{xx}) - \gamma (w + v) + S(x, t)$$

$$v_t = \alpha^2 v_{xx} - \gamma v + \{\alpha^2 w_{xx} - \gamma w - w_t\} + S(x, t)$$