Lecture 31: Sturm-Liouville Boundary Value Problems

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In this lecture we abstract the eigenvalue problems that we have found so useful thus far for solving the PDEs to a general class of boundary value problems that share a common set of properties. The so-called *Sturm-Liouville Problems* define a class of eigenvalue problems which include many of the previous problems as special cases. The $S-L$ Problem helps to identify those assumptions that are needed to define an eigenvalue problems with the properties that we require.

Key Concepts: Eigenvalue Problems, Sturm-Liouville Boundary Value Problems; Robin Boundary conditions.

Reference Section: Boyce and Di Prima Section 11.1 and 11.2

31 Solving the heat equation with Robin BC

31.1 Expansion in Robin Eigenfunctions

Assume that we can expand $f(x)$ in terms of $\phi_n(x)$:

$$f(x) = \sum_{n-1}^{\infty} c_n \phi_n(x) \tag{31.1}$$

$$\int_0^1 f(x)\sin(\mu_n x)\,dx = c_n \int_0^1 \left[\phi_n(x)\right]^2 dx \tag{31.2}$$

$$= c_n \frac{1}{\cdot 2}\left[1+\cos^2\mu_n\right] \tag{31.3}$$

Therefore

$$c_n = \frac{2}{[1+\cos^2\mu_n]}\int_0^1 f(x)\sin(\mu_n x)\,dx. \tag{31.4}$$

If $f(x) = x$ then

$$\begin{aligned}\int_0^1 x\sin(\mu_n x)\,dx &= -\tfrac{\cos(\mu_n x)}{\mu_n} - x\Big|_0^1 + \tfrac{1}{\mu_n}\int_0^1 \cos\mu_n x\,dx \\ &= -\tfrac{\cos(\mu_n)}{\mu_n} + \tfrac{\sin\mu_n x}{\mu_n^2}\Big|_0^1 \\ &= \tfrac{\sin\mu_n - \mu_n\cos\mu_n}{\mu_n^2} = 2\tfrac{\sin\mu_n}{\mu_n^2}.\end{aligned} \tag{31.5}$$

but $-\mu_n\cos\mu_n = \sin\mu_n$

Therefore

$$c_n = \frac{4\sin\mu_n}{\mu_n^2[1+\cos^2\mu_n]} \tag{31.6}$$

$$f(x) = 4\sum_{n=1}^{\infty}\frac{\sin\mu_n\sin(\mu_n x)}{\mu_n^2[1+\cos^2\mu_n]} \tag{31.7}$$

31.2 Solving the Heat Equation with Robin BC

$$u_t = \alpha^2 u_{xx} \quad 0 < x < 1 \tag{31.8}$$

$$u(0,t) = 1 \quad u_x(1,t) + u(1,t) = 0 \tag{31.9}$$

$$u(x,0) = f(x). \tag{31.10}$$

Look for a steady state solution $v(x)$

$$\left.\begin{array}{l} v''(x) = 0 \\ v(0) = 1 \quad v'(1) + v(1) = 0 \end{array}\right\} \tag{31.11}$$

$$\begin{array}{c} v = Ax + B \quad v(0) = B = 1 \quad v'(x) = A \quad v'(1) + v(1) = A + (A+1) = 0 \\ A = -1/2 \end{array} \tag{31.12}$$

Therefore

$$v(x) = 1 - x/2. \tag{31.13}$$

Now let $u(x,t) = v(x) + w(x,t)$

$$u_t = w_t = \alpha^2(\cancel{v''} + w_{xx}) \Rightarrow w_t = \alpha^2 w_{xx}$$

$$1 = u(0,t) = v(0) + w(0,t) = 1 + w(0,t) \Rightarrow w(0,t) = 0$$

$$\begin{array}{rcllcl} 0 = u_x(1,t) + u(1,t) & = & \{v'(1) \cancel{+} v(1)\} & + w_x(1,t) + w(1,t) & \Rightarrow & w_x(1,t) + w(1,t) = 0 \\ f(x) = u(x,0) & = & v(x) + w(x,0) & & \Rightarrow & w(x,0) = f(x) - v(x). \end{array}$$

Let

$$w(x,t) = X(x)T(t) \tag{31.14}$$

$$\frac{\dot{T}(t)}{\alpha^2 T(t)} = \frac{X''}{X} = -\mu^2 \tag{31.15}$$

$$T(t) = c\rho^{-\alpha^2\mu^2 t} \tag{31.16}$$

$$\left.\begin{array}{l} X'' + \mu^2 X = 0 \\ X(0) = 0 \quad X'(1) + X(1) = 0 \end{array}\right\} \quad \text{The } \mu_n \text{ are solutions of the transcendental equation: } \tan\mu_n = -\mu_n. \tag{31.17}$$

$$X_n(x) = \sin(\mu_n x) \tag{31.18}$$

$$w(x,t) = \sum_{n=1}^{\infty} c_n \rho^{-\alpha^2\mu_n^2 t} \sin(\mu_n x) \tag{31.19}$$

where

$$f(x) - v(x) = w(x, 0) = \sum_{n=1}^{\infty} c_n \sin(\mu_n x) \tag{31.20}$$

$$\Rightarrow c_n = \frac{2}{[1 + \cos^2 \mu_n]} \int_0^1 [f(x) - v(x)] \sin(\mu_n x)\, dx \tag{31.21}$$

$$u(x, t) = 1 - \frac{n?}{2} + \sum_{n=1}^{\infty} c_n \rho^{-\alpha^2 \mu_n^2 t} \sin(\mu_n x). \tag{31.22}$$