Topic: Introduction to Green's functions

(*Compiled 20 September 2012*)

In this lecture we provide a brief introduction to Green's Functions.

Key Concepts: Green's Functions, Linear Self-Adjoint Differential Operators,.

9 Introduction/Overview

9.1 Green's Function Example: A Loaded String

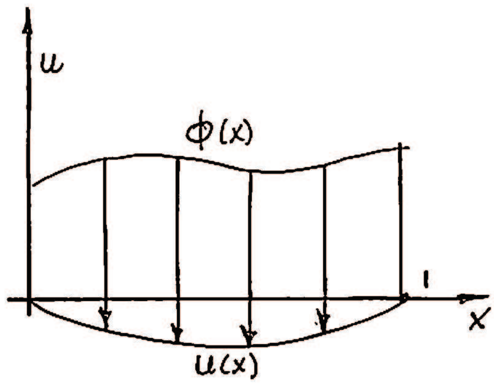

FIGURE 1. Model of a loaded string

Consider the forced boundary value problem

$$Lu = u''(x) = \phi(x) \qquad u(0) = 0 = u(1)$$

Physical Interpretation: $u(x)$ is the static deflection of a string stretched under unit tension between fixed endpoints and subject to a force distribution $\phi(x)$ Newtons per unit length shown in figure 1.
Question: Since this is a linear equation can we invert the differential operator $L = \frac{d^2}{dx^2}$ to obtain an expression for the solution in the form:

$$u(x) = T(x) \cdot \phi$$

9.1.1 *Method 1: Variation of Parameters*

Homogeneous eqn: $u'' = 0$ has solution $u(x) = c_1 x + c_2$.
Part Solution:

$$\begin{aligned} u(x) &= xv_1(x) + v_2(x) \\ u' &= v_1(x) + \{xv_1' + v_2'\} \\ u'' &= v_1' + \{\ \}' \\ \text{Therefore } Lu &= v_1' = \phi(x). \end{aligned}$$

Require $\{\ \} = 0$ since we need another constraint to determine $v_1 > v_2$ uniquely.

$$\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \phi(x) \end{bmatrix}$$

$$\begin{array}{rclrcl} v_1' &=& -\phi(x)/(-1) & \qquad v_2' &=& x\phi(x)(-1) \\ v_1 &=& \int_0^x \phi(s)ds & \qquad v_2 &=& -\int_0^x s\phi(s)ds \end{array}$$

$$\text{Therefore } u_p(x) = \left(\int_0^x \phi(s)ds\right) x - \int_0^x s\phi(s)ds.$$

$$u_c = c_1 x + c_2 + \left(\int_0^x \phi ds\right) x - \int_0^x s\phi(s)ds$$

$$u(0) = 0 \Rightarrow c_2 = 0, \quad u(1) = 0 \Rightarrow c_1 + \int_0^1 \phi ds \cdot 1 - \int_0^1 s\phi(s)ds = 0$$

$$\begin{aligned} u(x) &= \left(-\int_0^1 \phi ds + \int_0^1 s\phi ds\right) x + \left(\int_0^x \phi ds\right) x - \int_0^x s\phi(s)ds \\ &= \int_0^1 x(s-1)\phi(s)ds + \int_0^x s(x-1)\phi(s)ds \end{aligned}$$

$$\boxed{\text{Therefore } u(x) = \int_0^1 G(s,x)\phi(s)ds \text{ where } G(s,x) = \begin{cases} s(x-1) & s < x \\ x(s-1) & s > x \end{cases}.}$$

Physical Interpretation: $G(s,x)$ is the deflection at s due to a unit point load at x.

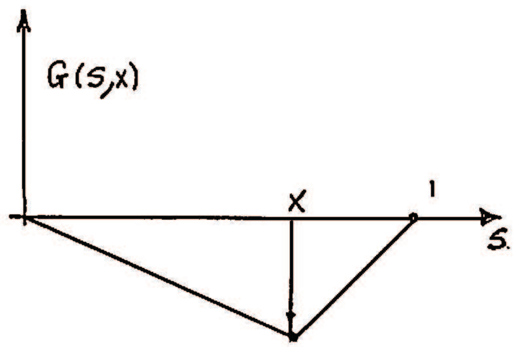

FIGURE 2. Displacement of a string due to a point loading

$$G(s,x) = \begin{cases} s(x-1) & s < x \\ x(s-1) & s > x \end{cases}$$

Physical Interpretation of reciprocity: $G(s,x) = G(x,s)$ Therefore deflection at s due to a unit point load at x = deflection at x due to a unit point load at s.

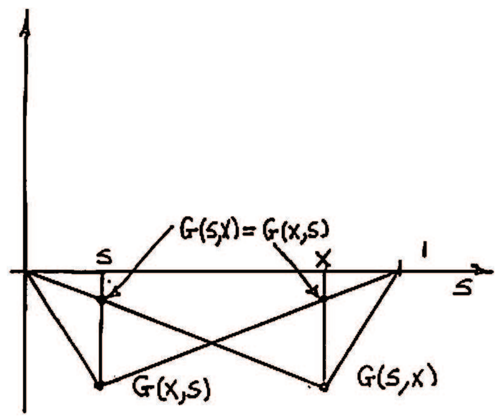

FIGURE 3. Physical interpretation of reciprocity

9.1.2 *Method 2: The Adjoint Operator*

$Lu = u'' = \phi \qquad u(0) = 0 = u(1)$

$$\int_0^1 vLuds = \int_0^1 vu''ds$$
$$= vu']_0^1 - \int_0^1 v'u'ds$$
$$= vu']_0^1 - v'u]_0^1 + \int_0^1 uv''ds$$
$$= [vu' - uv']_0^1 + \int_0^1 uL^*vds$$

Note: Since $L^* = L$, we say that L is formerly self-adjoint.

$$\int_0^1 v\phi ds = v(1)u'(1) - v(0)u'(0) - u(1)v'(1) + u(0)v'(0) + \int_0^1 u\frac{d^2}{ds^2}vds \tag{9.1}$$

$$= v(1)u'(1) - v(0)u'(0) + \int_0^1 u\frac{d^2}{ds^2}vds \tag{9.2}$$

Up till now other than being sufficiently differentiable, v has been arbitrary. How can we choose v so that we obtain an expression of the form:

$$u(x) = \int_0^1 v(s,x)\phi(s)ds \tag{9.3}$$

If v satisfies the following boundary value problem

$$\left.\begin{aligned} L^*v &= \tfrac{d^2}{ds^2}v(s,x) = \delta(s-x) \\ v(0) &= 0 = v(1) \end{aligned}\right\} \tag{9.4}$$

then (9.2) reduces to (9.3). How do we solve (9.4)?

Method A: direct integration

$$\begin{aligned} v_{ss} &= \delta(s-x) \qquad \text{Recall } H'(x) = s(x) \\ v_s &= H(s-x) + A \end{aligned}$$

$$\begin{aligned} v(s,x) &= \int H(s-x)ds + As \qquad s - x = \chi \\ &= \int H(\chi)d\chi + As \\ &= \chi H(\chi) + As + B \\ &= (s-x)H(s-x) + As + B \end{aligned}$$

$$0 = v(0, x) = B$$
$$0 = v(1, x) = (1 - x)H(1 - x) + A$$

$$\text{Therefore } A = (x - 1)H(1 - x) = (x - 1)$$
$$\text{Therefore } v(s, x) = (s - x)H(s - x) + s(x - 1)$$
$$= \begin{cases} s(x-1) & s < x \\ (s - x) + sx - s = x(s - 1) & s > x \end{cases}$$

Method B: Stitching in the region $s < x$ **and** $s > x$ $v_{ss} = 0$ thus:

$$v(s, x) = \begin{cases} A_- s + B_- = v_- & s < x \\ A_+ s + B_+ = v_+ & s > x \end{cases}$$

We have 4 constants and only two boundary conditions so we need some additional conditions to determine v.
Continuity at x

$$v(x_-, x) = v(x_+, x)$$

$$A_- x + B_- = A_+ ? + B_+ \tag{9.5}$$

Jump Condition at x

$$v_{ss} = \delta(s - x)$$
$$\int_{x-\varepsilon}^{x+\varepsilon} v_{ss} ds = \int_{x-\varepsilon}^{x+\varepsilon} \delta(s - x) dx = 1$$
$$[v_s]_{x-\varepsilon}^{x+\varepsilon} = 1$$

Therefore

$$A_+ - A_- = 1 \tag{9.6}$$
$$0 = v(0, x) = B_- \tag{9.7}$$
$$0 = v(1, x) = A_+ + B_+ \tag{9.8}$$

Therefore

$$\begin{bmatrix} x & (1-x) \\ -1 & 1 \end{bmatrix} \begin{bmatrix} A_- \\ A_+ \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A_- = -(1 - x)/1 = (x - 1), \quad A_+ = x$$

Therefore

$$v(s, x) = \begin{cases} s(x - 1) & s < x \\ x(s - 1) & s > x \end{cases}$$

9.2 Summary

Given a linear differential operator $lu = f + BC$ we will be looking for a Green's Function satisfying

$$L^*G = \delta(\varepsilon - x) + \text{ appropriate } BC$$

such that we can express the inverse operator for L in the form:

$$u(x) = \int_\Omega G(\varepsilon, x) f(\varepsilon) ds.$$

9.3 Applications

(1) Boundary integral methods – Heat Transfer, Fluid Flow, Elasticity, Electram?

(2) Tomography

Note: What is the analogue of the Green's Function in a discrete problem? Consider a linear operator $A : \mathbb{R}^N \to \mathbb{R}^N$ e.g. the matrix problem $Au = f$.

$$\begin{bmatrix} a_u & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

Suppose we solve each of the problems

$$A^T v_k = e_k = [0 \dots 1_k \dots 0]^T$$

Now define the matrix V whose columns comprise the v_k so that

$$V = \begin{bmatrix} \vdots & \vdots & \dots & \vdots \\ v_1 & v_2 & \cdots & v_n \\ \vdots & \vdots & \dots & \vdots \end{bmatrix}$$

$$v_k^T Au = u^T A^T v_k = u^T e_k = u_k \quad \text{since} \quad A^T v_k = e_k$$

$$u_k = v_k^T Au = v_k^T f \quad \text{since} \quad Au = f$$

$$u = \begin{bmatrix} \cdots & v_1 & \cdots \\ \cdots & v_2 & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & v_n & \cdots \end{bmatrix} f = V^T f = A^{-1} f.$$