Lecture 2: Error in polynomial approximation and interpolation with equally spaced sample points

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In this lecture we discuss the error that is made when a function is approximated by an interpolation polynomial. We also introduce shift, difference, and average operators that can be defined for the special case of equally spaced sample points.

Key Concepts: Polynomial Truncation Error, Interpolation formulae for equally spaced points, The Gregory-Newton interpolation formula, Difference approximations to the derivative.

2 Polynomial Approximation

2.1 Error Estimate for polynomial approximation

Question What is the error involved when we try to approximate $f(x)$ by a polynomial of degree N?

Theorem: If $f \in C^{N+1}[a, b]$ then

$$f(x) = p_N(x) + \frac{f^{(N+1)}(\xi)}{(N+1)!}(x - x_0)(x - x_1)\dots(x - x_N) \qquad \xi \in (x_0, x_N).$$

Lemma 1: Divided difference expression for the error.

$$e_N(\bar{x}) = f(\bar{x}) - p_N(\bar{x}) = f[x_0, x_1, \dots, x_N, \bar{x}] \prod_{j=0}^{N} (\bar{x} - x_j)(*) \text{ for any } \bar{x} \in [x_0, x_N].$$

Proof: If $\bar{x} = x_j$ then the formula $(*)$ holds.

If $\bar{x} \neq x_j \quad j = 0, \dots, N$, then consider the polynomial $p_{N+1}(x)$ that passes through $f(x_0), \dots, f(x_N)$ and $f(\bar{x})$. Then

$$f(\bar{x}) = p_{N+1}(\bar{x}) = p_N(\bar{x}) + f[x_0, \dots, x_N, \bar{x}] \prod_{j=0}^{N} (\bar{x} - x_j)$$

$$\therefore\ e_N(\bar{x}) = f(\bar{x}) - p_N(\bar{x}) = f[x_0, \dots, x_N, \bar{x}] \prod_{j=0}^{N} (\bar{x} - x_j)$$

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Lemma 2: (Like the Mean Value Theorem)

If f is continuous on $[x_0, x_k]$ and k times differentiable on (x_0, x_k) then there exists a $\xi \in (x_0, x_k)$ such that

$$f[x_0, x_1, \ldots, x_k] = f^{(k)}(\xi)/k!$$

Proof: $e_N(x) = f(x) - p_N(x)$ has $N+1$ roots in $[x_0, x_N]$, namely $x_0, x_1, \ldots, x_N$.

$$\text{Rolle } \Rightarrow e_n' \text{ has } N \text{ roots } \Rightarrow \ldots \Rightarrow e_N^{(N)} \text{ has 1 root in } (x_0, x_N)$$
$$\therefore \exists\, \xi \in (x_0, x_N) \text{ such that } e_N^{(N)}(\xi) = f^{(N)}(\xi) - f[x_0, x_1, \ldots, x_N]N! = 0$$
$$\exists\, \xi \in (x_0, x_N) : f[x_0, x_1, \ldots, x_N] = f^{(N)}(\xi)/N!$$

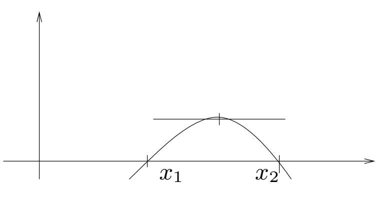
■

Proof of Theorem:

By Lemma 1: $e_N(x) = f[x_0, \ldots, x_N, x] \prod_{j=0}^{N} (x - x_j)$.

By Lemma 2: $\exists\, \xi \in (x_0, x_N) : e_N(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!} \prod_{j=0}^{N} (x - x_j)$. ■

Recall Rolle's Theorem:



g CONT ON $[x_1, x_2]$
g DIFF ON (x_1, x_2)
$\exists\, \xi \in (x_1, x_2) : g'(\xi) = 0$

2.2 Polynomial approximation for equally spaced meshpoints

Assume $x_k = a + kh$ where $h = \frac{b-a}{N}$, $k = 0, \ldots, N$

Mesh Operators: We now define the following difference, shift and averaging operators that can be applied to the sequence $\{f_n\}$.

Forward difference operator:

$$\begin{aligned} \Delta f_n &= f_{n+1} - f_n \\ \Delta^2 f_n &= \Delta f_{n+1} - \Delta f_n = f_{n+2} - 2f_{n+1} + f_n \end{aligned}$$

Backward difference operator:

$$\begin{aligned} \nabla f_n &= f_n - f_{n-1} \\ \nabla^2 f_n &= \nabla f_n - \nabla f_{n-1} = f_n - 2f_{n-1} + f_{n-2} \end{aligned}$$

Central difference operator:

$$\begin{aligned} \delta f_n &= f_{n+1/2} - f_{n-1/2} \\ \delta^2 f_n &= \delta f_{n+1/2} - \delta f_{n-1/2} = f_{n+1} - 2f_n + f_{n-1} \end{aligned}$$

Shift operator:

$$E f_n = f_{n+1}$$

Average operator:

$$\mu f_n = \frac{1}{2}\left\{f_{n+1/2} + f_{n-1/2}\right\}$$

Derivative operator:

$$D f_n = \frac{d}{dx} f\Big|_{x=x_n}$$

Relationship between the shift and difference operators:

$$\Delta = (E-1) \qquad E = (1+\Delta)$$
$$\nabla = (1-E^{-1}) \qquad E^{-1} = (1-\nabla)$$

Gregory-Newton Interpolation formula:

$$f_p = E^{p-i} f_i = (1+\Delta)^{p-i} f_i$$
$$= \left\{ 1 + \frac{(p-i)}{1!}\Delta + \frac{(p-i)(p-i-1)}{2!}\Delta^2 + \ldots, \right\} f_i$$

$$\boxed{f_p = \sum_{k=0}^{m} \binom{p-i}{k} \Delta^k f_i}$$

Gregory-Newton Interpolation formula

Eg: Derive the identity $s_N = \sum_{i=1}^{N} i = \frac{N(N+1)}{2}$

f_n	Δ	Δ^2	Δ^3
$s_0 = 0$	1	1	0
$s_1 = 1$	2	1	0
$s_2 = 3$	3	1	0
$s_3 = 6$	4	1	
$s_4 = 10$			

$$\therefore \quad s_N = \left\{ 1 + N\Delta + \frac{N(N-1)}{2!}\Delta^2 \right\} s_0$$
$$= 0 + N \cdot 1 + \frac{N(N-1)}{2} \cdot 1 = \frac{2N+N^2-N}{2} = \frac{N(N+1)}{2}$$

Using backward differences:

$$f_{n-p} = E^{-p} f_n = (1-\nabla)^p f_n = \sum_{k=0}^{m} (-1)^k \binom{p}{k} \nabla^k f_n$$
$$f_{n+s} = E^{s} f_n = (1-\nabla)^{-s} f_n = \sum_{k=0}^{m} (-1)^k \binom{-s}{k} \nabla^k f_n$$

$$\text{where } \binom{y}{k} = \begin{cases} \frac{y(y-1)\ldots(y-k+1)}{k!} & k > 0 \\ 1 & k = 0. \end{cases}$$

Example 2:

$$f(x) = x^3 + 2x + 1$$

x_i	$f(x_i)$	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$
0	1	3	6	6
1	4	9	12	6
2	13	21	18	
3	34	39		
4	73			

x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$
0	1	12	48	48
2	13	60	96	48
4	73	156	144	
6	229	300		
8	529			

$$\begin{aligned}
E^p f_0 &= \left(1 + p\Delta + \frac{p^2-p}{2}\Delta^2 + \left(\frac{p^3-3p^2+2p}{6}\right)\Delta^3\right) f_0 \\
&= 1 + p\cdot 3 + \frac{p^2-p}{2}\cdot 6 + \frac{p^3-3p^2+2p}{6}\cdot 6 \\
&= 1 + \cancel{3p} + \cancel{3p^2} - \cancel{3p} + p^3 - \cancel{3p^2} + 2p \\
&= p^3 + 2p + 1
\end{aligned}$$

Note: in general $p = \left(\frac{x-x_0}{h}\right)$ or $x = x_0 + ph$

$$\begin{aligned}
E^p f_0 &= 1 + p\cdot 12 + \frac{(p^2-p)}{2}\cdot 48 + \left(\frac{p^3-3p^2+2p}{6}\right)\cdot 48 = 1 + 12p + \cancel{24p^2} - 24p + 8p^3 - \cancel{24p^2} + 16p \\
&= 1 + 4p + 8p^3 = 1 + 4\left(\frac{x}{2}\right) + 8\left(\frac{x}{2}\right)^3 = 1 + 2x + x^3
\end{aligned}$$

$$p = \left(\frac{x-x_0}{h}\right) = \frac{x-0}{2}$$

2.2.1 *Taylor Series and numerical differentiation*

(Basically differentiate polynomial interpolants)

$$\begin{aligned}
y_{n+1} &= y_n + hDy_n + h^2\frac{D^2}{2!}y_n + \ldots = \left[1 + hD + \frac{h^2D^2}{2!} + \ldots\right]y_n = e^{hD}y_n \\
\therefore E &= e^{hD} \\
hD &= \ln E = \ln(1+\Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 + \ldots \\
&= -\ln(1-\nabla) = \nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \ldots \\
\delta y_n &= \left(E^{1/2} - E^{-1/2}\right)y_n \\
&= 2\left\{\frac{e^{hD/2} - e^{-hD/2}}{2}\right\}y_n = 2\sinh\left(\frac{hD}{2}\right)y_n \\
&\Rightarrow hD = 2\sinh^{-1}\left(\frac{\delta}{2}\right) = 2\left\{\left(\frac{\delta}{2}\right) - \frac{1}{2^3 3}\delta^3 + \ldots\right\}
\end{aligned}$$

$$\therefore hDy_n \approx \left\{\begin{array}{ll} \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 + \ldots & \\ \nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \ldots & \\ \quad\mu\delta & -\frac{1}{3}\mu\delta^3 + \ldots \end{array}\right\} y_n$$

$$h^2D^2y_n \approx \left\{\begin{array}{ll} \Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 + \ldots & \\ \nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \ldots & \\ \quad\delta^2 & -\frac{1}{12}\delta^4 + \end{array}\right\} y_n$$

$$h^3D^3y_n \approx \left\{\begin{array}{ll} \Delta^3 - \frac{3}{2}\Delta^4 + \frac{7}{4}\Delta^5 - \ldots & \\ \nabla^3 + \frac{3}{2}\nabla^4 + \frac{7}{4}\nabla^5 + \ldots & \\ \quad\mu\delta^3 & -\frac{1}{4}\mu\delta^5 + \ldots \end{array}\right.$$

$$h^4D^4y_n \approx \left\{\begin{array}{ll} \Delta^4 - 2\Delta^5 + \frac{17}{6}\Delta^6 - \ldots & \\ \nabla^4 + 2\nabla^5 + \frac{17}{6}\Delta^6 + \ldots & \\ \quad\delta^4 & -\frac{1}{6}\delta^6 + \end{array}\right.$$

To determine the error terms involved in finite difference approximations consider the following expansions.

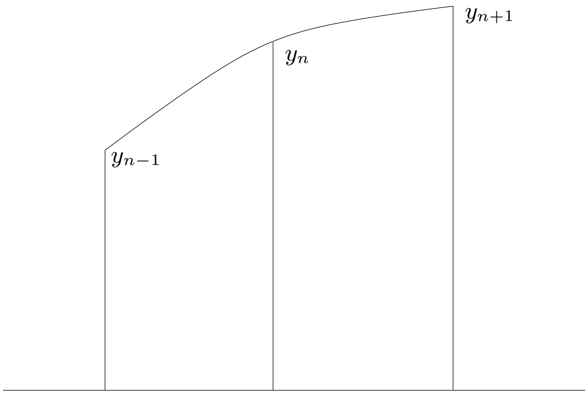
$$\begin{aligned}
y_{n\pm1} &= y_n \pm hy_n' + \frac{h^2}{2}y_n'' + \ldots \\
y_{n+2} &= y_n + 2hy_n' + 2h^2y_n'' + \ldots
\end{aligned}$$

Discretization or Truncation Error

Eg. $hDy_n \approx \Delta y_n = y_{n+1} - y_n = hy'_n + \frac{h^2}{2}y''(\xi_1)$ $\quad 0(h)$ forward

$\approx \nabla y_n \quad y_n - y_{n-1} = hy'_n - \frac{h^2}{2}y''(\xi_2)$ $\quad 0(h)$backward

$\approx \mu\delta y_n = \frac{y_{n+1}-y_{n-1}}{2} = hy'_n + \frac{h^3}{6}y^{(3)}(\xi_3)$ $\quad 0(h^2)$central



Similarly:

$$
\begin{aligned}
h^2D^2y_n \approx \Delta^2 y_n &= y_{n+2} - 2y_{n+1} + y_n \\
&= \cancel{y_n} + \cancel{2h}y'_n + 2h^2y''_n + \frac{8h^3}{6}y^{(3)} + \ldots \\
&\quad -2\left\{\cancel{y_n} + \cancel{h}y'_n + \frac{h^2}{2}y''_n + \frac{h^3}{6}y^{(3)} + \ldots\right. \\
&\quad +\cancel{y_n} \\
&= h^2y''_n + h^3y^{(3)}(\xi) \qquad 0(h) \\
\approx \delta^2 y_n &= y_{n+1} - 2y_n + y_{n-1} \\
&= h^2y''_n + \frac{h^4}{12}y^{(4)}(\xi) \qquad 0(h^2)
\end{aligned}
$$