Lecture 3: The Runge Phenomenon and Piecewise Polynomial Interpolation

(*Compiled 16 August 2017*)

In this lecture we consider the dangers of high degree polynomial interpolation and the spurious oscillations that can occur - as is illustrated by Runge's classic example. We discuss the remedies for this, including: optimal distribution of sample points at the zeros of the Chebyshev polynomials; and piecewise polynomial interpolation in which the oscillations are limited by restricting the degree of the interpolating polynomials that are applied on subintervals of the domain and stitched together.

Key Concepts: The Runge Phenomenon, Approximation by Chebyshev Polynomials, Piecewise polynomial Interpolation.

3 High order polynomial Interpolation and Piecewise Polynomial Interpolation

3.1 The Runge Phenomenon

There can be problems with high degree polynomial interpolants particularly in the neighborhood of singularities of the function $f(x)$ as is illustrated by this classic example due to Runge. Consider the polynomial interpolant that passes through the function $f(x) = \dfrac{1}{1+25x^2}$ at $n = 11$ equidistant points on the interval $[-1, 1]$.

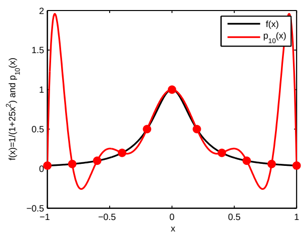


FIGURE 1. Plot of $f(x) = 1/(1 + 25x^2)$ and its polynomial interpolant through 11 equally spaced points

Note the oscillations in the interpolant which renders it basically useless for interpolation, as an approximation for the derivative, or for the purposes of numerical integration.

Solutions to the problem of interpolating over many points.

- Smooth the wrinkles in the interpolating polynomial by *fitting* a lower degree polynomial – no longer interpolation.
- Restrict ourselves to a string of lower degree polynomials each of which are only applied over one or two subintervals—*use piecewise polynomial interpolation.*
- Choose the interpolation points more judiciously.

3.2 Chebyshev interpolation- Minimax Optimization

Question: Is it possible to choose the interpolation points $\{x_i\}_{i=0}^N$ so that the maximum absolute error (i.e. $||e_n(x)||_\infty$) is minimized?

Recall: $e_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\dots(x-x_n) \qquad \xi \in (a,b)$.

For convenience we consider the interval $[-1,1]$. There is no loss of generality in this assumption as the transformation $x = \frac{t(b-a)+(a+b)}{2}$ can be used to transform the problem $x \in [a,b]$ into one in which the independent variable is $t \in [-1,1]$.

Important Properties of the Chebyshev Polynomials:

(1) **Definition** Let $z = e^{i\theta}$ be a point on the unit circle. The associated x coordinate is $x = cos\theta$ or $\theta = \cos^{-1} x$ where $x \in [-1,1]$. Define the nth degree Chebyshev polynomial to be $T_n(x) = \cos n\theta$. Thus $T_0(x) = Re(z^0) = cos0 = 1$, $T_1(x) = Re(z^1) = cos\theta = (z+z^{-1})/2 = x$, $T_2(x) = Re(z^2) = cos2\theta = (z^2+z^{-2})/2 = \frac{1}{2}(z+z^{-1})^2 - 1 = 2x^2 - 1, \dots$.

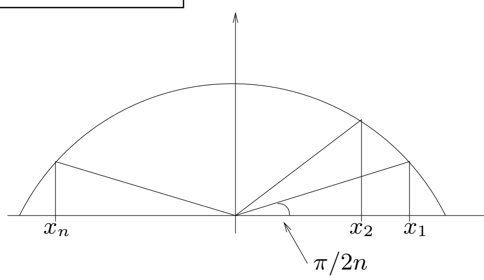
(2) **Recursion**: The identity $\cos n\theta = 2\cos\theta\cos(n-1)\theta - \cos(n-2)\theta$ implies the recursion $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$.

Starting with $T_0(x) = 1$ and $T_1(x) = x$ the recursion yields $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x, \dots$. Note that the leading coefficient of $T_n(x)$ is 2^{n-1}.

(3) **Orthogonality**: $\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_m(x)T_n(x)dx = \delta_{mn}(\pi/2)$.

(4) **Max/Min values and roots**:

- **Roots**: $T_n(x) = \cos n\theta = 0$ when $n\theta = (2k+1)\frac{\pi}{2} \quad k = 0, \dots, n-1 \Rightarrow \boxed{x_k = \cos\left[\frac{(2k+1)\pi}{2n}\right]}$



- **Max/Min**: There are $n-1$ extrema between the n roots (Rollé's Theorem). In addition

$$T_n(-1) = \cos n\left(\cos^{-1}(-1)\right) = \cos(n\pi) = (-1)^n$$
$$T_n(+1) = \cos n\left(\cos^{-1}(1)\right) = \cos(n2\pi) = 1$$

$\therefore T_n(x)$ has $n+1$ extreme values on $[-1,+1]$ which are either -1 or $+1$.

In order to minimize the maximum absolute error $\max_{x\in[-1,1]} |f(x) - p_n(x)|$ we must choose the $\{x_i\}$ so that

$$\max_{x\in[-1,1]} |(x-x_0)\dots(x-x_n)|$$

is minimized since we have no control over the term $\frac{f^{(n+1)}(\xi)}{(n+1)!}$ which may be regarded as a constant for our purposes. If we choose $x_0, x_1, \dots, x_n$ to be the zeros of $T_{n+1}(x)$ then

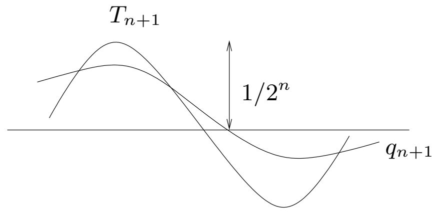
$$(x-x_0)(x-x_1)\dots(x-x_n) = \frac{T_{n+1}(x)}{2^n} : \text{ where } \mathrm{x_k} = \cos\left[\frac{(2\mathrm{k}+1)\pi}{2(\mathrm{n}+1)}\right] \quad \mathrm{k} = 0, 1, \dots, \mathrm{n}.$$

Claim: $\frac{T_{n+1}(x)}{2^n}$ is the polynomial of degree $(n+1)$ with leading coefficient 1 that has the smallest $||\cdot||_\infty$ value over the interval $[-1,1]$.

Proof: Assume q_{n+1} is a polynomial of degree $n+1$ with leading coefficient 1 that achieves a lower $||\cdot||_\infty$ norm, i.e. $||q_{n+1}||_\infty \leq ||T_{n+1}||_\infty$.

Now $||T_{n+1}/2^n||_\infty = 1/2^n$ is achieved $n+2$ times within $[-1,1]$. By definition $|q_{n+1}(x)| < 1/2^n$ at each of the $n+2$ extreme points.

Thus $D(x) = \frac{T_{n+1}}{2^n} - q_{n+1}$ is a polynomial of degree $\leq n$ and has the same sign as T_{n+1} at each of the $n+2$ extreme points.



$\Rightarrow D(x)$ must change sign $n+1$ times on $[-1,1]$ which is impossible for a polynomial of degree $\leq n$.$\Rightarrow$ contradiction.

■

Conclusions:

(1) If we choose the $\{x_k\}$ to be the Chebyschev points then $||f(x) - p_n(x)||_\infty$ is the smallest for all polynomials of degree n.

(2) In the Chebyschev case the error is more uniformly distributed over the interval $[-1,1]$ than for any other polynomial.

(3) **Spectral convergence:** If the $(N+1)$ sample points for the interpolation polynomial $p_N(x)$ are chosen at the roots of the Chebyshev polynomials $x_k = \cos\left[\frac{(2k+1)\pi}{2N}\right]$, then

$$\begin{aligned} e_N(x) &= f(x) - p_N(x) \\ &= \frac{f^{(N+1)}(\xi)}{(N+1)!}(x-x_0)\cdots(x-x_N) \\ &= \frac{f^{(N+1)}(\xi)}{(N+1)!}\frac{T_{N+1}(x)}{2^N} \end{aligned}$$

Thus taking the absolute value of both sides

$$\begin{aligned} |e_N(x)| &= \left|\frac{f^{(N+1)}(\xi)}{(N+1)!}\frac{T_{N+1}(x)}{2^N}\right| \\ &\leq \frac{\left\|f^{(N+1)}\right\|_\infty}{(N+1)!}\frac{|T_{N+1}(x)|}{2^N} \\ &\leq \frac{\left\|f^{(N+1)}\right\|_\infty}{2^N(N+1)!} \end{aligned}$$

Thus the error decreases exponentially with N - a property known as *spectral convergence.*

3.3 Piecewise polynomial interpolation

Idea: Limit the oscillations of high degree polynomials by stringing together lower degree polynomial interpolants.

3.3.1 *Piecewise linear interpolation*

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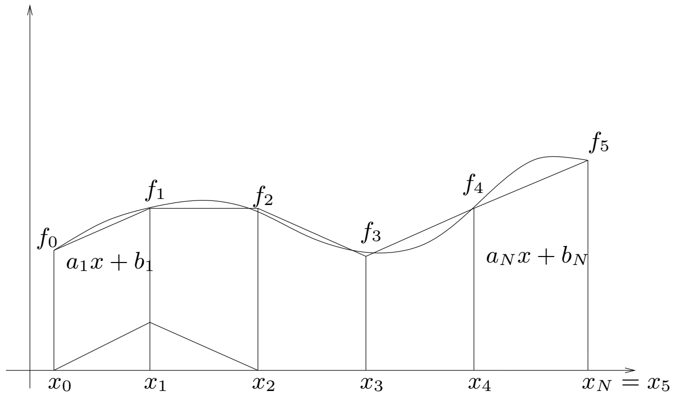


Degree of freedom analysis:

$$\left.\begin{array}{ll} N \text{ intervals} & \\ a_ix+b_i & \text{2 coefficients for interval} \end{array}\right\} \Rightarrow 2N \text{ unknowns}$$

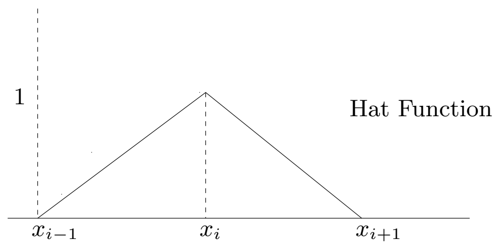
Impose continuity between interior nodes $\Rightarrow N-1$ constraints

$$2N-(N-1) = N+1 \quad \begin{array}{l}\text{degrees of freedom which can be} \\ \text{determined by specifying } f \text{ as } N+1 \text{ points } x_0,\ldots,x_n.\end{array}$$

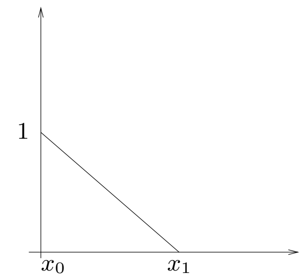
Convenient Lagrange basis function representation of the PWL interpolants of f:

Let

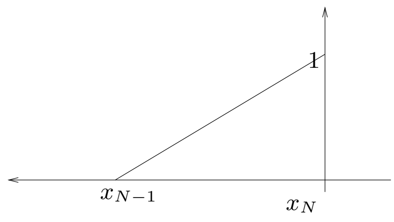
$$N_i^1(x) = \begin{cases} \left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right) & x \in [x_{i-1}, x_i] \\ \left(\frac{x_{i+1}-x}{x_{i+1}-x_i}\right) & x \in [x_i, x_{i+1}] \\ 0 & x \notin [x_{i-1}, x_{i+1}] \end{cases}$$



$$N_0^1(x) = \begin{cases} \left(\frac{x_1-x}{x_1-x_0}\right) & x \in [x_0, x_1] \\ 0 & x \notin [x_0, x_1] \end{cases}$$



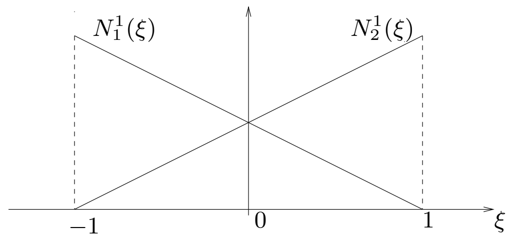
$$N_n^1(x) = \begin{cases} 0 & x \notin [x_{N-1}, x_N] \\ \left(\frac{x-x_{N-1}}{x_N-x_{N-1}}\right) & x \in [x_{N-1}, x_N] \end{cases}$$



Then $p_{i,N}(x) = \sum_{i=0}^{N} f_i N_i^1(x) \approx f(x)$. We notice that $N_i^1(x_j) = \delta_{ij}$ so that the basis functions are zero outside the interval (x_{i-1}, x_{i+1}) – we say that such *basis functions have local support.*

Representation on a canonical interval:

Sometimes it is more convenient to perform calculations by representing the piecewise linear basis functions on a canonical interval: $[-1, 1]$. On the canonical interval the basis functions assume the form.:



Canonical Interval

$$N_1^1(\xi) = \frac{1}{2}(1-\xi) \qquad N_2^1(\xi) = \frac{1}{2}(1+\xi)$$

$$\text{or} \quad N_a^1(\xi) = \frac{1}{2}(1+\xi_a\xi) \qquad \xi_1 = -1 \qquad \xi_2 = +1$$

Note: $N_a^1(\xi_b) = \delta_{ab}$ and $x(\xi) = \sum_{a=1}^{2} x_a N_a^1(\xi)$

Error involved: Recall $e_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x - x_j)$ for polynomial interpolants.

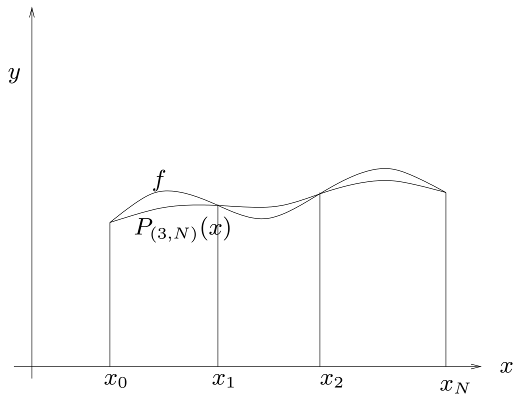
$$\max_{x\in[x_i,x_{i+1}]} |e_1(x)|$$
$$= \max_{x\in[x_i,x_{i+1}]} \left| \frac{f''(s)}{2}(x-x_i)(x-x_{i+1}) \right| \leq \frac{1}{2}||f''||_\infty \max_{x\in[x_i,x_{i+1}]} |(x-x_i)(x-x_{i+1})| \leq \frac{h^2}{8}||f''||_\infty.$$

Using:

$$w(x) = (x-x_i)(x-x_{i+1}) = x^2 - (x_i + x_{i+1})x + x_i x_{i+1}$$

$$w'(x) = 2x - (x_i + x_{i+1}) = 0 \Rightarrow \quad x = \frac{(x_i + x_{i+1})}{2} \quad \& \quad w\left(\frac{x_i + x_{i+1}}{2}\right) = \left(\frac{x_{i+1} - x_i}{2}\right)\left(\frac{x_i - x_{i+1}}{2}\right)$$

3.3.2 *Piecewise quadratic interpolation*



Degree of freedom analysis:

$$N \text{ Subintervals} \qquad 3 \text{ coefficients for quadratic}$$
$$3N \text{ DOF}$$

Constraints

$$(1) \quad \left.\begin{array}{l}\text{Continuity at interior points}\\ \text{Continuity of derivative at interior points}\end{array}\right\} \Rightarrow 2(N-1) \text{ constraints}$$

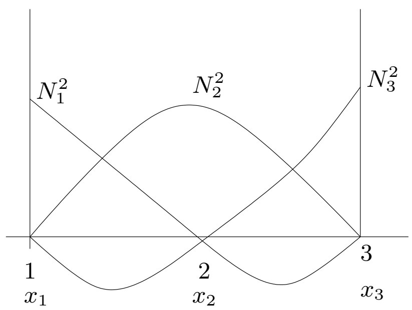
$$\begin{aligned}\text{Remaining DOF } &= 3N - 2(N-1) = N + 2 = (N+1) + 1\\ &= \text{function values at } N+1 \text{ nodes}\\ &\quad \text{and 1 extra condition (?)}\end{aligned}$$

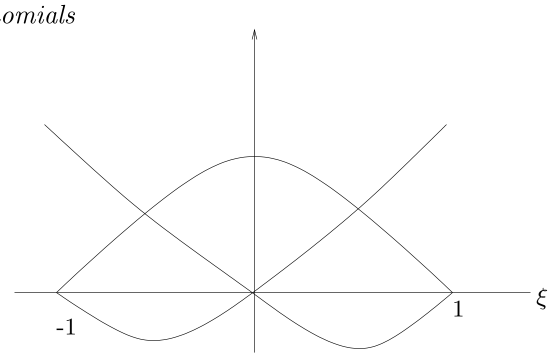
(2) Continuity at interior nodes $\Rightarrow N - 1$

$$\begin{aligned}\text{Remaining DOF } &= 3N - (N-1) = 2N + 1 = (N+1) + N\\ &= \text{ function values at } (N+1) \text{ nodes}\\ &\quad +1 \text{ function value within each interval}\end{aligned}$$

These are called *quadratic Lagrange interpolants*.

Lagrange basis function representation for piecewise quadratic polynomials





On Canonical Interval

$$N_1^2(x) = \frac{(x-x_2)}{(x_1-x_2)}\frac{(x-x_3)}{(x_1-x_3)} \qquad N_1(\xi) = \frac{1}{2}\xi(\xi-1)$$

$$N_2^2(x) = \frac{(x-x_1)}{(x_2-x_1)}\frac{(x-x_3)}{(x_2-x_3)} \qquad N_2(\xi) = 1 - \xi^2$$

$$N_3^2(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} \qquad N_3(\xi) = \frac{1}{2}\xi(\xi+1)$$

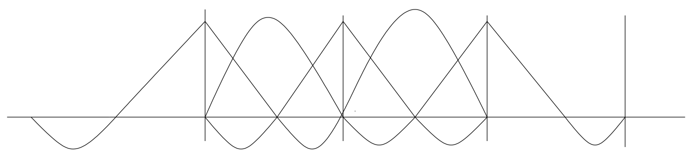
Note:

(1) $N_i^2(x_j) = \delta_{ij}$ • On a canonical interval $[-1, 1]$.

(2) $\sum_{i=1}^{N} N_i(x) = 1$ $\quad N_1 + N_2 + N_3 = \frac{1}{2}\cancel{\xi^2} - \frac{1}{2}\cancel{\xi} + 1 - \cancel{\xi^2} + \frac{1}{2}\cancel{\xi^2} + \frac{1}{2}\cancel{\xi} = 1$

Must be true as we must be able to represent a constant function exactly. We can now obtain a global representation of the interpolants by numbering all the basis functions:

$$f(x) \sim \sum_{i=0}^{n} f_i N_i^2(x)$$



3.4 Finite elements in 2D

Interpolation in 2D:

$$u^h(x,y) = \sum_{i=1}^{N} N_i(x,y)u_i \qquad\qquad N_i(x_j,y_i) = \delta_{ij}$$

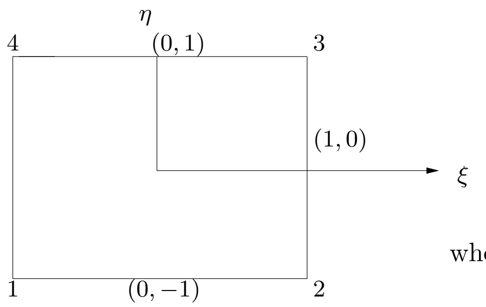
How do we construct basis functions? Can we still map onto canonical elements?

ISOPARAMETRIC ELEMENTS (most commonly used)

IDEA 1: Same basis functions are used to transform from canonical elements to actual elements in the mesh as those that are used to represent the unknown solution.

IDEA 2: Use products of 1D basis functions to construct 2D basis functions.

1. Bilinear elements:



$$\mathbf{x}(\xi) = \sum_{a=1}^{4} \mathbf{x}_a N_a(\boldsymbol{\xi}) \qquad \text{where } N_a(\boldsymbol{\xi}_b) = \delta_{ab} \qquad \begin{array}{c}\boldsymbol{\xi} = (\xi,\eta)\\ \mathbf{x}_a = (x_a, y_a)\end{array}$$

$$u^h(\boldsymbol{\xi}) = \sum_{a=1}^{4} u_a N_a(\boldsymbol{\xi})$$

where

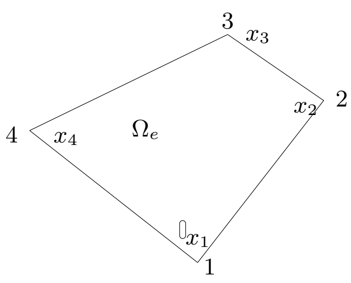
$$N_a(\boldsymbol{\xi}) = \frac{1}{4}(1+\xi^a\xi)(1+\eta^a\eta) = N_a(\xi)N_a(\eta)$$

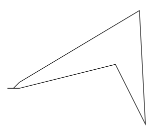
$$N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta) = N_1(\xi)N_1(\eta)$$

$$N_2(\xi,\eta) = N_2(\xi)N_1(\eta) = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3(\xi,\eta) = N_2(\xi)N_2(\eta) = \frac{1}{4}(1+\xi)(1+\eta)$$

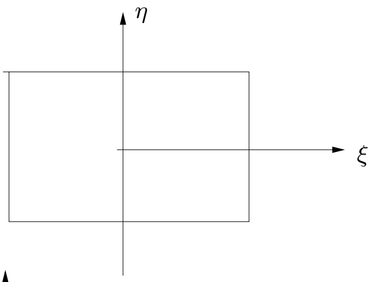
$$N_4(\xi,\eta) = N_1(\xi)N_2(\eta) = \frac{1}{4}(1-\xi)(1+\eta)$$



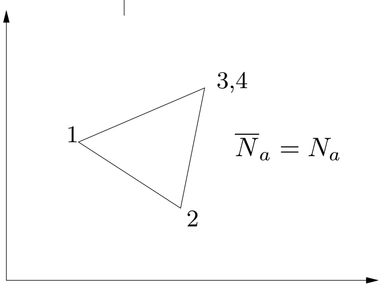


AVOID BAD DISTORTIONS

2. Triangle as a degenerate rectangle:



$$\mathbf{x} = \sum_{a=1}^{4} N_a \mathbf{x}^a = N_1\mathbf{x}_1 + N_2\mathbf{x}_2 + \{N_3 + N_4\}\mathbf{x}_3$$
$$= \sum_{a=1}^{3} \overline{N}_{\dot{a}}(\boldsymbol{\xi}) x_a$$



where

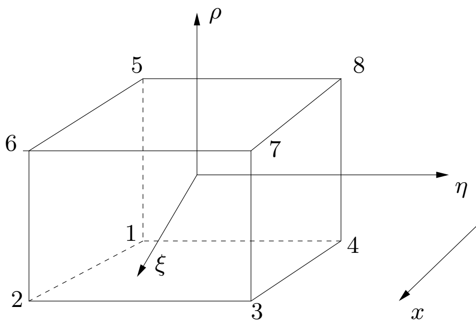
$$\overline{N}_a = N_a \qquad a = 1, 2,$$

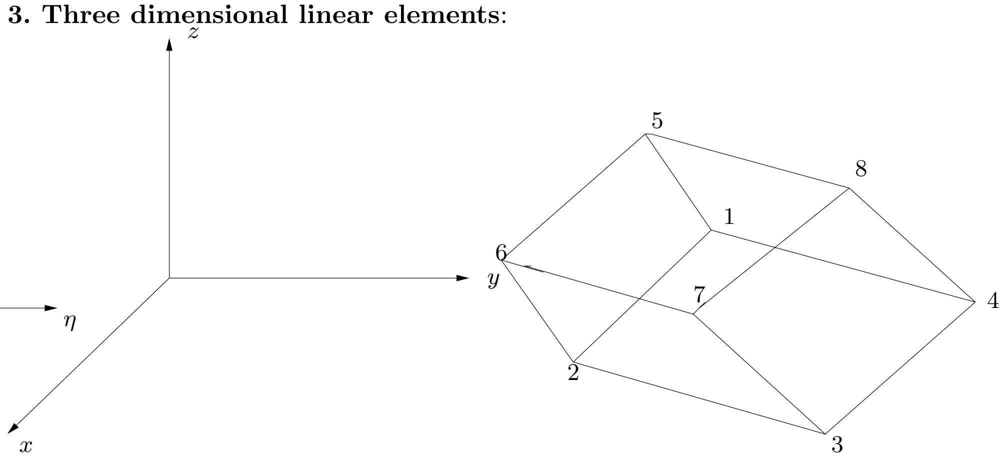
$$\begin{aligned}\overline{N}_3(\boldsymbol{\xi}) &= N_3(\xi,\eta) + N_4(\xi,\eta)\\ &= \frac{1}{4}(1+\xi)(1+\eta) + \frac{1}{4}(1-\xi)(1+\eta)\\ &= \frac{1}{2}(1+\eta)\end{aligned}$$

(-) Not very good because derivatives can be piecewise constant.

(+) Triangular tessellations are very easy.

3. Three dimensional linear elements:



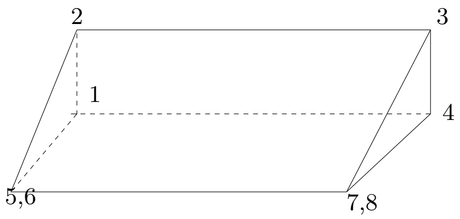


$$\mathbf{x} = \sum_{d=1}^{8} N_a(\boldsymbol{\xi})\mathbf{x}_a$$

$$u^h = \sum_{a=1}^{8} N_a(\xi) u_a$$

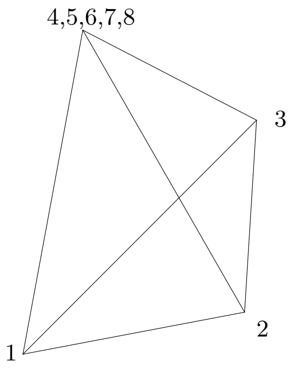
$$N_a(\xi) = N_a(\xi)N_a(\eta)N_a(\rho) = \frac{1}{8}(1+\xi_a\xi)(1+\eta_a\eta)(1+\rho_a\rho)$$

Wedge elements:



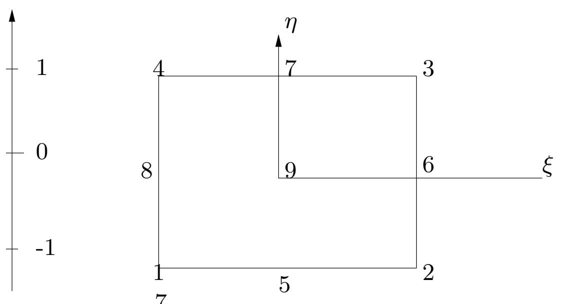
$$\overline{N}_5 = N_5 + N_6$$
$$\overline{N}_6 = N_7 + N_8$$

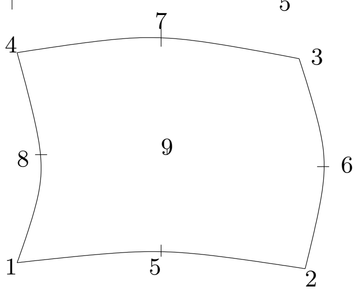
Tetrahedral elements:



$$\overline{N}_4 = N_4 + \ldots + N_8$$

4. Biquadratic elements:

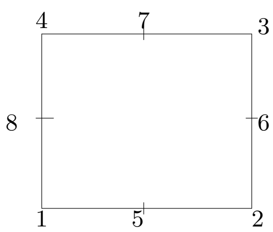




9-node Lagrange ELT.

$$N_1^2(\xi,\eta) = N_1^2(\xi)N_1^2(\eta) = \frac{1}{4}\xi(\xi-1)\eta(\eta-1)$$
$$N_5^2(\xi,\eta) = N_2^2(\xi)N_1^2(\eta)$$
$$N_9^2(\xi,\eta) = N_2^2(\xi)N_2^2(\eta)$$

8-Node serendipity element:



$$N_a^s(\xi,\eta) = \frac{1}{4}(1+\xi_a\xi)(1+\eta_a\eta)(\xi_a\xi+\eta_a\eta-1) \qquad a = 1,2,3,4$$
$$N_a^s(\xi,\eta) = \frac{\xi_a^2}{2}(1+\xi_a\xi)(1-\eta^2) + \frac{\eta_a^2}{2}(1+\eta_a\eta)(1-\xi^2) \qquad a = 5,6,7,8$$