Hydraulic Fractures: multiscale phenomena, asymptotic and numerical solutions

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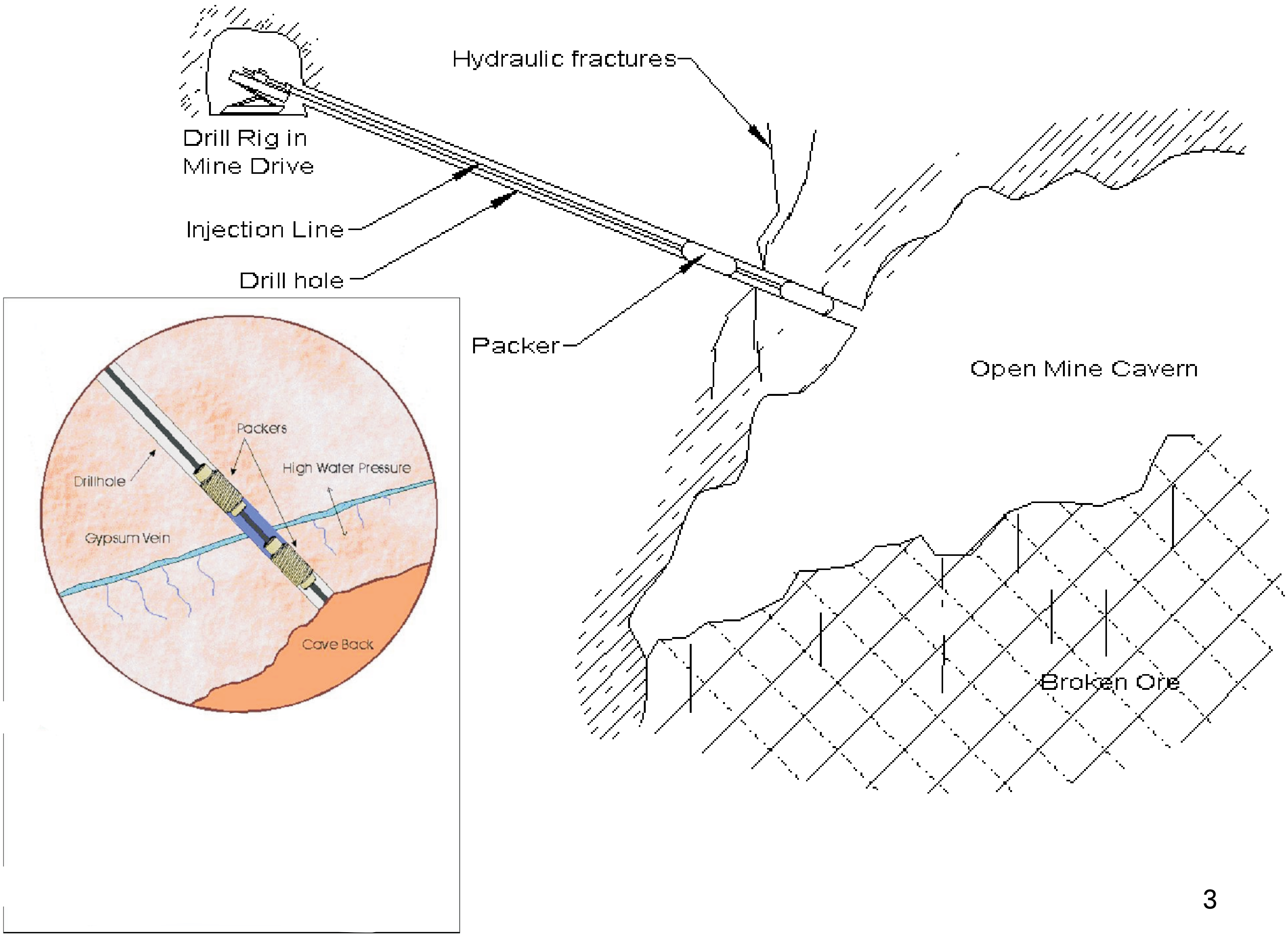
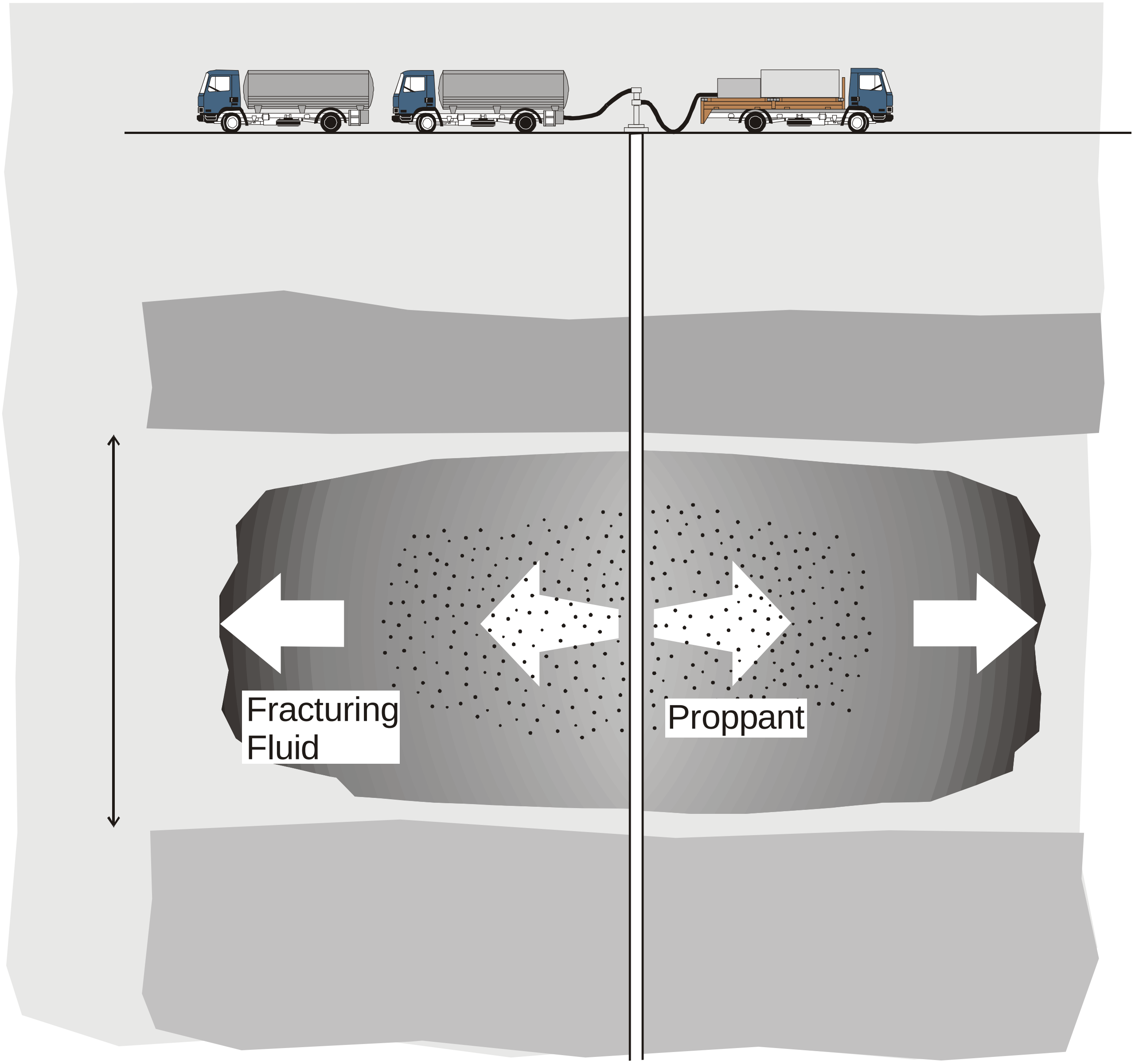
Outline

- Examples of hydraulic fractures
- Governing equations, scaling, asymptotic solutions 1-2D and 2-3D
- Solving The coupled Equations
  - MultiGrid Approach
  - Incomplete LU Factorization of Localized Jacobian
- Solving the free boundary problem
  - Existing methods: VOF, Level Set and K_I matching
  - Tip asymptote and Eikonal boundary value problem
  - Setting the tip volumes using the tip asymptotes
  - Coupled equations
- Numerical examples
  - M-vertex and K-vertex
  - Viscous crack propagating in a variable *in situ* stress
  - Stress jump solutions

HF Examples - block caving



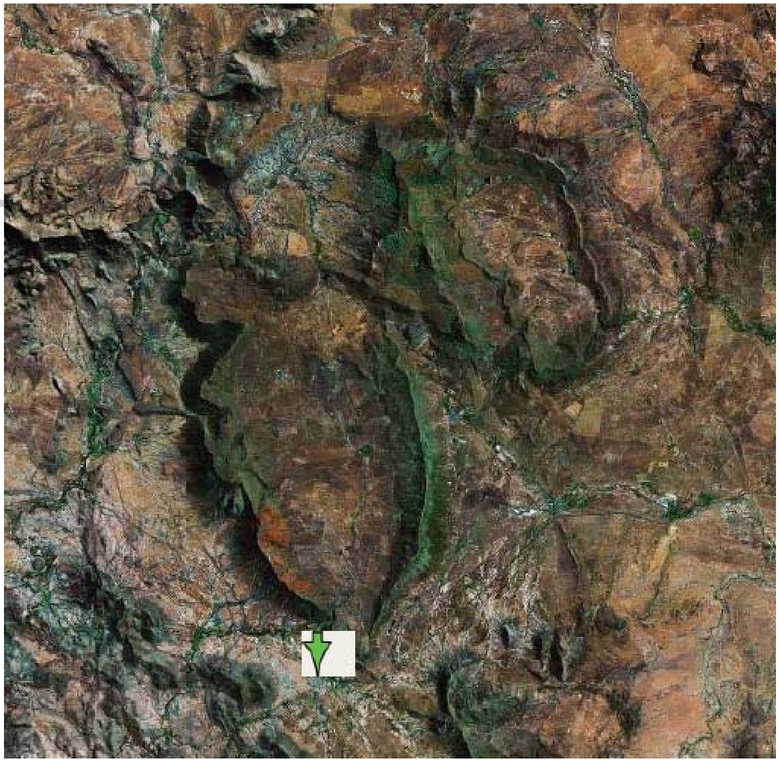
Oil well stimulation



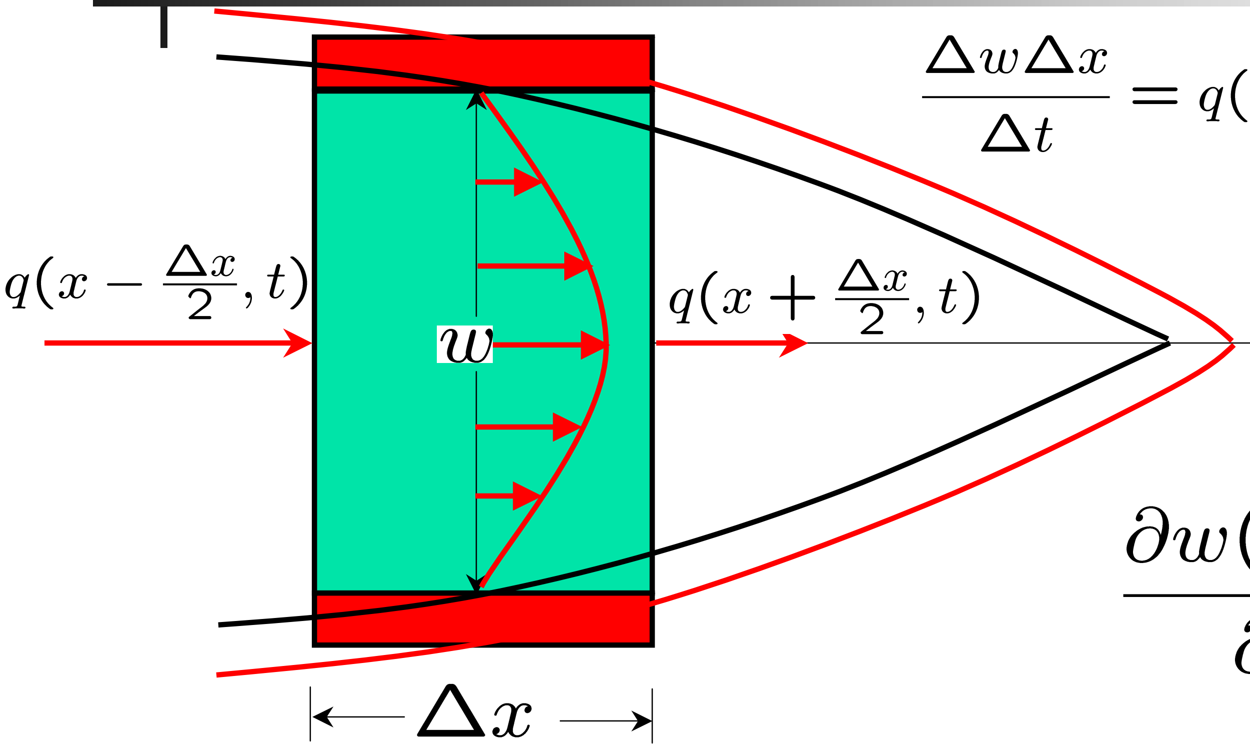
Quarries

Magma flow


Windhoek
Walvisbaai
Botswana
Selebi-Phikwe
Messina
Pietersburg
Inhambane
Gaborone
Pretória
Witbank
Barberton
Maputo
Johannesburg
Ermelo
Swaziland
Lüderitz
Vryburg
Orkney
Sasolburg
Standerton
Kroonstad
Newcastle
Odendaalsrus
Harrismith
Dundee
Upington
Kimberley
Bloemfontein
Lesotho
Pietermaritzburg
South Africa
Middelburg
Tarkastad
Beaufort West
Cradock
Bisho
East London
Paarl
Oudtshoorn
Grahamstown
Robertson
George
Cape Town
Port Elizabeth



Model EQ 1: Conservation of mass



$$\frac{\Delta w \Delta x}{\Delta t} = q(x - \frac{\Delta x}{2}, t) - q(x + \frac{\Delta x}{2}, t)$$

$$\Downarrow \lim_{\Delta x,\ \Delta t \to 0}$$

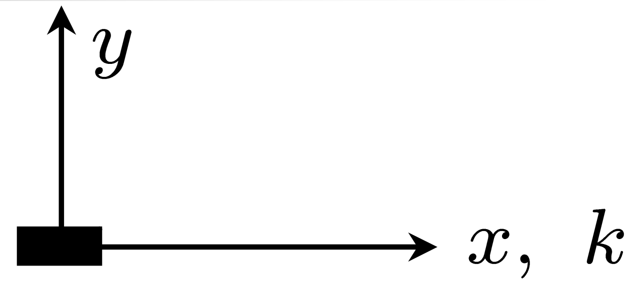
$$\frac{\partial w(x,t)}{\partial t} = -\frac{\partial q(x,t)}{\partial x}$$

$$v_x = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left(\frac{w^2}{4} - y^2\right)$$

$$q(x,t) = -\frac{w^3}{12\mu}\frac{\partial p}{\partial x}$$

$$\boxed{\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(\frac{w^3}{\mu'}\frac{\partial p}{\partial x}\right)}$$

Model EQ 2: The elasticity equation



$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2G\epsilon_{ij}$$

$$\sigma_{ij,j} + f_i = 0$$

$$[\sigma_{yy}] = 0, \;\; [\sigma_{xy}] = 0, \;\; [u_y] = \delta(x), \;\; [u_x] = 0$$

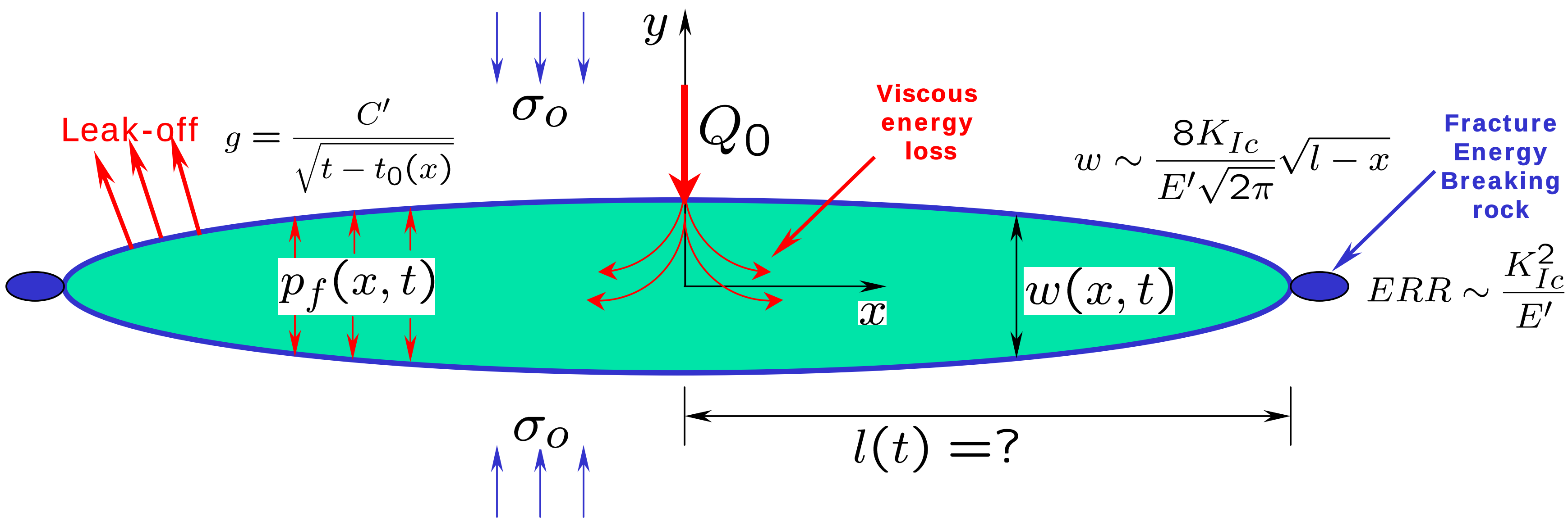
$$\overset{FT_x}{\Rightarrow} \quad \hat{\sigma}_{yy} = -\frac{E'}{4}(k^2|y| + |k|)e^{-|ky|}$$

$$\sigma_{yy} = \frac{E'}{8\pi}\left[\frac{2}{r^2} + \frac{8y^2}{r^4} - \frac{16y^4}{r^6}\right]$$

$$\lim_{y \to 0} \quad \hat{\sigma}_{yy} = -\frac{E'}{4}|k| \quad \sigma_{yy} = \frac{E'}{4\pi}\left[\frac{1}{x^2}\right]$$

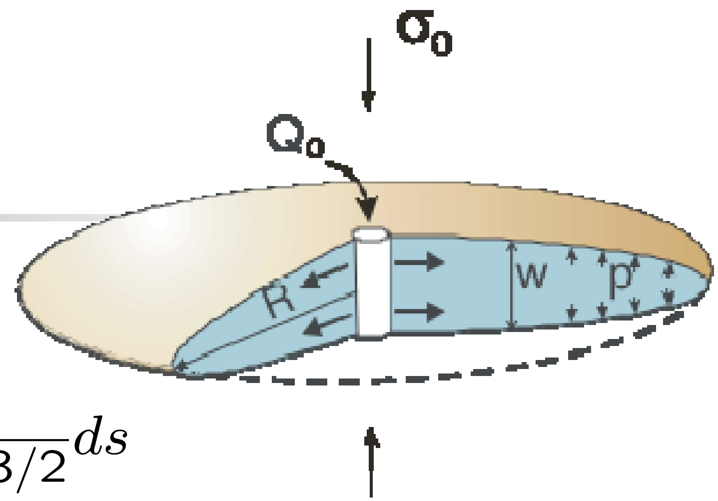
1-2D model and physical processes

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(\frac{w^3}{\mu'}\frac{\partial p_f}{\partial x}\right) + Q_0\delta(x) - g; \quad w^3\frac{\partial p_f}{\partial x}\Big]_{\pm l} = 0$$



$$p_f(x,t) - \sigma_o = -\frac{E'}{4\pi}\int_{-l(t)}^{l(t)} \frac{\partial w}{\partial \xi}\frac{d\xi}{\xi - x}; \quad w(\pm l, t) = 0$$

2-3D HF Equations



- Elasticity (non-locality)

$$p_f - \sigma_o = -\frac{E'}{8\pi}\int_{S(t)} \frac{w(x',y',t)}{\left[(x'-x)^2 + (y'-y)^2\right]^{3/2}} ds$$

$$\Rightarrow \quad p_f - \sigma_o = Cw$$

- Lubrication (non-linearity)

$$\frac{\partial w}{\partial t} = \nabla \cdot \left(\frac{w^3}{\mu'}\nabla p_f\right) - \frac{C'}{\sqrt{t - t_0(x,y)}} + Q(t)\delta(x,y)$$

$$\frac{\partial w}{\partial t} = A(w)Cw + S$$

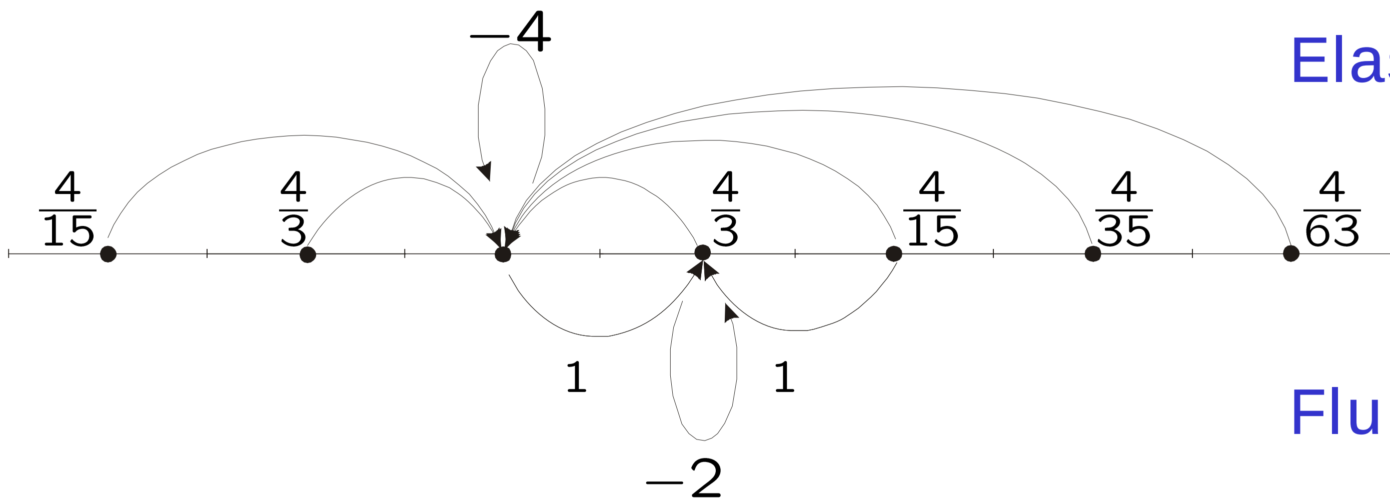
$$\Rightarrow \quad \frac{\Delta w}{\Delta t} = A(w)p_f + S$$

- Boundary conditions at moving front (free boundary)

$$\lim_{s\to 0} \frac{w}{s^{1/2}} = \frac{K'}{E'}, \qquad \lim_{s\to 0} w^3 \frac{\partial p_f}{\partial s} = 0 \qquad v = \lim_{s\to 0} -\frac{1}{\mu'} w^2 \nabla p$$

Coupled equations – a model problem

Elasticity



Fluid Flow

n-2 n-1 n n+1 n+2

$$\lambda \int_{-l}^{l} \frac{w(s,t)}{(s-x)^2} ds = p \xrightarrow{C} \frac{\lambda}{\Delta x}\sum_{n=1}^{N} \frac{w_n}{(m-n)^2-\frac{1}{4}} = p_m$$

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}(D(w)\frac{\partial p}{\partial x}) \xrightarrow{A} Ap_n = \frac{\bar{D}}{\Delta x^2}\left(p_{n+1} - 2p_n + p_{n-1}\right)$$

Conditioning of the Jacobian

- Evolution equation for w:

$$\left.\begin{aligned} p &= Cw \\ \frac{\partial w}{\partial t} &= A(w)p \end{aligned}\right\} \Rightarrow \frac{\partial w}{\partial t} = A(w)Cw$$

- Eigenvalues of AC:

$$\widehat{C}_k = \frac{\lambda}{\Delta x}\sum_{k=-\infty}^{\infty}\frac{e^{ikm\Delta x}}{m^2-\frac{1}{4}} = \frac{\lambda}{\Delta x}\left[2\pi\,\mathrm{sin}\left(\frac{|k|\Delta x}{2}\right)\right]$$

$$\widehat{A}_k = \frac{\bar{D}}{\Delta x^2}(e^{ik\Delta x}-2+e^{-ik\Delta x}) = -\frac{4\bar{D}}{\Delta x^2}\mathrm{sin}^2\left(\frac{k\Delta x}{2}\right)$$

Explicit Scheme

$$\widehat{A}_k\widehat{C}_k = -\frac{8\pi|\lambda|\bar{D}}{\Delta x^3}\mathrm{sin}^3\left(\frac{|k|\Delta x}{2}\right) \Rightarrow \Delta t < \frac{\Delta x^3}{8\pi|\lambda|\bar{D}}$$

Use implicit time stepping

$$(I-\Delta t AC)w = f$$

General first order iterative method

- Consider solving: $ACw = f$

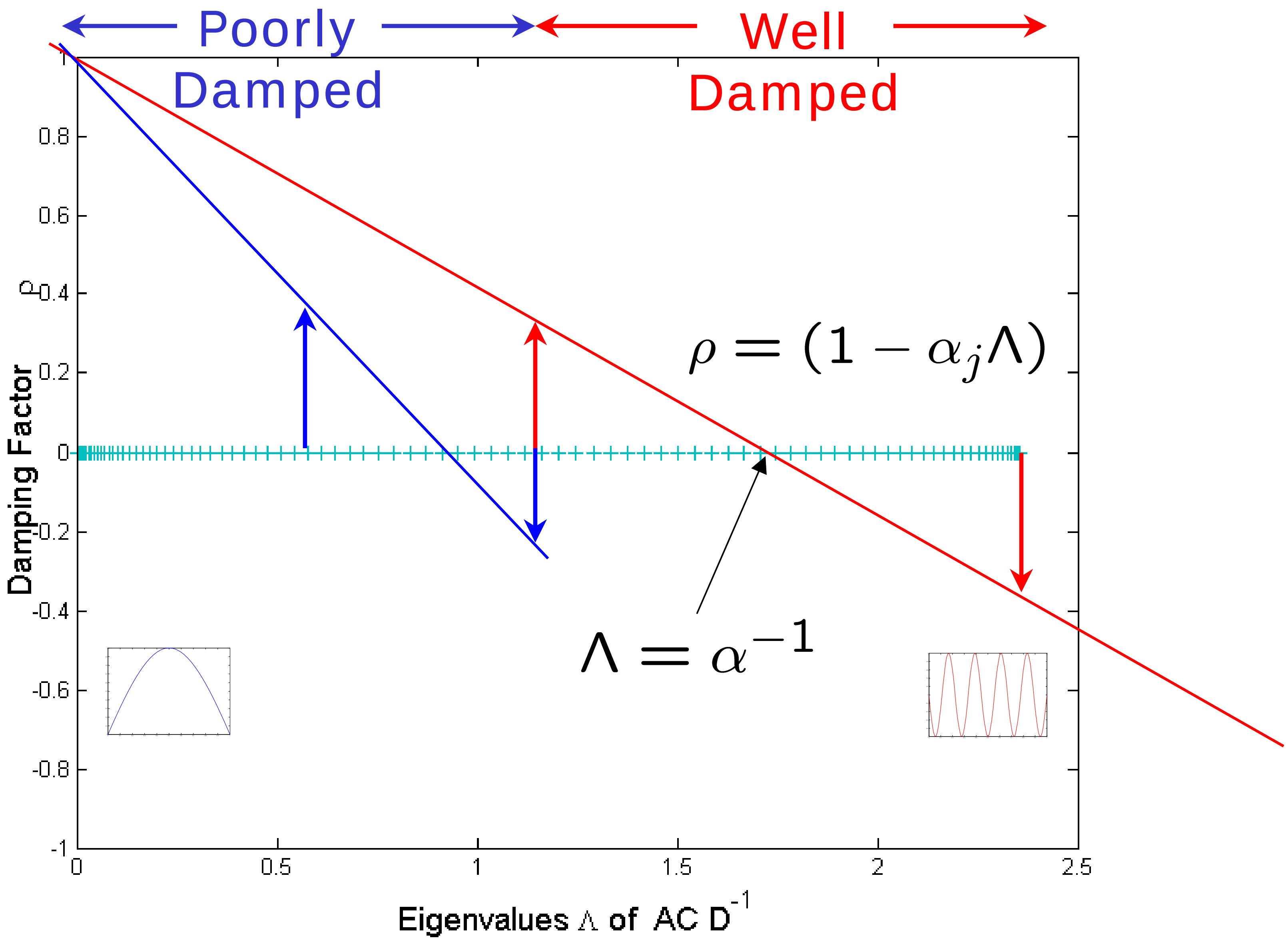
using the iterative method

$$w_{k+1} = w_k + \alpha B^{-1} \underbrace{(f - ACw_k)}_{r_k}$$

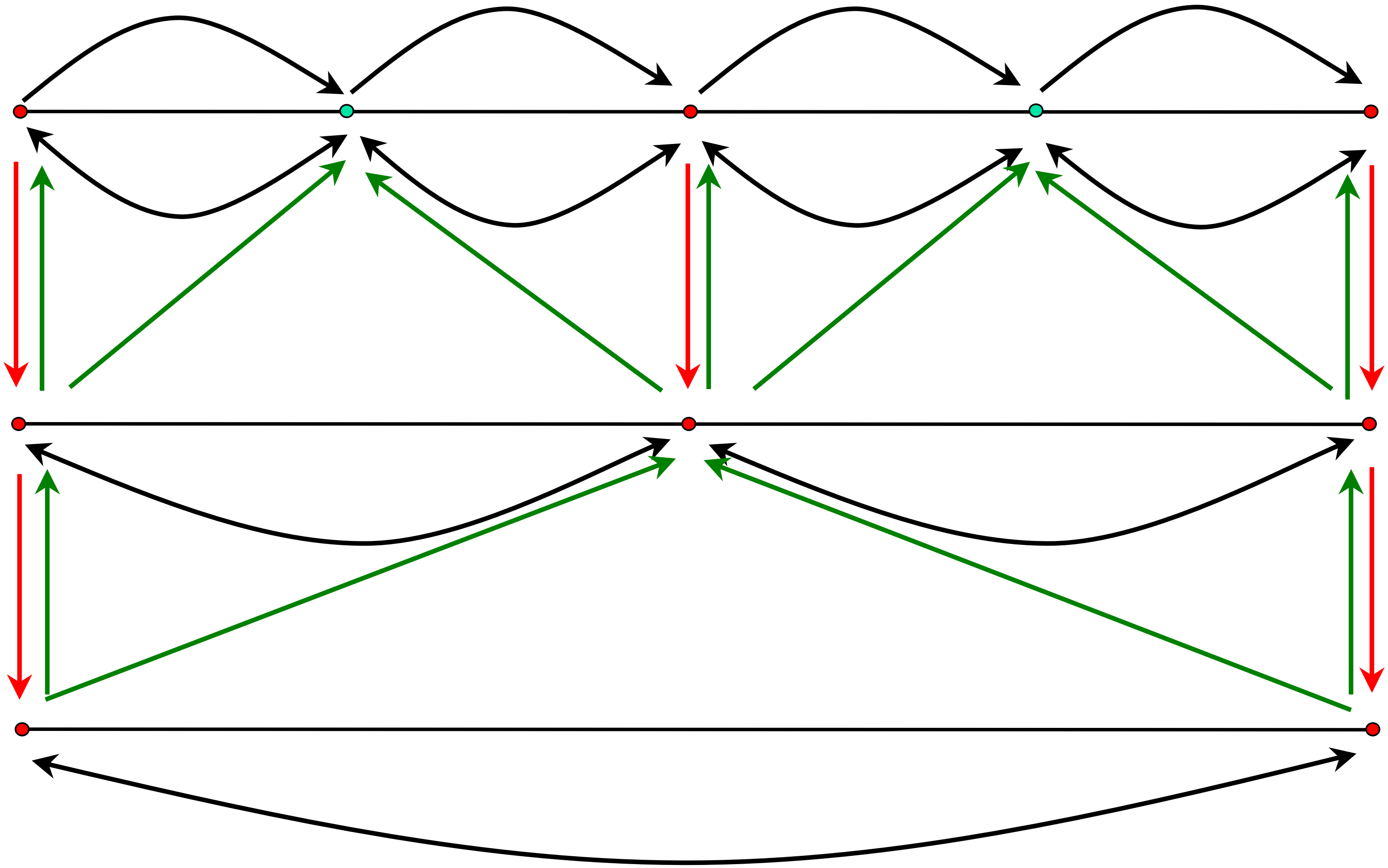
- Residual errors are damped according to

$$\begin{aligned} r_{k+1} &= f - ACw_{k+1} \\ &= f - AC(w_k + \alpha B^{-1} r_k) \\ &= \underbrace{(I - \alpha ACB^{-1})}_{\rho} r_k \end{aligned}$$

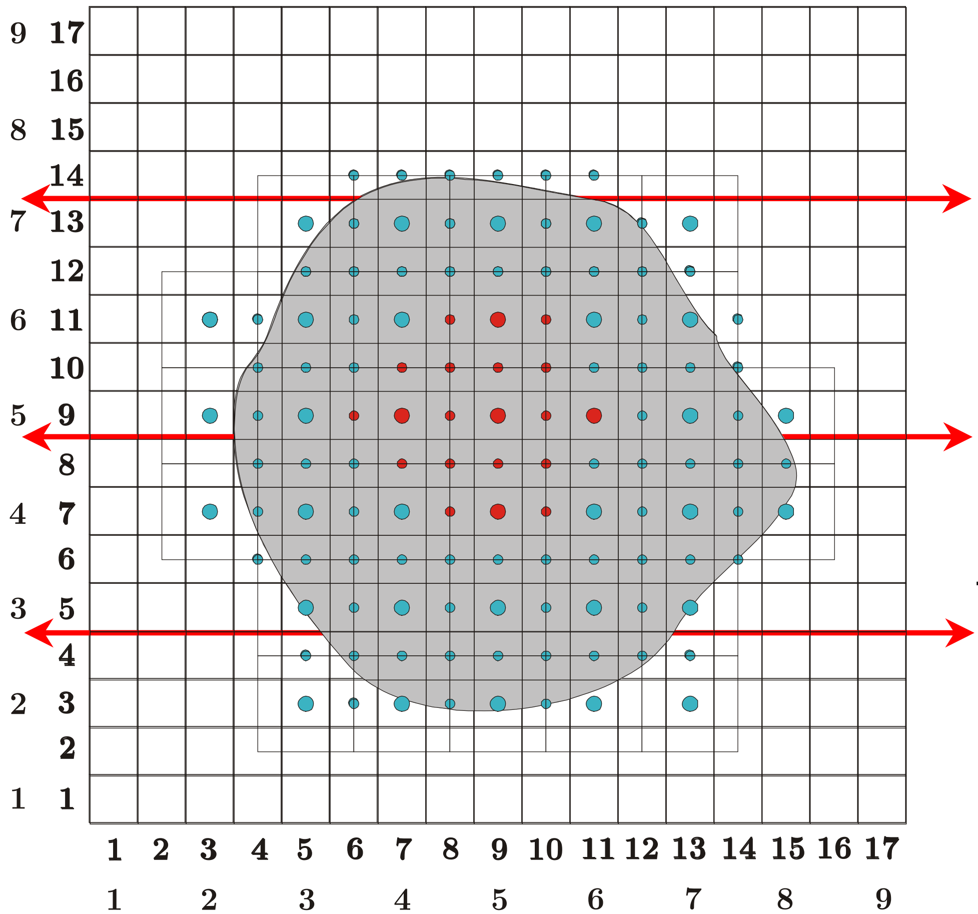
Damping factor



Multigrid Methods - an analogy



MG approach for coupled HF Equations



$$Cw = p; \quad w \geq w_c$$

$$+$$

$$\frac{\partial w}{\partial t} = A(w)p + S$$

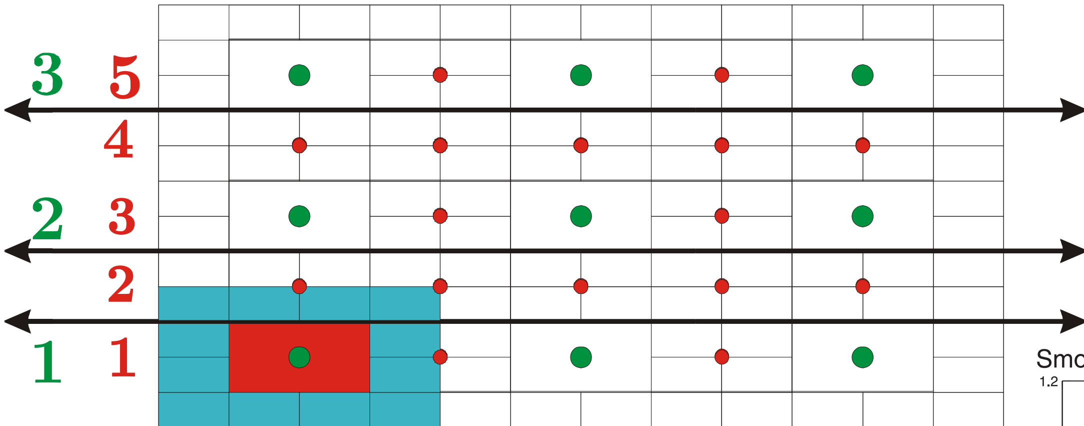
$$\Downarrow$$

$$\frac{\partial w}{\partial t} = A(w)Cw + S$$

$$(I - \Delta t AC)w = f$$

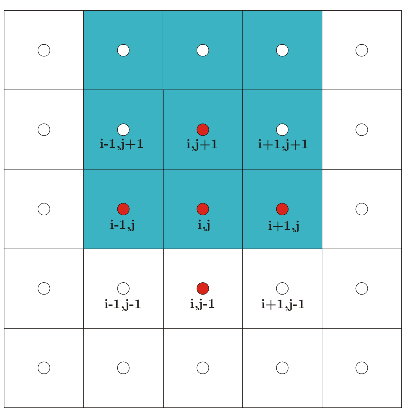
MG preconditioning of $\frac{\Delta w}{\Delta t} = A(w)Cw + S$

- C coarsening using dual mesh

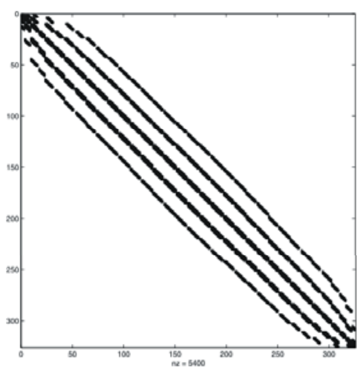


$$C_{ij}^{2h} = C_{ij}^{h} + \sum_{k \in N_j} C_{ik}^{\frac{h}{2}}$$

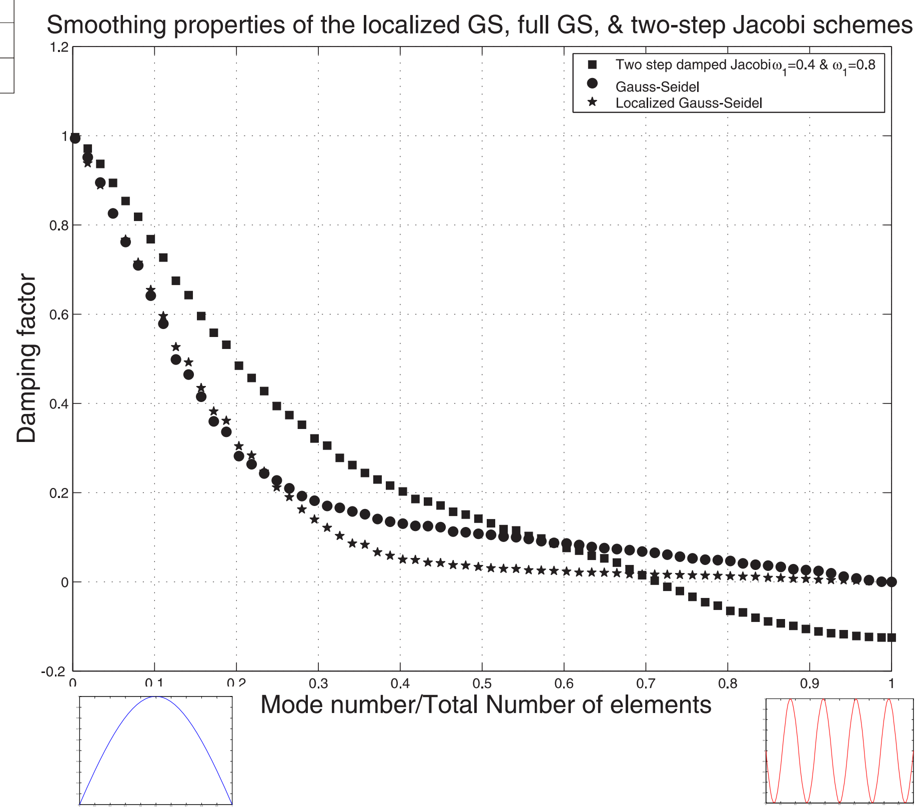
- Localized GS smoother



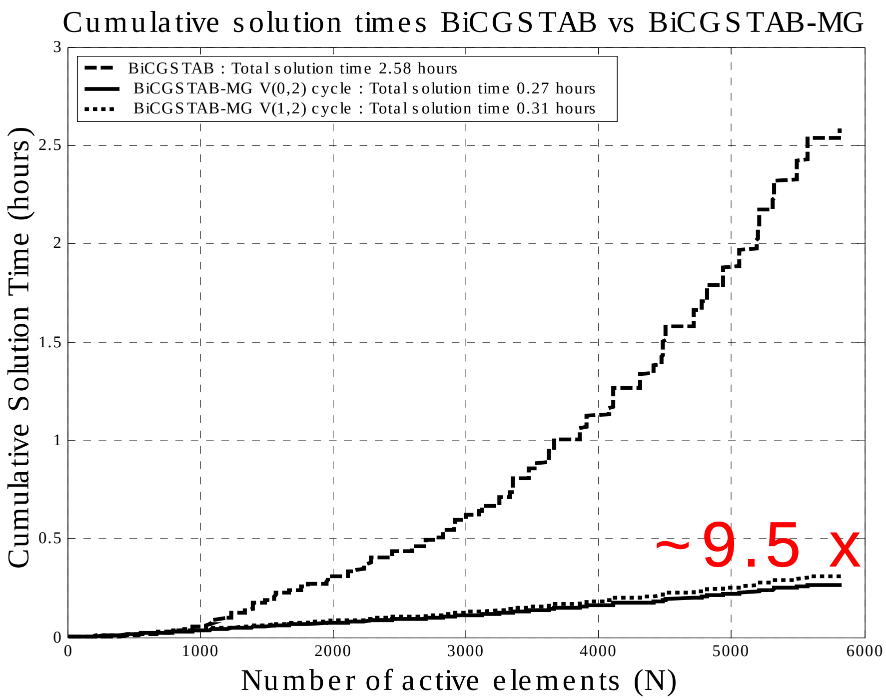
$\Rightarrow$

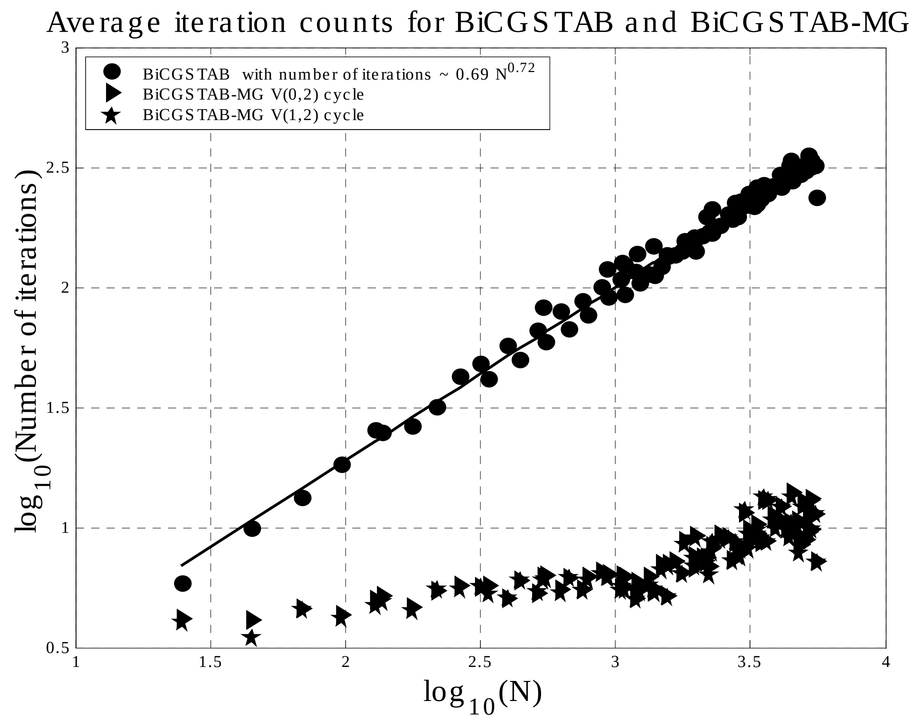


$$[AC]_{MN} \approx \sum_{K \in \mathcal{N}_M^5} A_{MK} C_{KN} \text{ when } N \in \mathcal{N}_K^9$$



Performance of MG Preconditioner



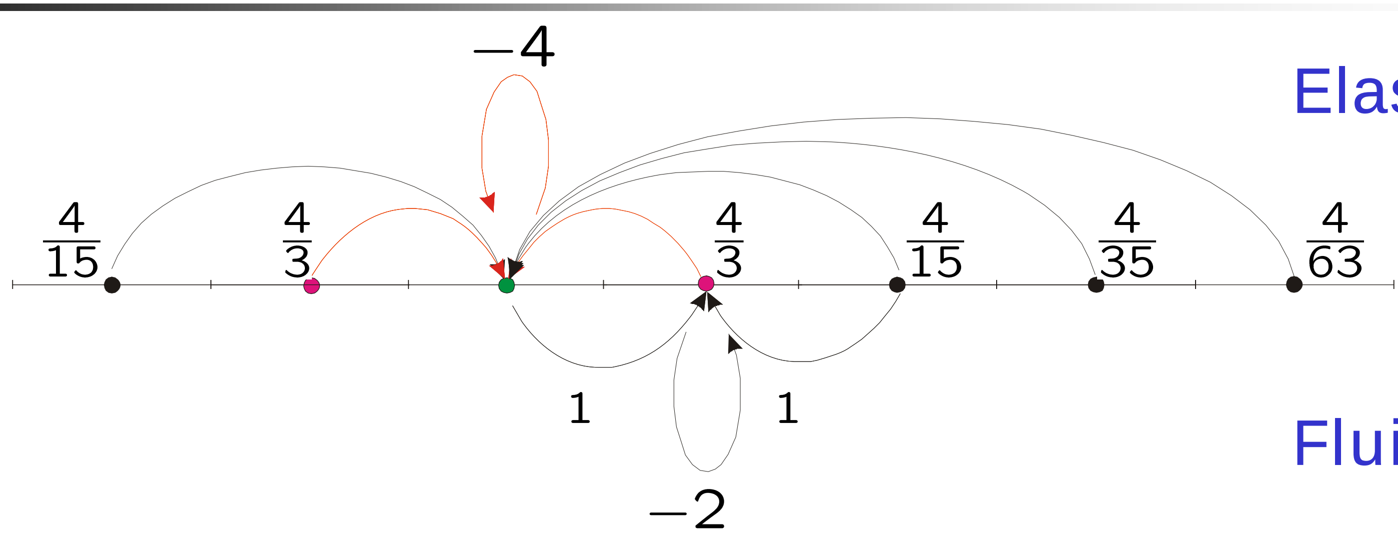


Incomplete LU factorization

```
function [A] = lu(A)
[n,n]=size(A);
for j=1:n
  for k=1:j-1
    for i=k+1:n
      A(i,j)=A(i,j)-A(i,k)*A(k,j);
    end
  end
  for i=j+1:n
    A(i,j)=A(i,j)/A(j,j);
  end
end
return
```

```
function [A] = basicILUc(A)
[n,n]=size(A);
for j=1:n
 for k=1:j-1
  if A(k,j) ~=0,
   for i=k+1:n
    if A(i,j) ~=0,
     A(i,j)=A(i,j)-A(i,k)*A(k,j);
    end
   end
  end
 end
 for i=j+1:n
  if A(i,j) ~=0,
   A(i,j)=A(i,j)/A(j,j);
  end
 end
end
return
```

Local Jacobian & Fourier Analysis



Elasticity

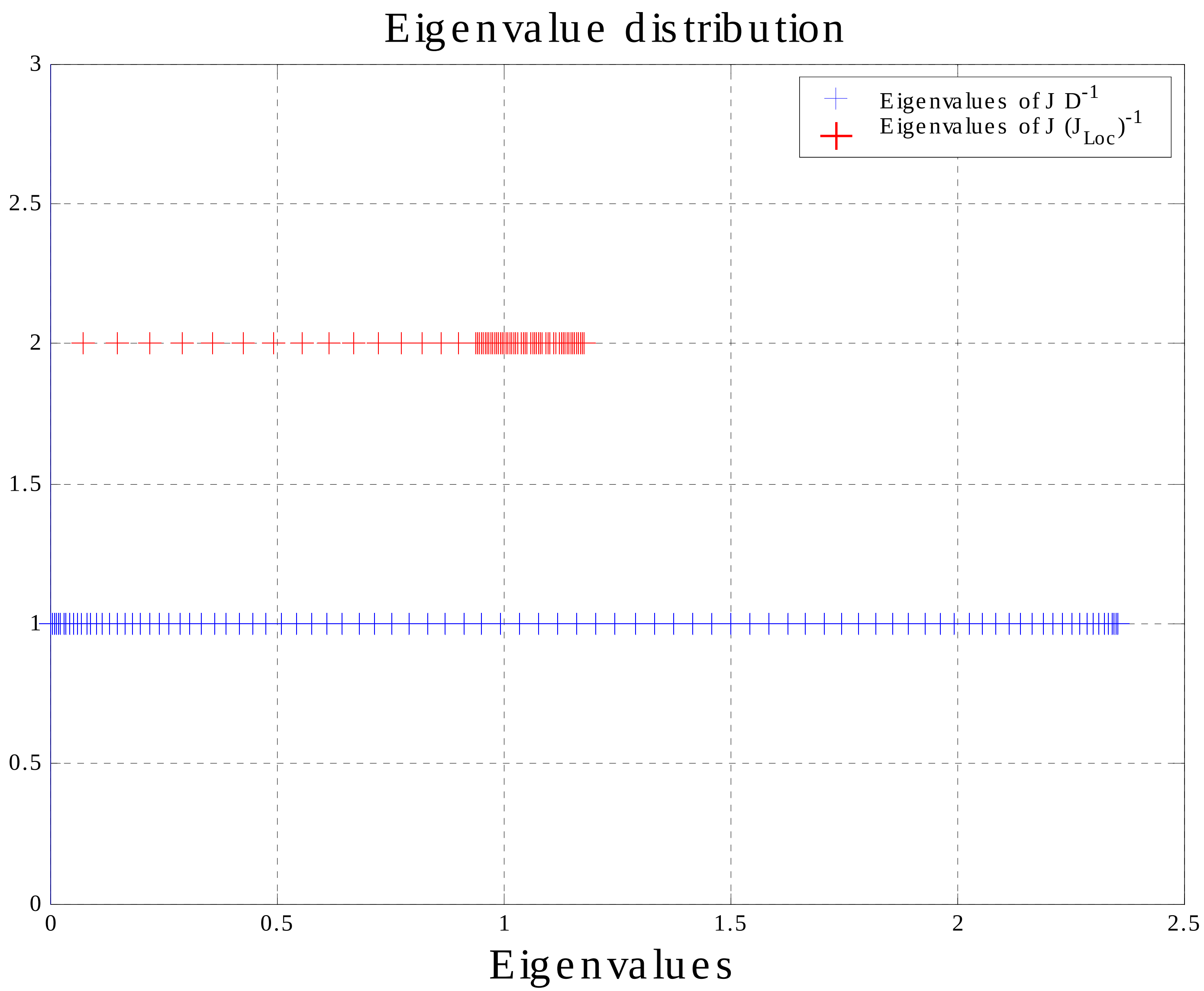
Fluid Flow

n-2 n-1 n n+1 n+2

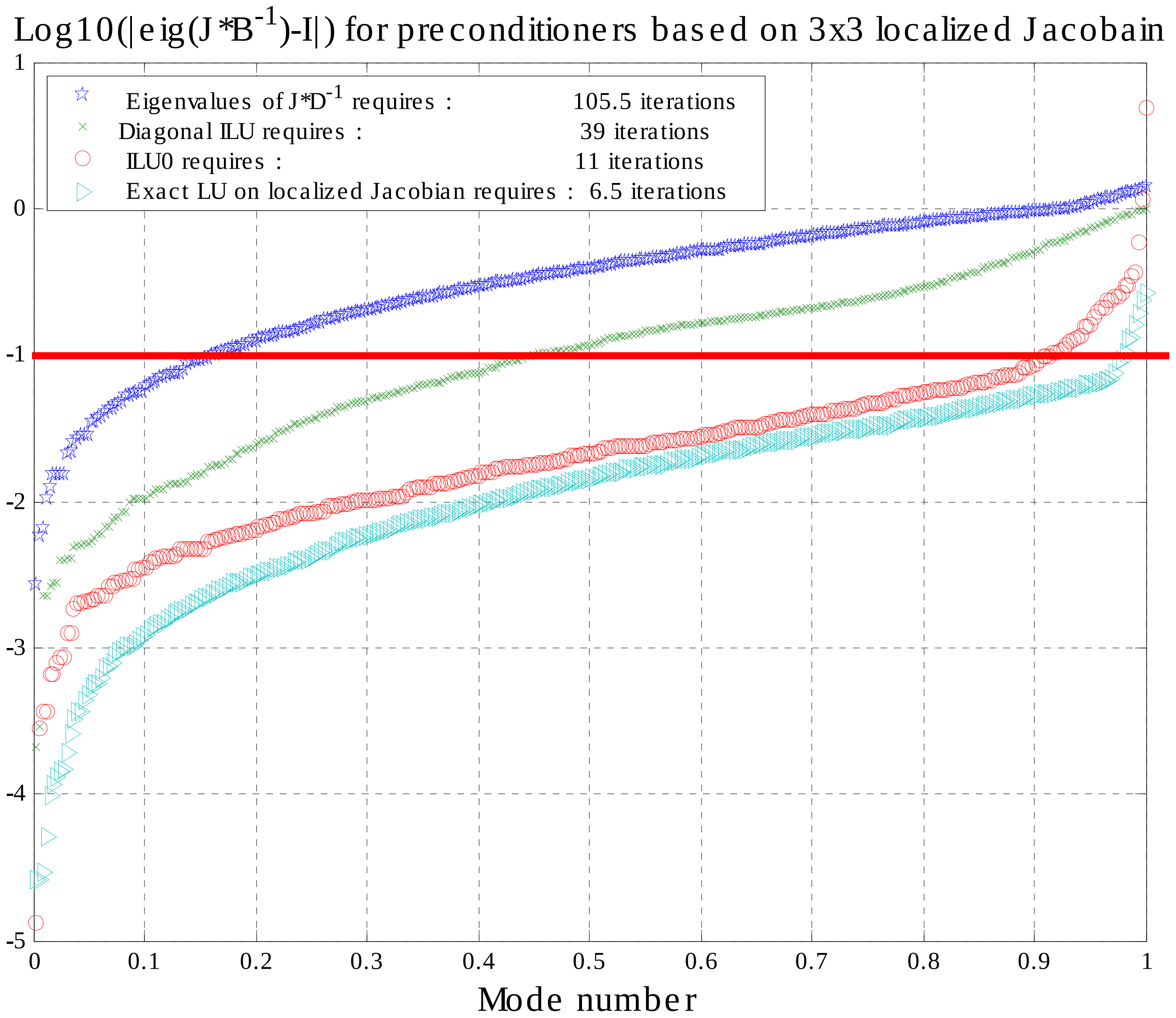
$$C_{Loc} = \frac{\lambda}{\Delta x}\left[\frac{4}{3} \quad -4 \quad \frac{4}{3}\right] \xrightarrow{*e^{ik\Delta x}} \hat{C}_{Loc,k} = \frac{4\lambda}{3\Delta x}(2\cos k\Delta x - 3)$$

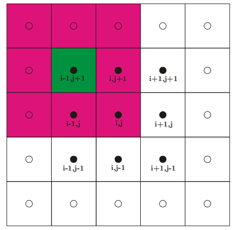
$$A = \frac{\bar{D}}{\Delta x^2}[1 \quad -2 \quad 1] \xrightarrow{*e^{ik\Delta x}} \hat{A}_k = -\frac{4\bar{D}}{\Delta x^2}\sin^2\left(\frac{k\Delta x}{2}\right)$$

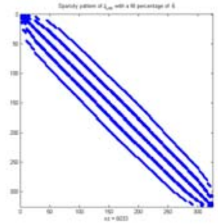
Spectrum of AC $(AC_{Loc})^{-1}$



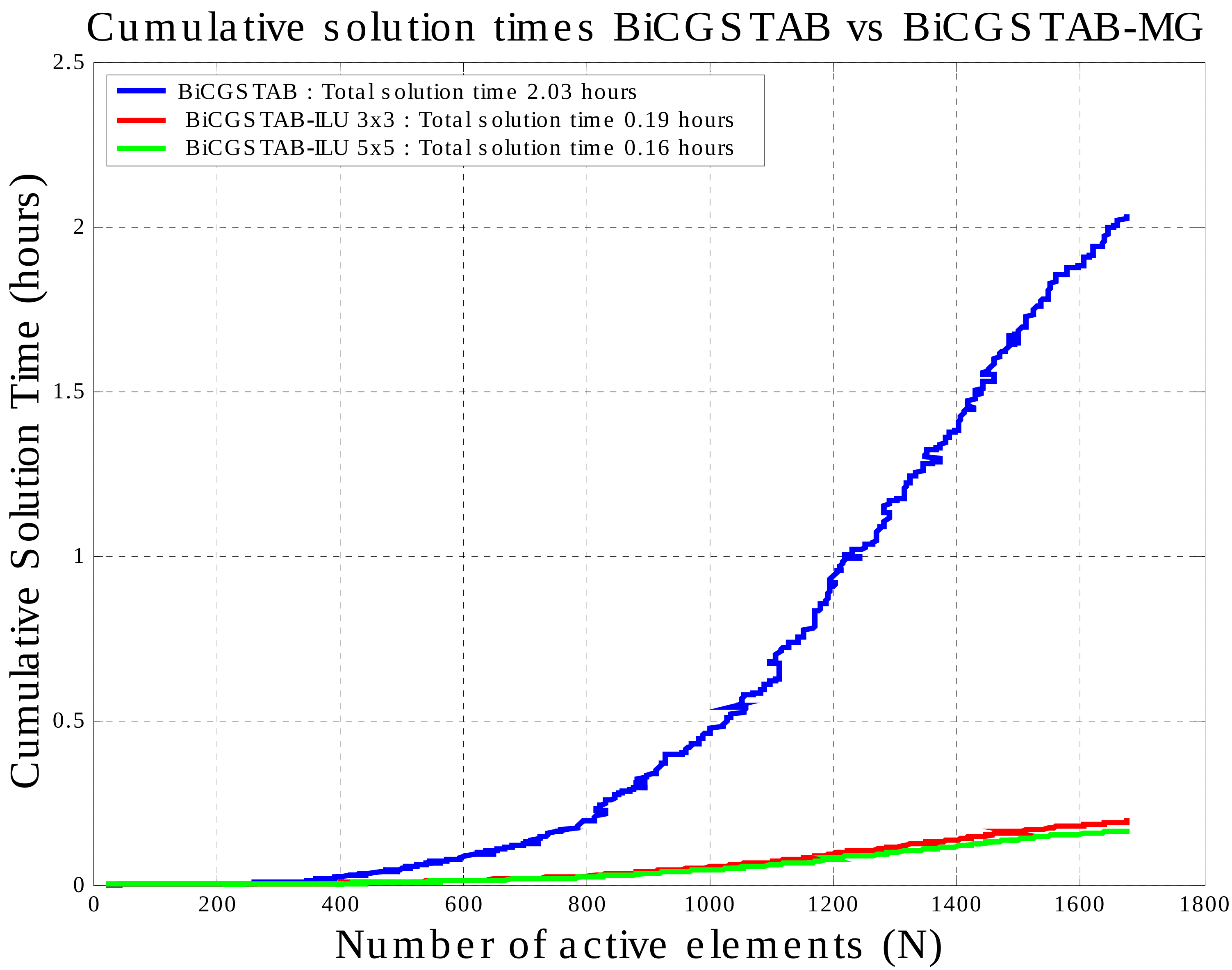
Eigenvalues of preconditioners



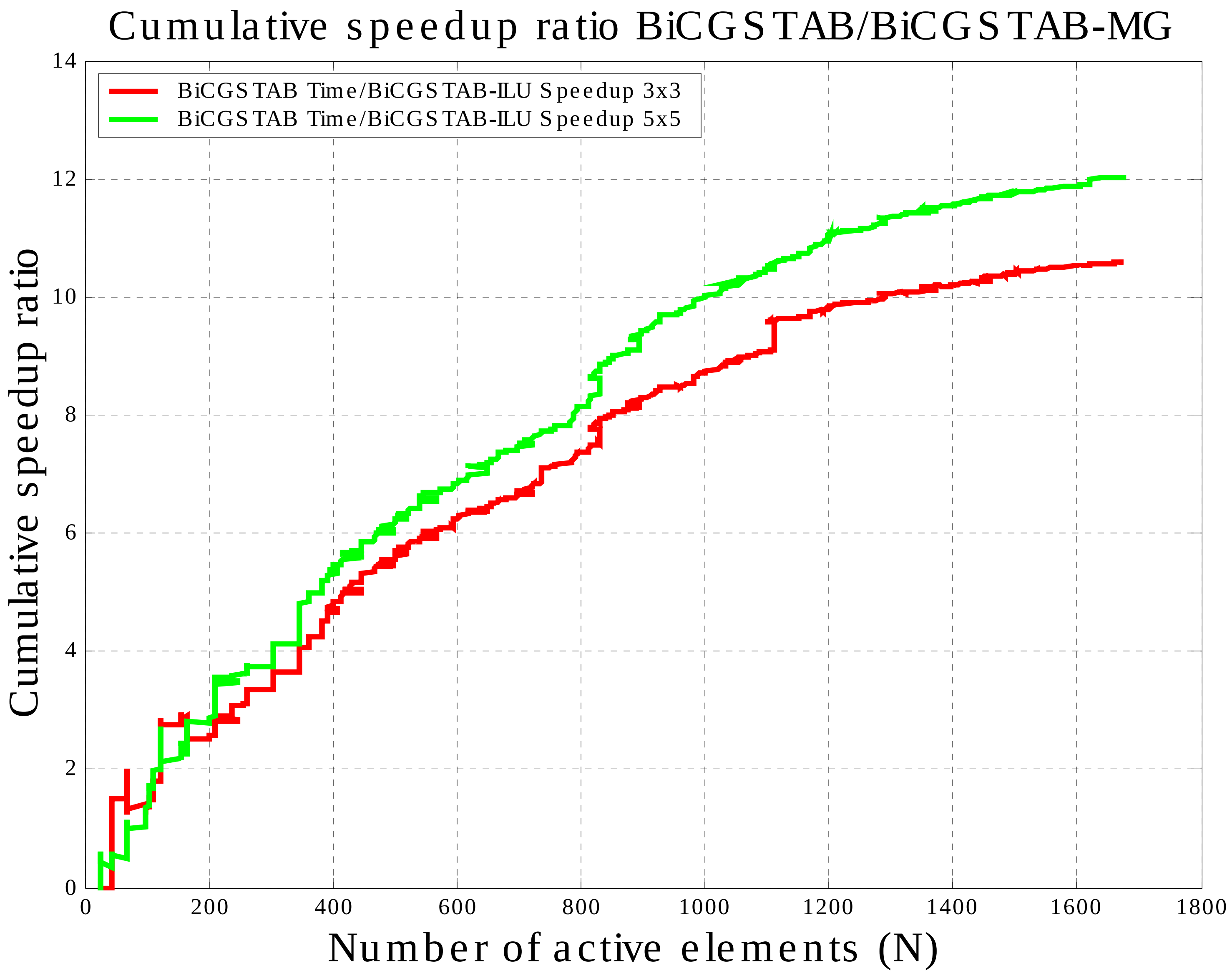




Numerical results for stress jump



Numerical results for stress jump



Scaling

- Rescale the physical quantities as follows:

$$x = L_* \chi, \qquad y = L_* \zeta, \qquad t = T_* \tau$$
$$w = W_* \Omega, \qquad p_f = P_* \Pi_f, \qquad Q = Q_0 \psi(\tau), \;\; \sigma = \sigma_0 \phi(\chi, \zeta)$$

- Dimensionless groups:

$$\mathcal{G}_c = \frac{C' L_*^2}{Q_0 T_*^{1/2}}, \quad \mathcal{G}_e = \frac{L_* P_*}{E' W_*}, \quad \mathcal{G}_m = \frac{\mu' Q_0}{P_* W_*^3}, \quad \mathcal{G}_k = \frac{K' L_*^{1/2}}{E' W_*}, \quad \mathcal{G}_v = \frac{Q_0 T_*}{L_*^2 W_*}$$

- Scaled Equations

$$\Pi(\chi, \zeta) = -\frac{1}{8\pi \boxed{\mathcal{G}_e}} \int_S \frac{\Omega(\chi', \zeta')}{[(\chi' - \chi)^2 + (\zeta' - \zeta)^2]^{\frac{3}{2}}} d\chi' d\zeta'$$

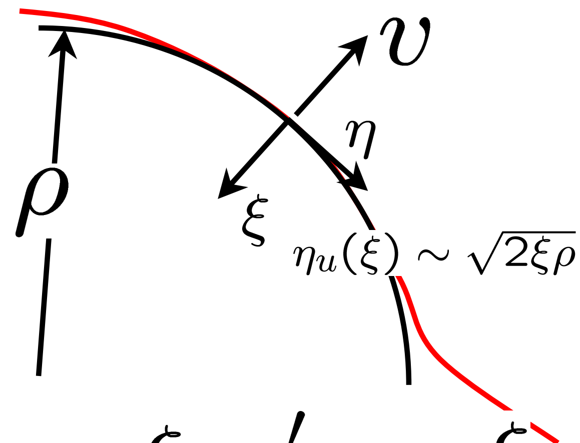
$$\frac{1}{\boxed{\mathcal{G}_v}} \frac{\partial \Omega}{\partial \tau} = \frac{1}{\boxed{\mathcal{G}_m}} \nabla \cdot \left(\Omega^3 \nabla \Pi \right) - \frac{\boxed{\mathcal{G}_c}}{\sqrt{\tau - \tau_0}} + \delta(\chi, \zeta) \psi(\tau)$$

$$\lim_{\xi \to 0} \frac{\Omega}{\xi^{1/2}} = \boxed{\mathcal{G}_k}, \qquad \lim_{\xi \to 0} \Omega^3 \frac{\partial \Pi_f}{\partial \xi} = 0, \qquad v = \lim_{\xi \to 0} \Omega^2 \frac{\partial \Pi_f}{\partial \xi}$$

Reduced eq near a smooth front

$$I(\xi,\eta) = \int_0^a i(\xi,\xi')\,d\xi'$$

$$(\xi-\rho)^2+\eta^2=\rho^2$$



$$i(\xi,\xi') = \int_{\eta_l(\xi')}^{\eta_u(\xi')} \frac{\Omega(\xi',\eta')}{[(\xi'-\xi)^2+(\eta'-\eta)^2]^{\frac{3}{2}}}d\eta'$$

$$\sim \xi^{-2} \int_{\eta_l(u\xi)/\xi}^{\eta_u(u\xi)/\xi} \frac{\Omega(u\xi,v\xi)}{[(u-1)^2+v^2]^{\frac{3}{2}}}dv$$

$$\xi' = u\xi \quad \eta' = v\xi$$

$$\sim \frac{\Omega_s(u\xi)}{\xi^2} \int_{-\sqrt{2u\rho/\xi}}^{\sqrt{2u\rho/\xi}} \frac{dv}{[(u-1)^2+v^2]^{\frac{3}{2}}}$$

$$\xi/\rho \ll 1 \quad \& \quad \eta/\rho \ll 1$$

$$\eta_l(\xi) \sim -\sqrt{2\xi\rho},\ \eta_u(\xi) \sim \sqrt{2\xi\rho}$$

$$\Omega(u\xi,v\xi) \overset{\xi\to 0}{\sim} \Omega_s(u\xi)$$

$$\sim \frac{2\Omega_s(u\xi)}{\xi^2(u-1)^2}$$

$$I(\xi,0) = 2\int_0^{a/\xi} \Omega_s(u\xi)\frac{du}{\xi^2(u-1)^2} \overset{\xi\to 0}{\sim} 2\int_0^{\infty} \frac{d\Omega_s(\xi')}{d\xi'}\frac{d\xi'}{(\xi'-\xi)}$$

Tip asymptotic behaviour $\Omega = A\xi^{\alpha}$

- Elasticity:

$$\Pi = -\frac{\mathcal{G}_e}{4\pi}\int_0^{\infty}\frac{\mathrm{d}\Omega}{\mathrm{d}\xi'}\frac{\mathrm{d}\xi'}{(\xi'-\xi)} = -\frac{\mathcal{G}_e A}{4\pi}\int_0^{\infty}\frac{s^{\alpha-1}\mathrm{d}s}{(s-\xi)} = \frac{\mathcal{G}_e A}{4\pi}\pi\cot(\pi\alpha)\xi^{\alpha-1}$$

- Lubrication: $\Omega(\hat{\xi},\hat{\eta},\tau) \approx \Omega(V\tau-\hat{\xi})$ $\quad \tau-\tau_0(\xi)\approx\frac{\xi}{V}$

$$\frac{\Omega^3}{\mathcal{G}_m}\frac{\mathrm{d}\Pi}{\mathrm{d}\xi} = \frac{1}{\mathcal{G}_v}V\Omega + 2\mathcal{G}_c V^{1/2}\xi^{1/2}$$

$$\xi^{4\alpha-2} \qquad \xi^{\alpha} \qquad \xi^{1/2}$$

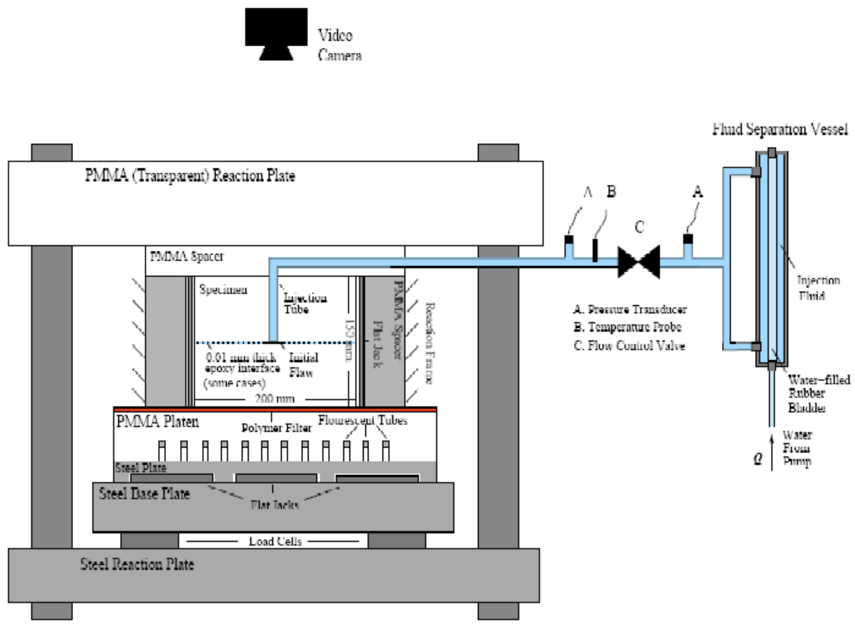
$$K\ \ vertex: \quad \Omega\sim\xi^{1/2}\Rightarrow\Pi\sim log\xi$$

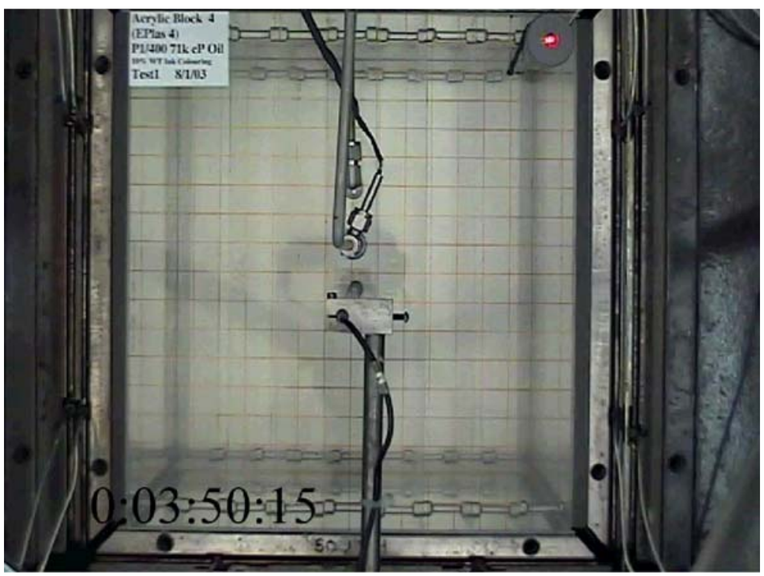
$$M\ \ vertex: 4\alpha-2=\alpha\ \Rightarrow\Omega\sim\beta_{m0}V^{1/3}\xi^{2/3}$$

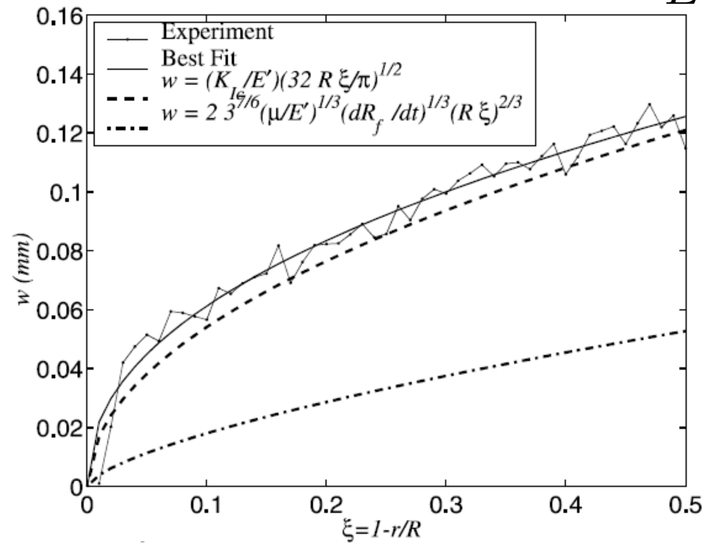
$$\tilde{M}\ \ vertex: 4\alpha-2=1/2\ \Rightarrow\Omega\sim\beta_{\tilde{m}0}V^{1/8}\xi^{5/8}$$

HF experiment
(*Bunger & Jeffrey* CSIRO)

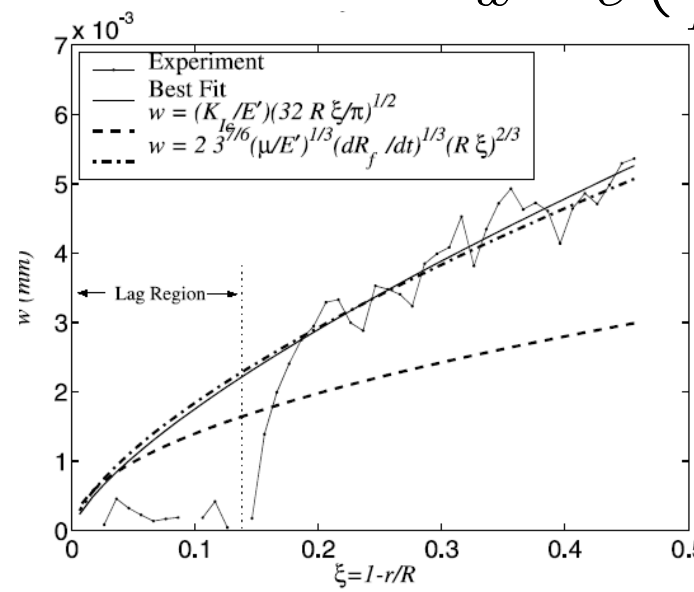
$$w \sim \frac{K'}{E'} s^{1/2}$$





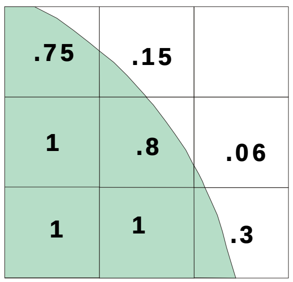


$$w \sim c\,\left(\frac{\mu' v}{E'}\right)^{1/3} s^{2/3}$$



How do we find the fracture front?

- VOF method



$$\frac{\partial \chi}{\partial t} + \mathbf{v} \cdot \nabla \chi = 0$$

$$\nabla \cdot \mathbf{v} = 0$$

$$A \frac{\partial}{\partial t}\left(\frac{1}{A}\int_A \chi dA\right) = -\int_A \nabla \cdot (\chi \mathbf{v})\, dA$$

$$\frac{\partial F}{\partial t} = -\frac{1}{A}\int_{\partial A} \chi v_n dl$$

Problem: we need an accurate

- Level set method

-move contours of a surface

$$\phi(x, y, t) = 0$$

- differentiate

$$\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi = 0$$

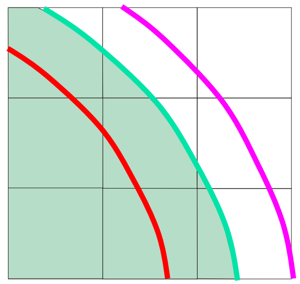
$$\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \mathbf{n} |\nabla \phi| = 0$$

$$\frac{\partial \phi}{\partial t} + v_n |\nabla \phi| = 0$$

$$v = \lim_{\xi \to 0} \Omega^2 \frac{\partial \Pi_f}{\partial \xi}$$

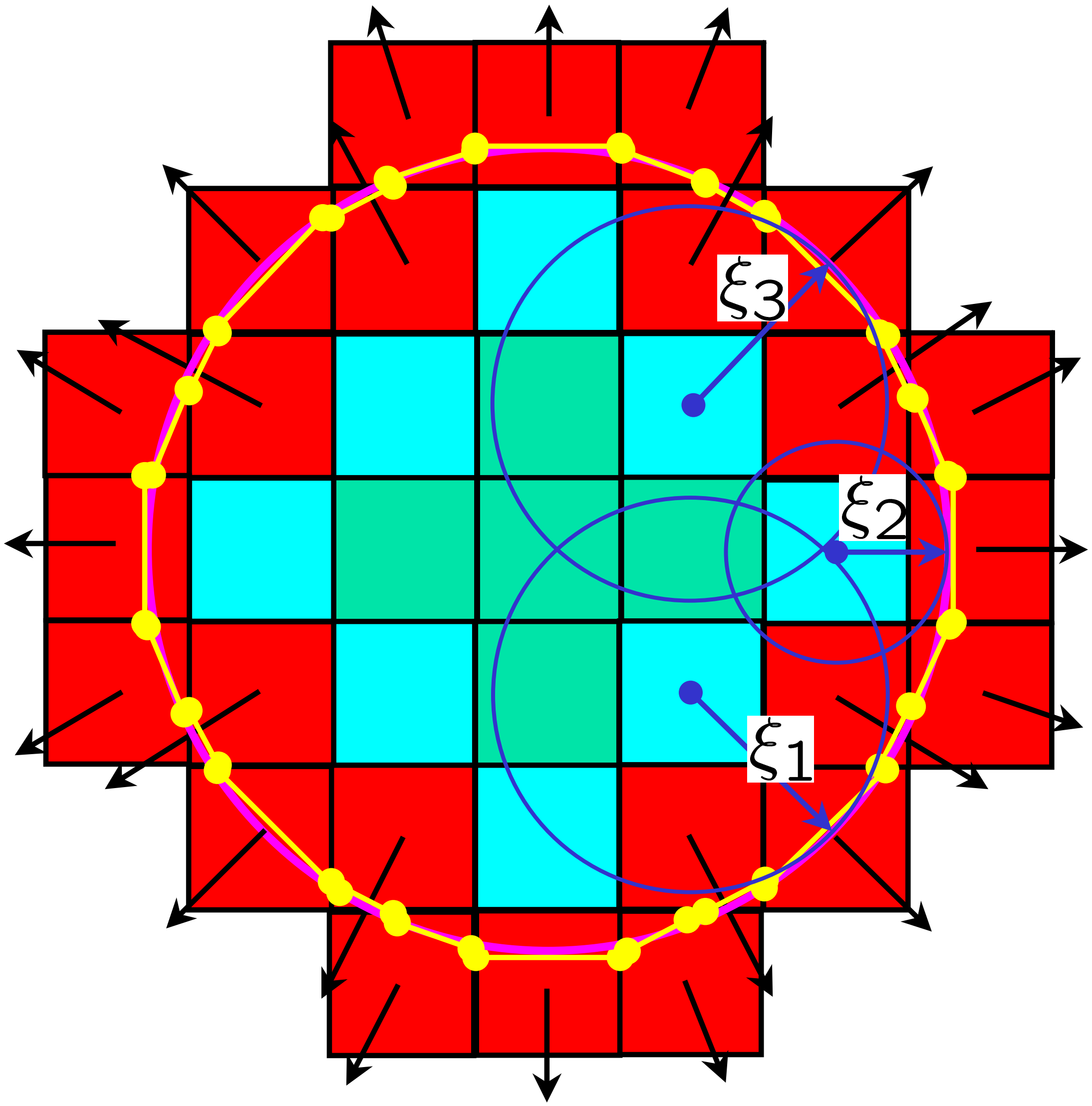
Why not use stress intensity factor?

- Move front till SIF = K_{Ic}
  - Find the SIF numerically
  - if $K_I > K_{Ic}$ move forward
  - if $K_I < K_{Ic}$ move back
  - if $K_I = K_{Ic}$ stay



- Problems
  - Accurate calculation of K_I
  - How far do you move?
  - What about viscous and leak-off dominated propagation regimes

Divide elements into tip & channel



w^T tip elt

w^C channel elt

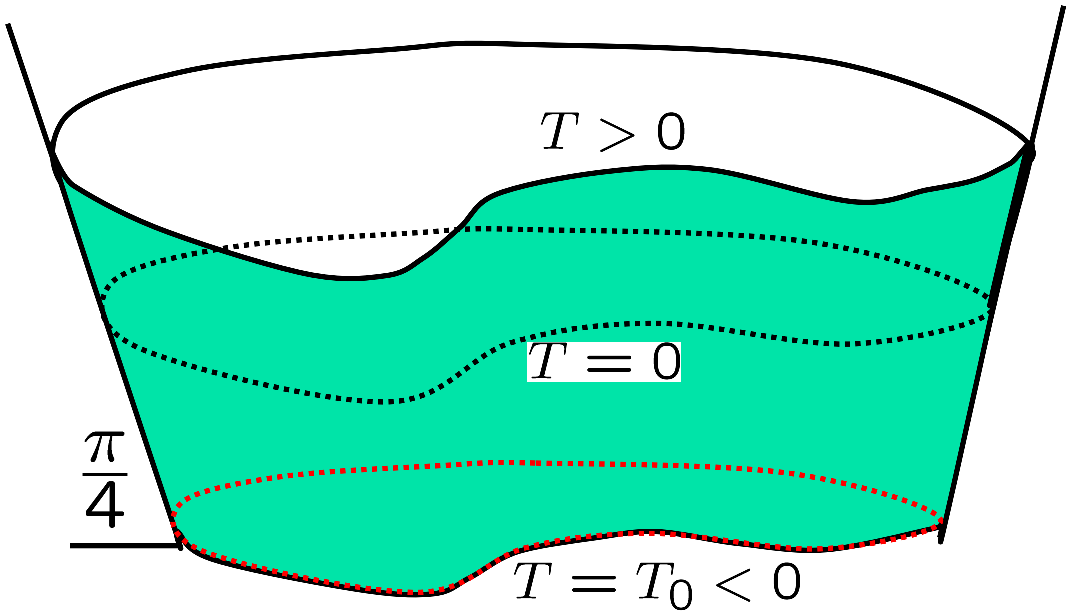
At survey points

$$\Omega = \xi^{1/2}$$

$$\xi = \Omega^2$$

Eikonal solution surfaces

The signed distance function



$$|\nabla T| = 1$$

$$T = T_0$$

Solving the Eikonal BVP

- The Eikonal Equation $|\nabla \mathcal{T}(\chi,\zeta)| = 1$
- First order discretization

$$\left[\max\left(\frac{\mathcal{T}_{ij}-\mathcal{T}_{i-1j}}{\Delta\chi}, \frac{\mathcal{T}_{ij}-\mathcal{T}_{i+1j}}{\Delta\chi}, 0\right)\right]^2 + \left[\max\left(\frac{\mathcal{T}_{ij}-\mathcal{T}_{ij-1}}{\Delta\zeta}, \frac{\mathcal{T}_{ij}-\mathcal{T}_{ij+1}}{\Delta\zeta}, 0\right)\right]^2 = 1$$

- Reduction to a quadratic

Let $\mathcal{T}_1 = \min\left(\mathcal{T}_{i-1j}, \mathcal{T}_{i+1j}\right)$ and $\mathcal{T}_2 = \min\left(\mathcal{T}_{ij-1}, \mathcal{T}_{ij+1}\right)$

$$\left[\max\left(\frac{\mathcal{T}-\mathcal{T}_1}{\Delta\chi}, 0\right)\right]^2 + \left[\max\left(\frac{\mathcal{T}-\mathcal{T}_2}{\Delta\zeta}, 0\right)\right]^2 = 1$$

- Solution to the quadratic equation

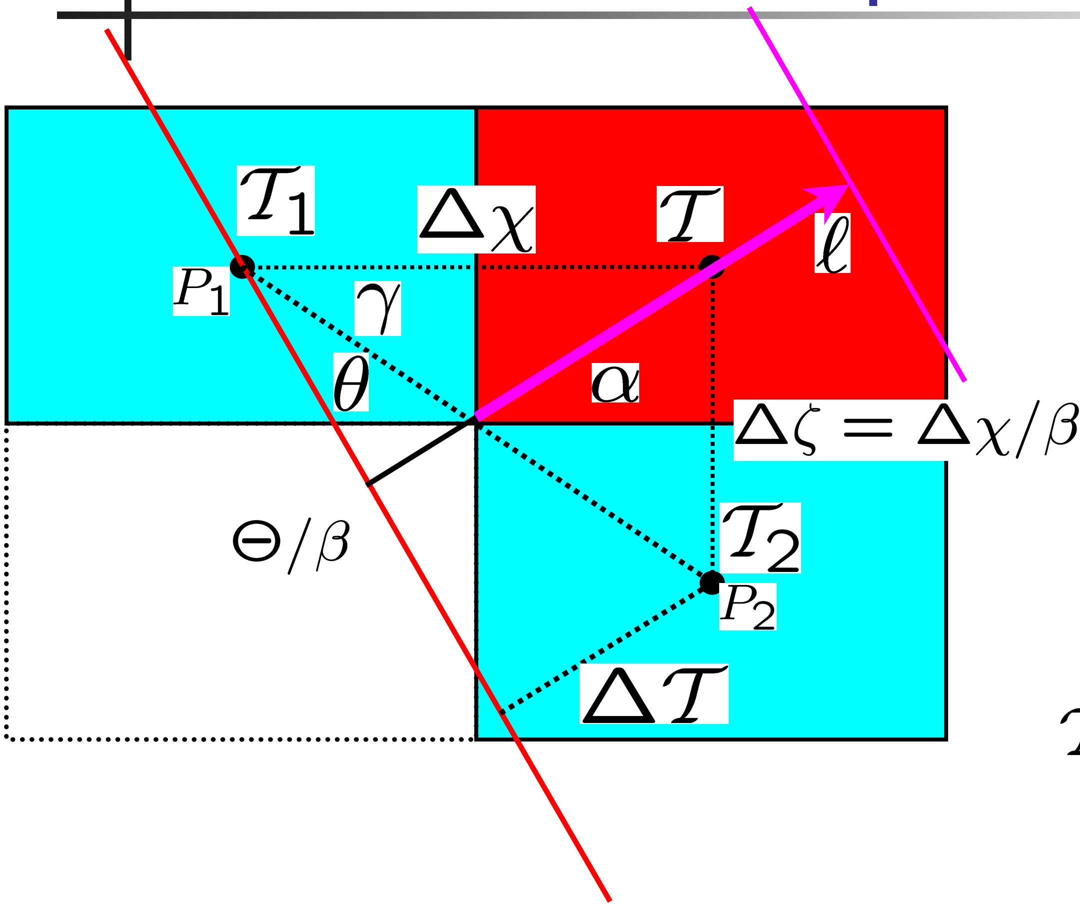
$$\mathcal{T} = \frac{\mathcal{T}_1+\beta^2\mathcal{T}_2+\Theta}{1+\beta^2}$$

$$\Theta = \sqrt{\Delta\chi^2(1+\beta^2) - \beta^2\Delta\mathcal{T}^2}$$

$$\Delta\mathcal{T} = \mathcal{T}_2 - \mathcal{T}_1$$

$$\Delta\chi = \beta\Delta\zeta$$

Geometric interpretation



$$\overline{P_1P_2} = \frac{\Delta\chi}{\beta}\sqrt{(1+\beta^2)}$$

$$\Theta = \sqrt{\Delta\chi^2(1+\beta^2) - \beta^2\Delta\mathcal{T}^2}$$

$$\mathcal{T} = \mathcal{T}_1 + \Delta\chi\sin(\theta+\gamma)$$

$$= \frac{\mathcal{T}_1+\beta^2\mathcal{T}_2+\Theta}{1+\beta^2}$$

$$\ell = -\left(\frac{\mathcal{T}_1+\mathcal{T}_2}{2}\right) \qquad \tan\alpha = \frac{\beta(\Theta-\Delta\mathcal{T})}{\Theta+\beta^2\Delta\mathcal{T}}$$

Initializing $\mathcal{T}(\chi,\zeta)$ from tip asymptotes

- General asymptote: $\Omega \sim \mathcal{W}(\xi; v)$

$$\mathcal{T}(\chi,\zeta) = -\xi \sim -\mathcal{W}^{-1}(\Omega; v)$$

- K-vertex: $\Omega \sim \xi^{\frac{1}{2}}$

$$\mathcal{T}(\chi,\zeta) = -\xi \sim -\Omega^2$$

- M-vertex: $\Omega \sim \beta_{m0}\, v^{1/3}\xi^{2/3}$

$$\mathcal{T}(\chi,\zeta) = -\xi \sim -\left(\frac{\Omega}{\beta_{m0}\, v^{1/3}}\right)^{\frac{3}{2}}$$

$$v = -\frac{\mathcal{T}-\mathcal{T}_0}{\Delta\tau} \quad \Rightarrow \quad \mathcal{T}^3 - \mathcal{T}_0\mathcal{T}^2 + b = 0$$

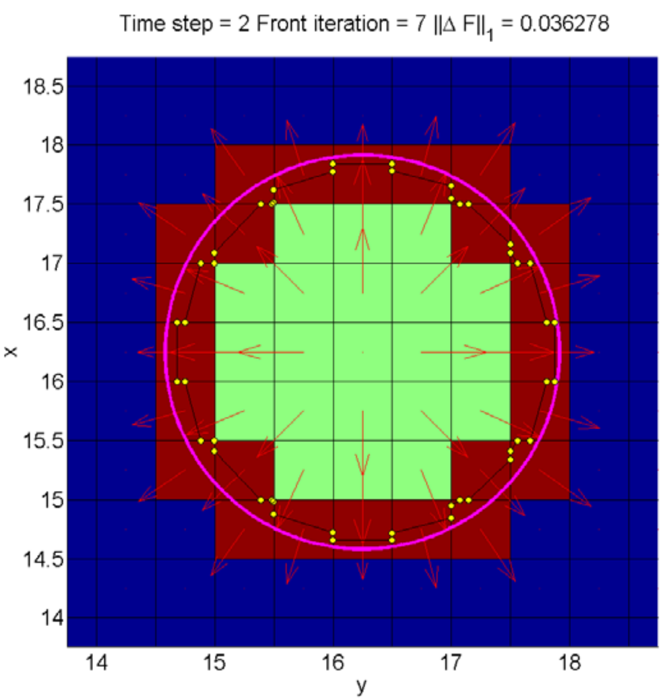
$$b = \Delta\tau \left(\frac{\Omega}{\beta_{m0}}\right)^3 > 0$$

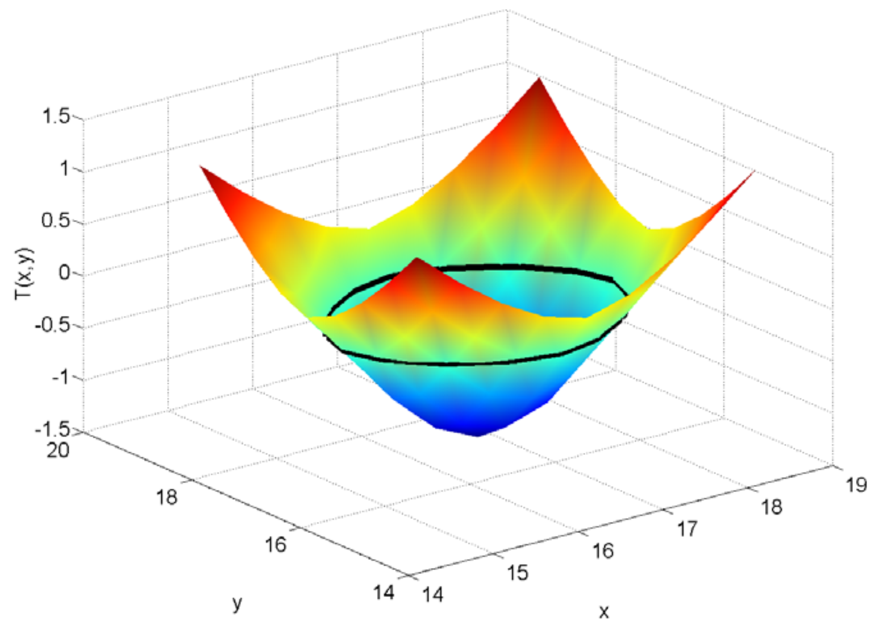
Sample signed distance function

$\mathcal{T}(\chi,\zeta)$ $\Delta\chi = 0.5 = \Delta\zeta$

χ (rows), ζ (columns)

1.44	1.09	0.78	0.53	0.41	0.52	0.75	1.05	1.4
1.09	0.7	0.34	0.05	-0.09	0.03	0.31	0.66	1.05
0.79	0.35	-0.07	-0.43	-0.59	-0.46	-0.11	0.3	0.74
0.55	0.07	-0.42	-0.83	-0.97	-0.86	-0.47	0.01	0.5
0.43	-0.07	-0.57	-0.96	-1.16	-0.99	-0.63	-0.13	0.37
0.55	0.07	-0.42	-0.83	-0.97	-0.86	-0.47	0.01	0.5
0.79	0.35	-0.07	-0.43	-0.59	-0.46	-0.11	0.3	0.74
1.09	0.7	0.34	0.05	-0.09	0.03	0.31	0.66	1.05
1.44	1.09	0.78	0.53	0.41	0.52	0.75	1.05	1.4





Time stepping and front evolution

Time step loop: $\tau \leftarrow \tau + \Delta\tau$

Front iteration loop:

Coupled Solution

$$\begin{bmatrix} I - \Delta\tau A^{cc} C^{cc} & -\Delta\tau A^{ct} \\ -\Delta\tau A^{tc} C^{cc} & -\Delta\tau A^{tt} \end{bmatrix} \begin{bmatrix} \Delta\Omega^{c} \\ \Pi^{t} \end{bmatrix} = \begin{bmatrix} r^{c} \\ r^{t} \end{bmatrix}$$

end

set $\mathcal{T}(\chi, \zeta) = -\xi \sim -\mathcal{W}^{-1}(\Omega; \upsilon)$ in

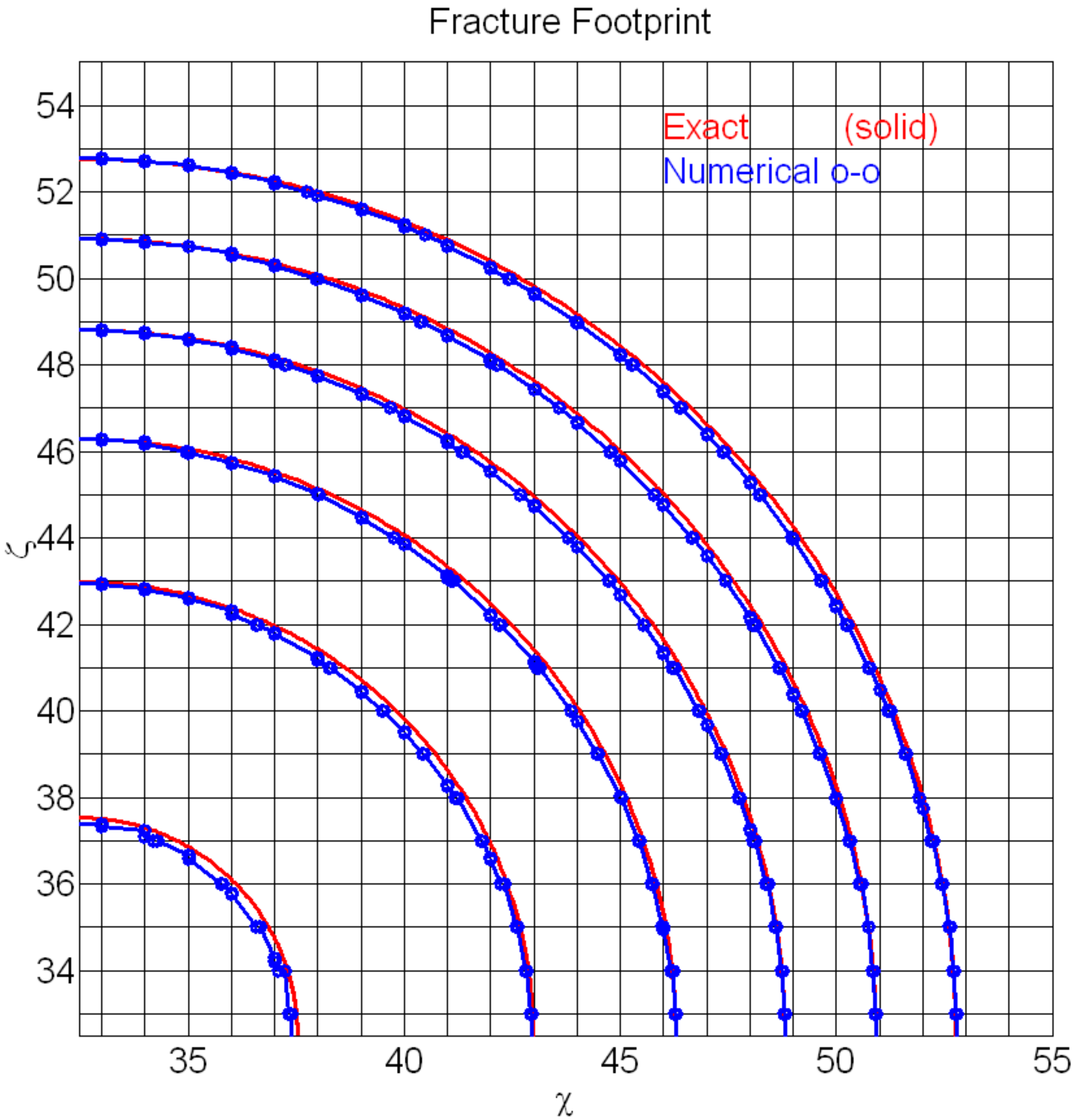
use FMM to solve $|\nabla \mathcal{T}(\chi, \zeta)| = 1$

Locate front $\ell, \alpha, \Omega^{t} = V^{t}(\ell)/\Delta A$

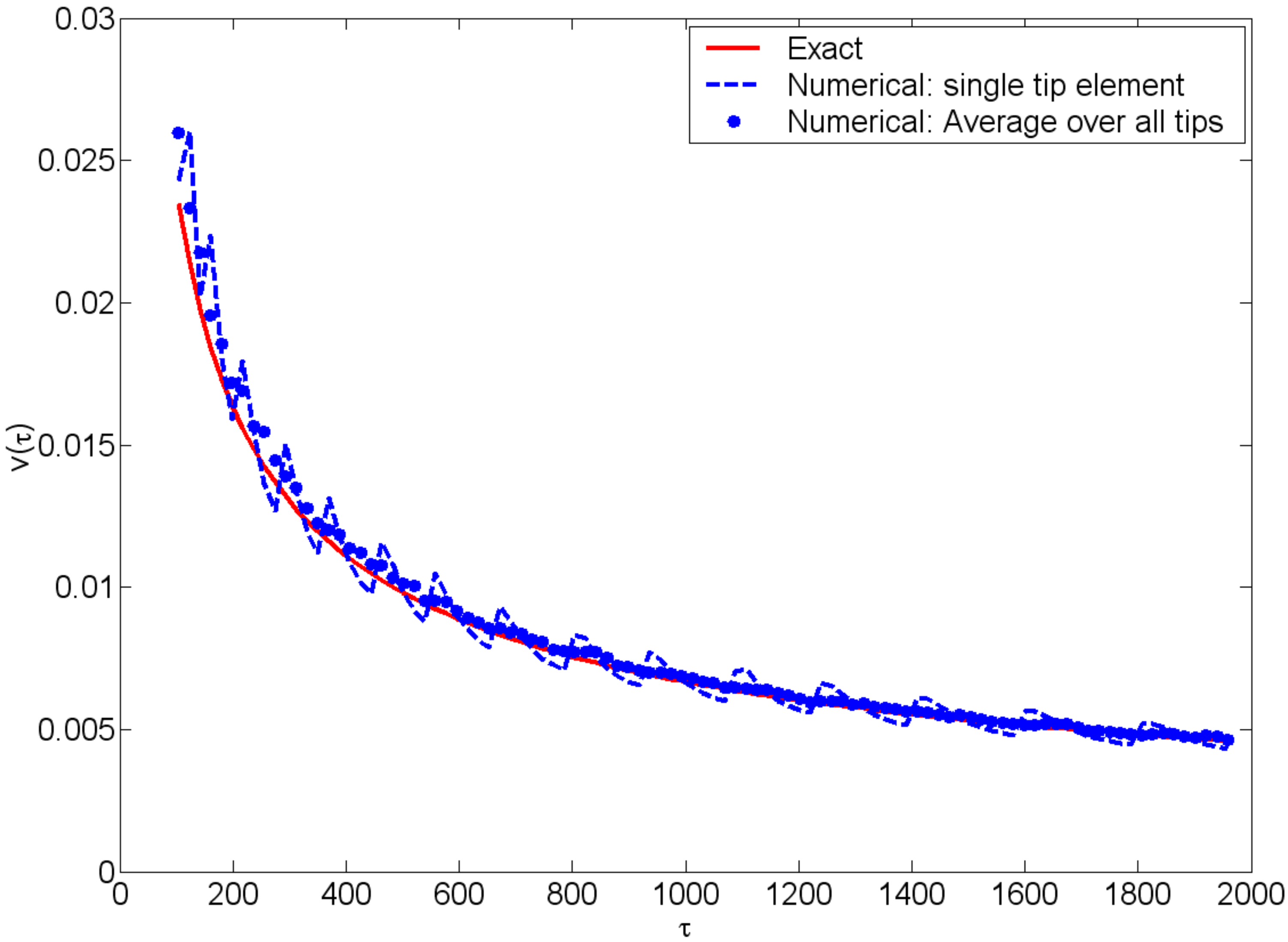
next front iteration

next time step

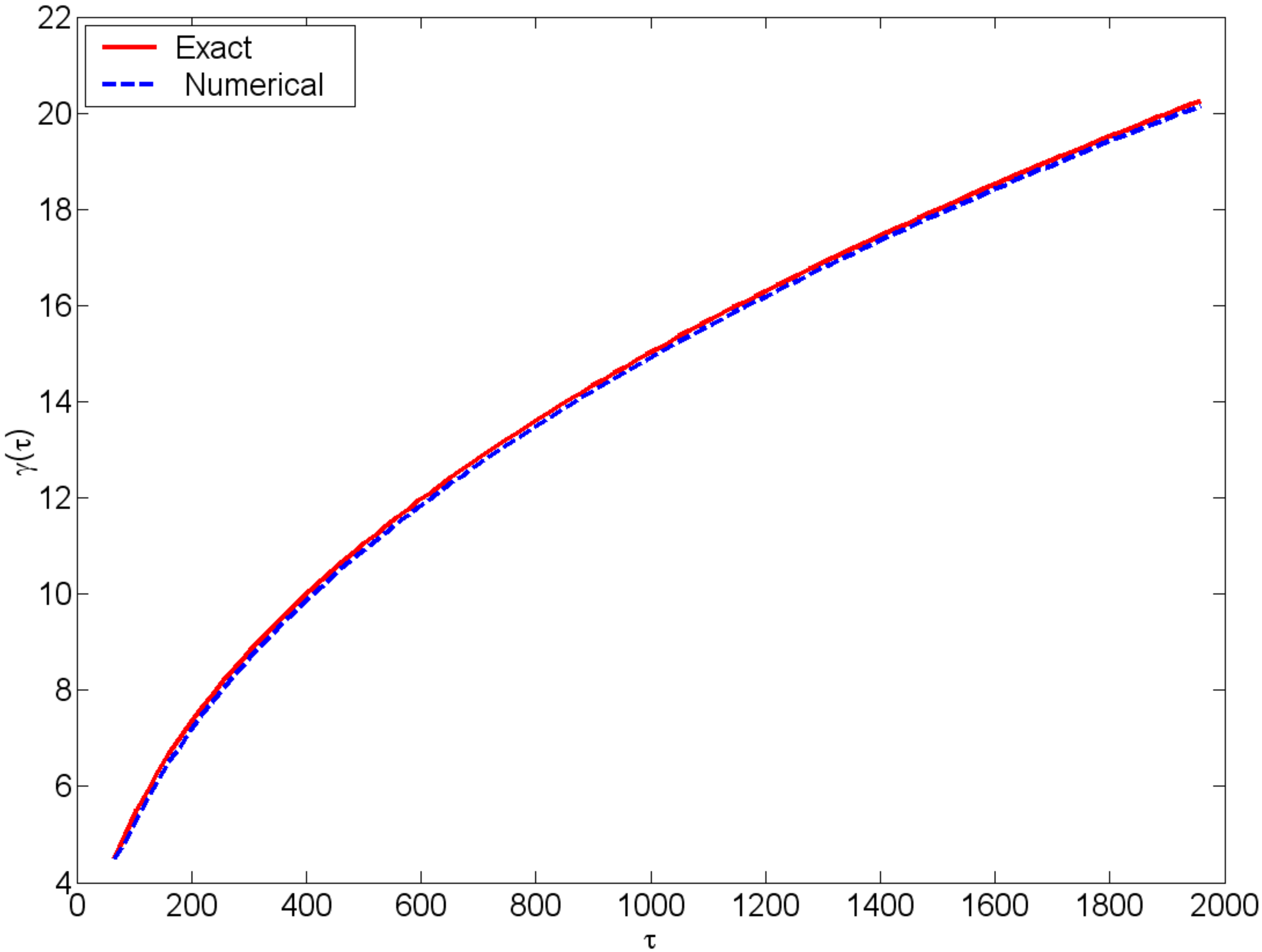
M-vertex radial sol'n – footprints



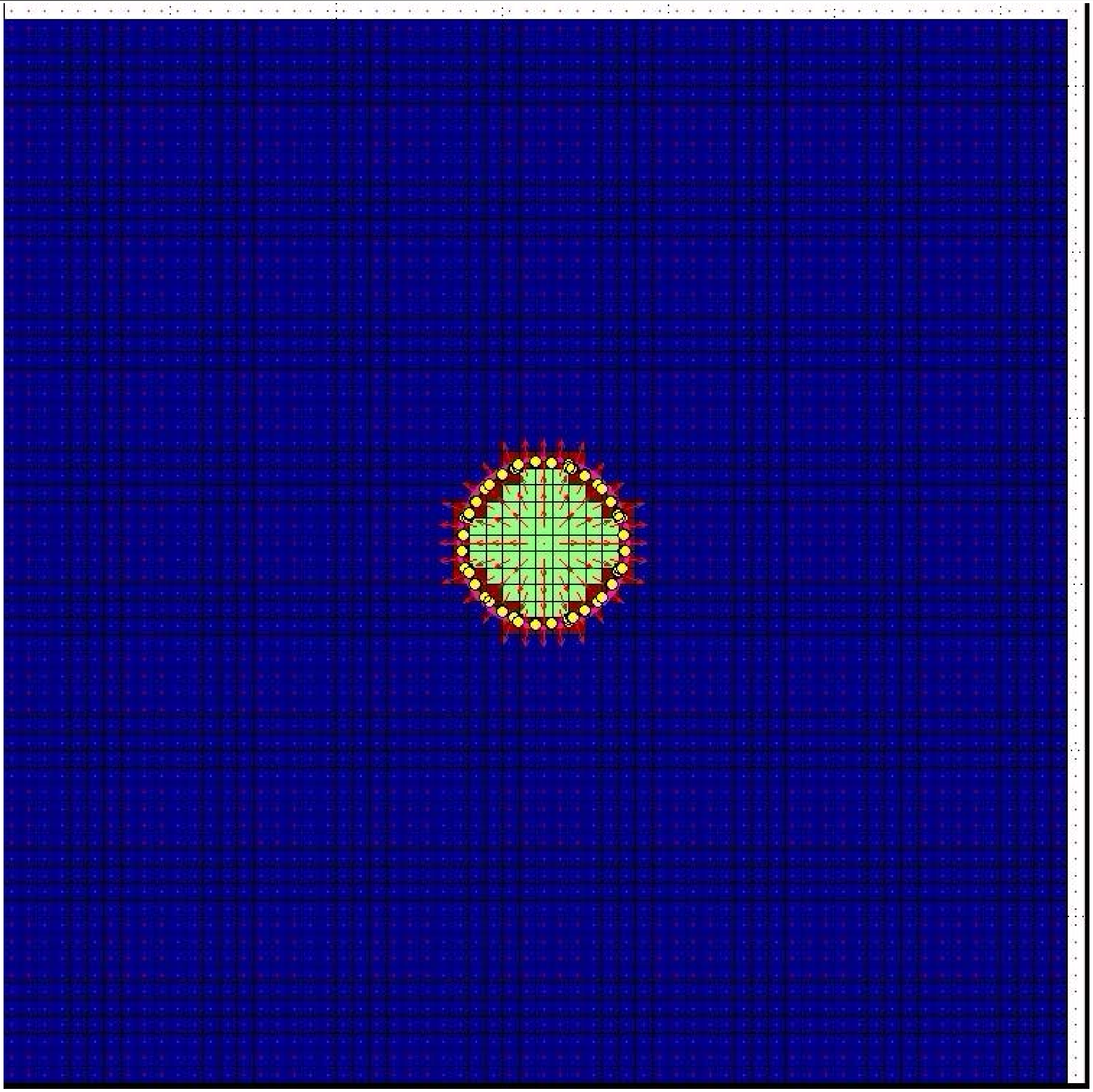
M-Vertex radial solution – front speed



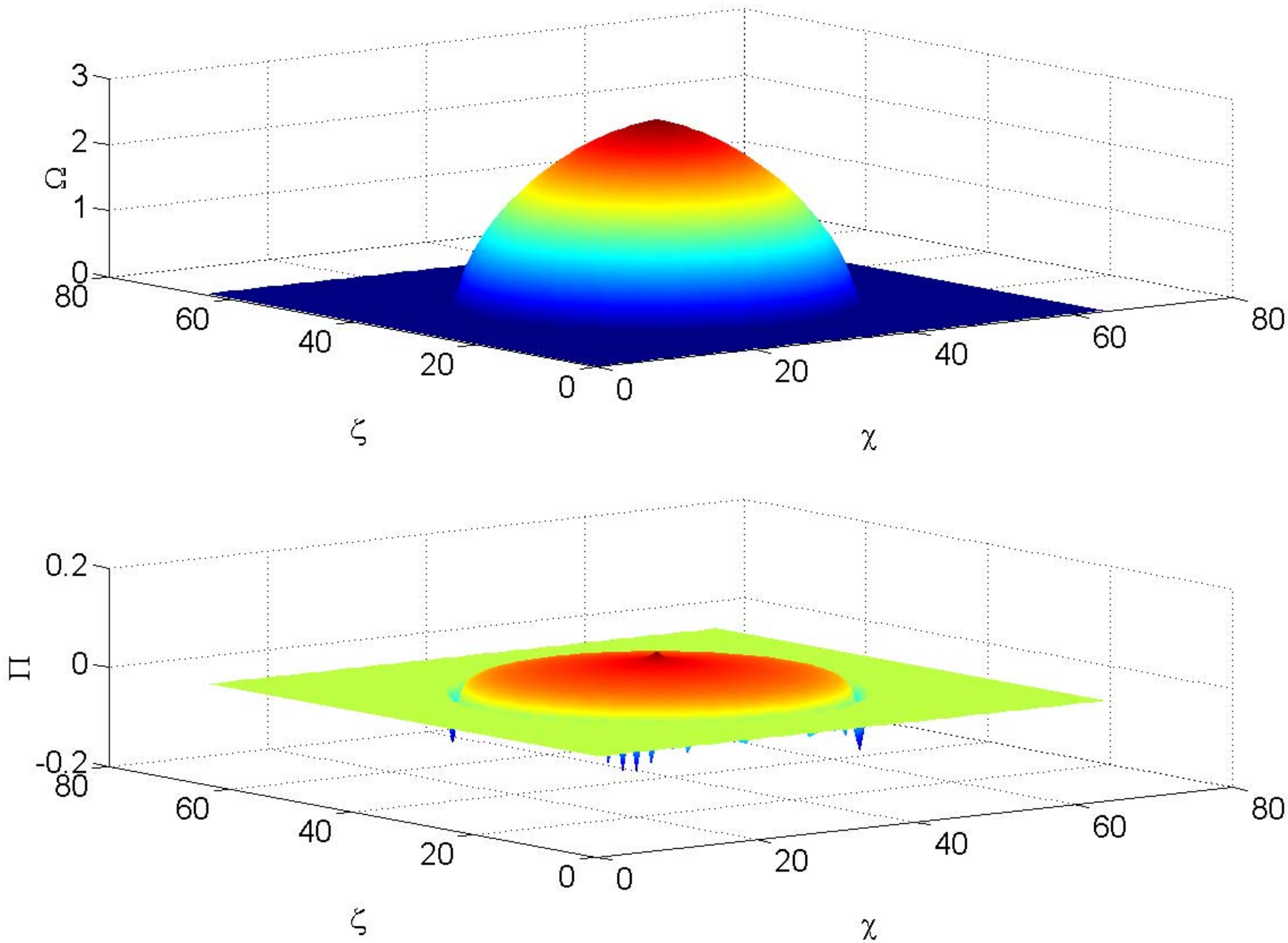
M-vertex radial solution - $\gamma(\tau)$



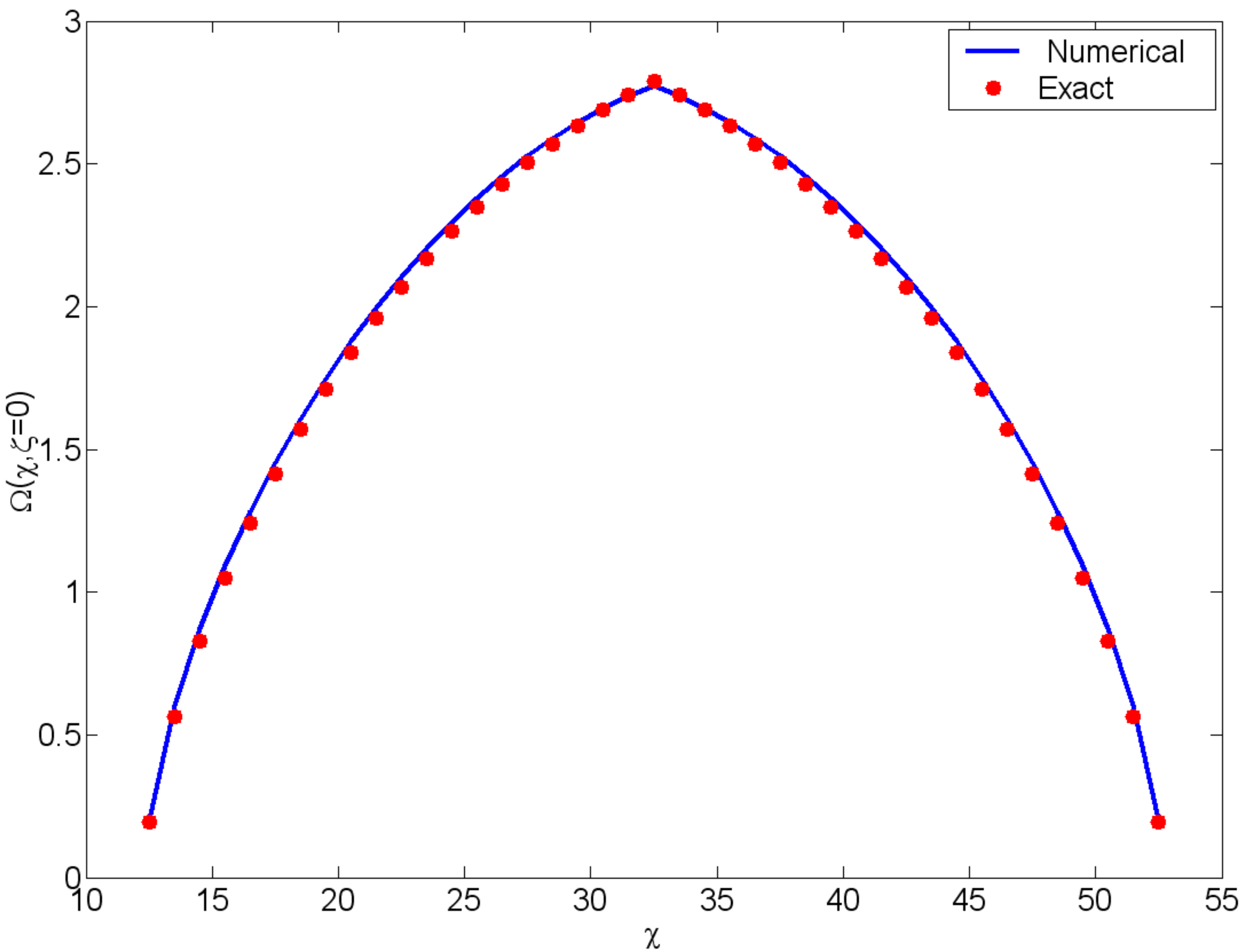
M-Vertex Footprint evolution – ILSA



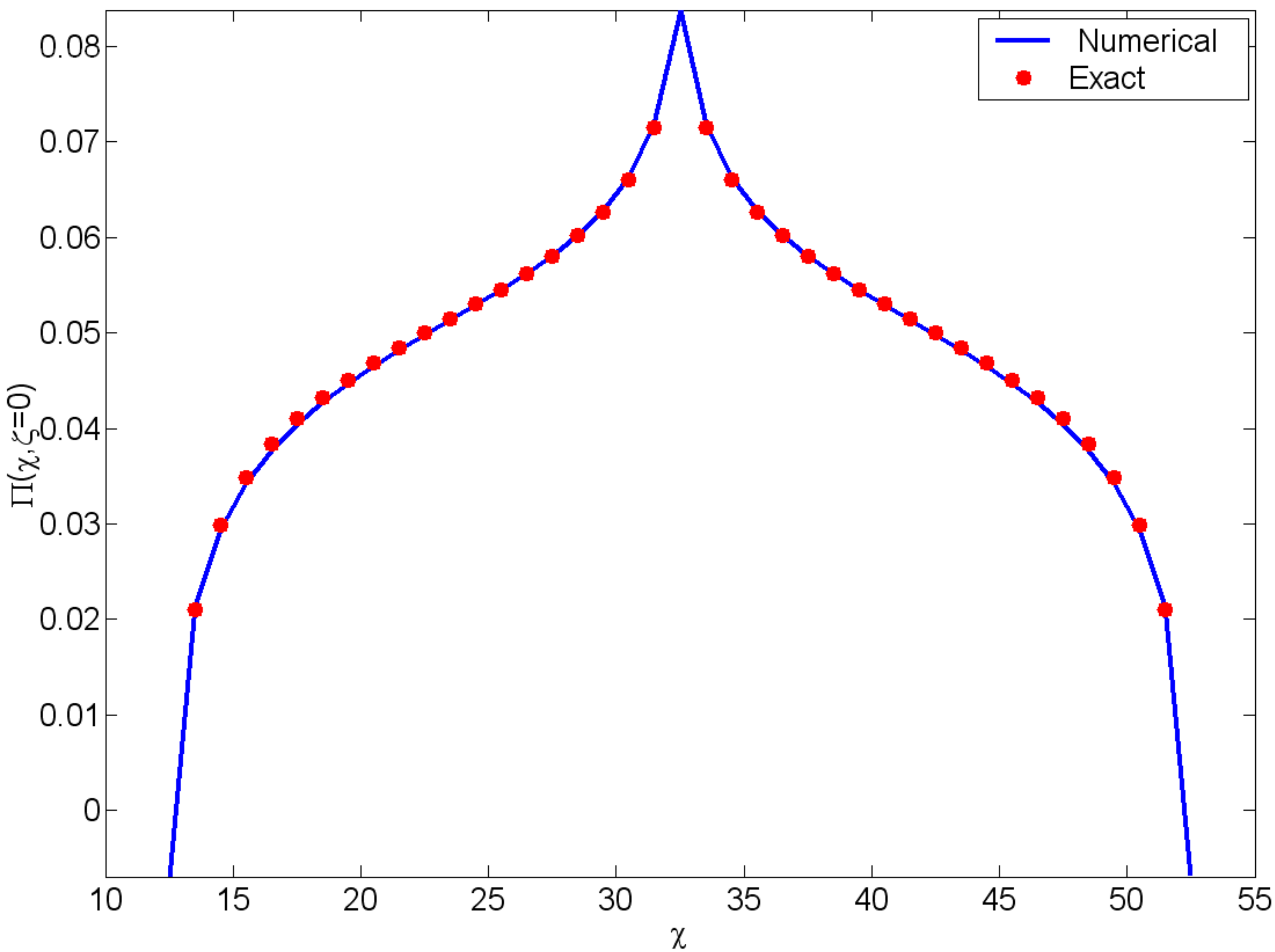
M-vertex radial soln:width & pressure



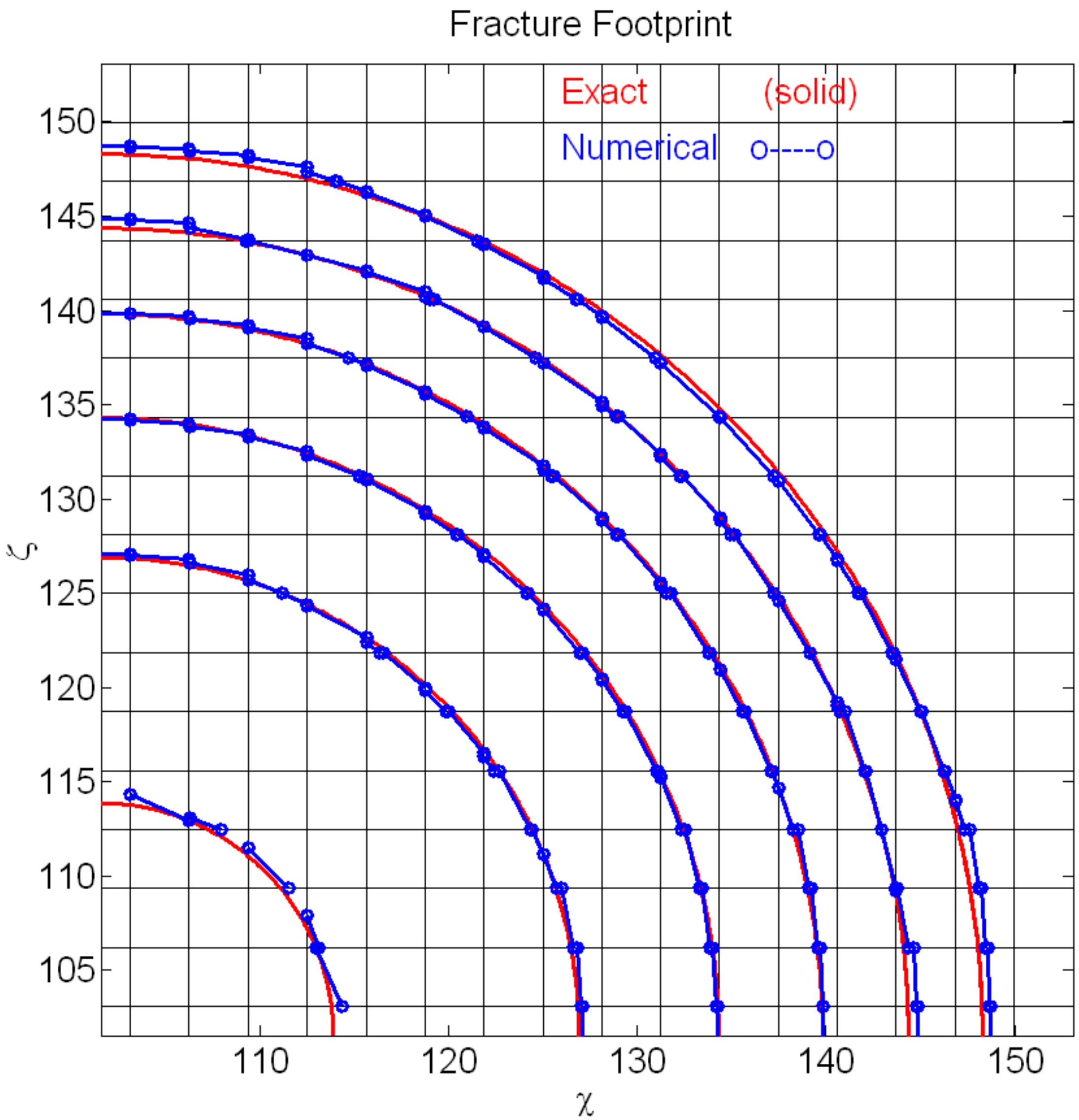
M-vertex width



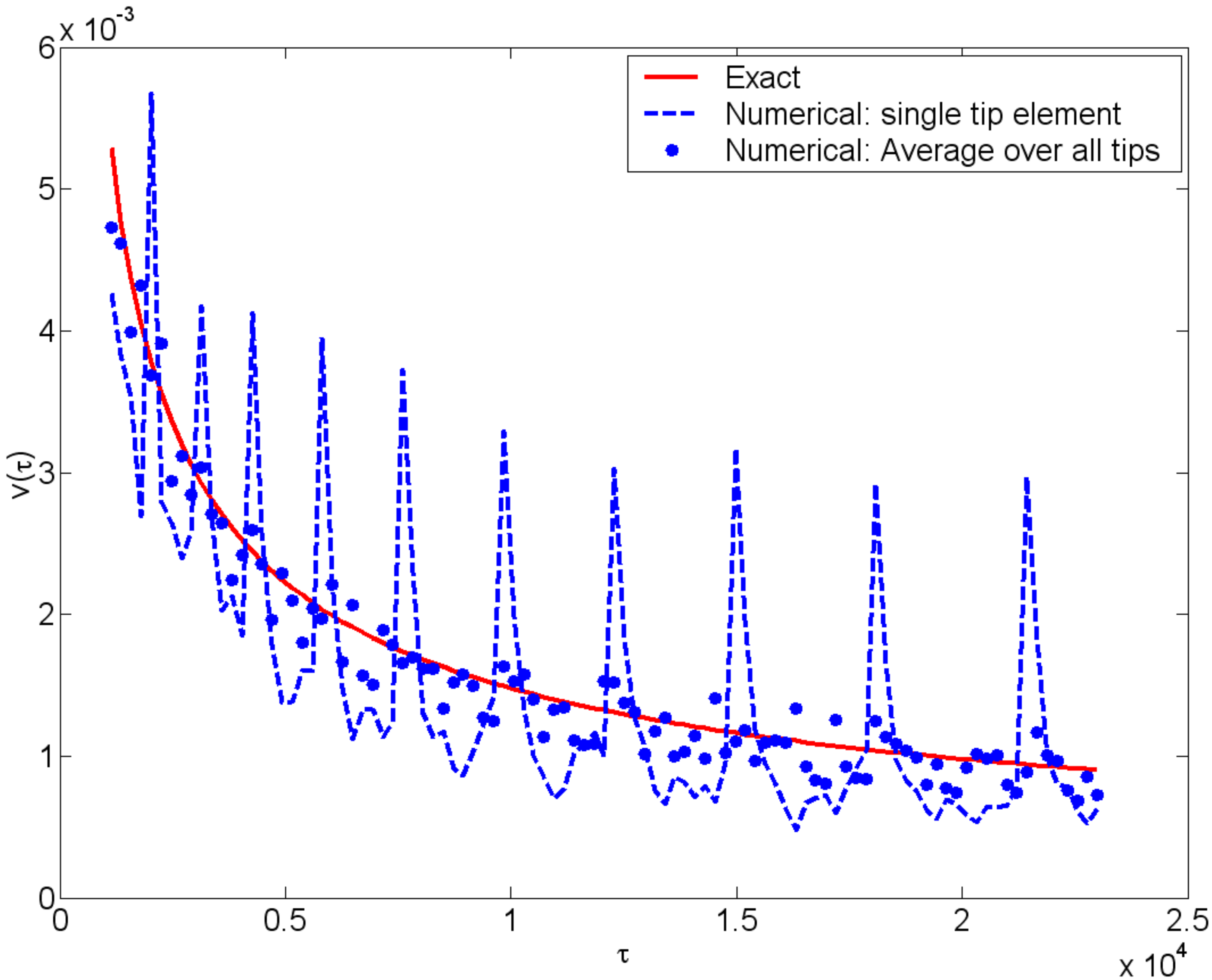
M-vertex pressure



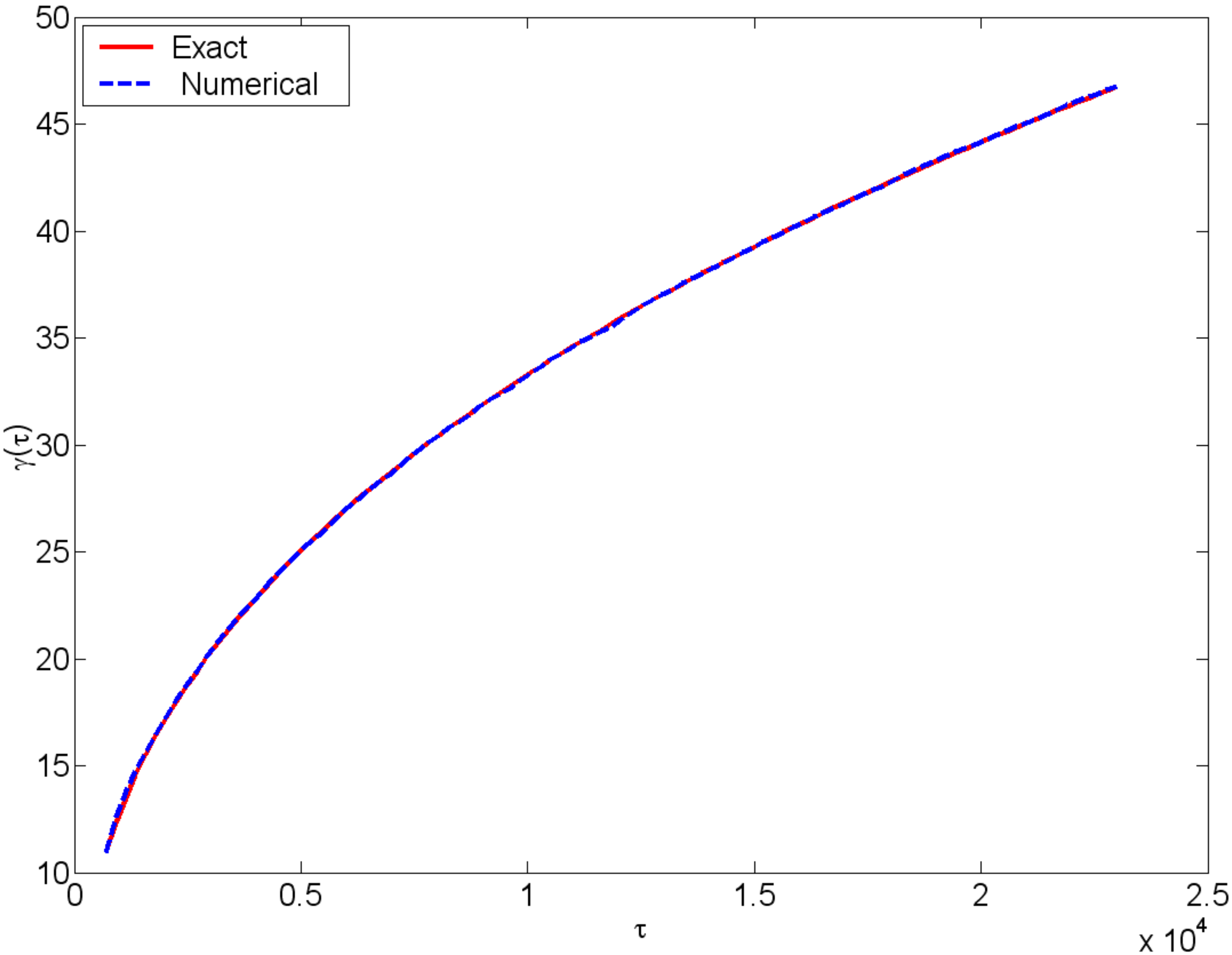
K-vertex radial sol'n – footprints



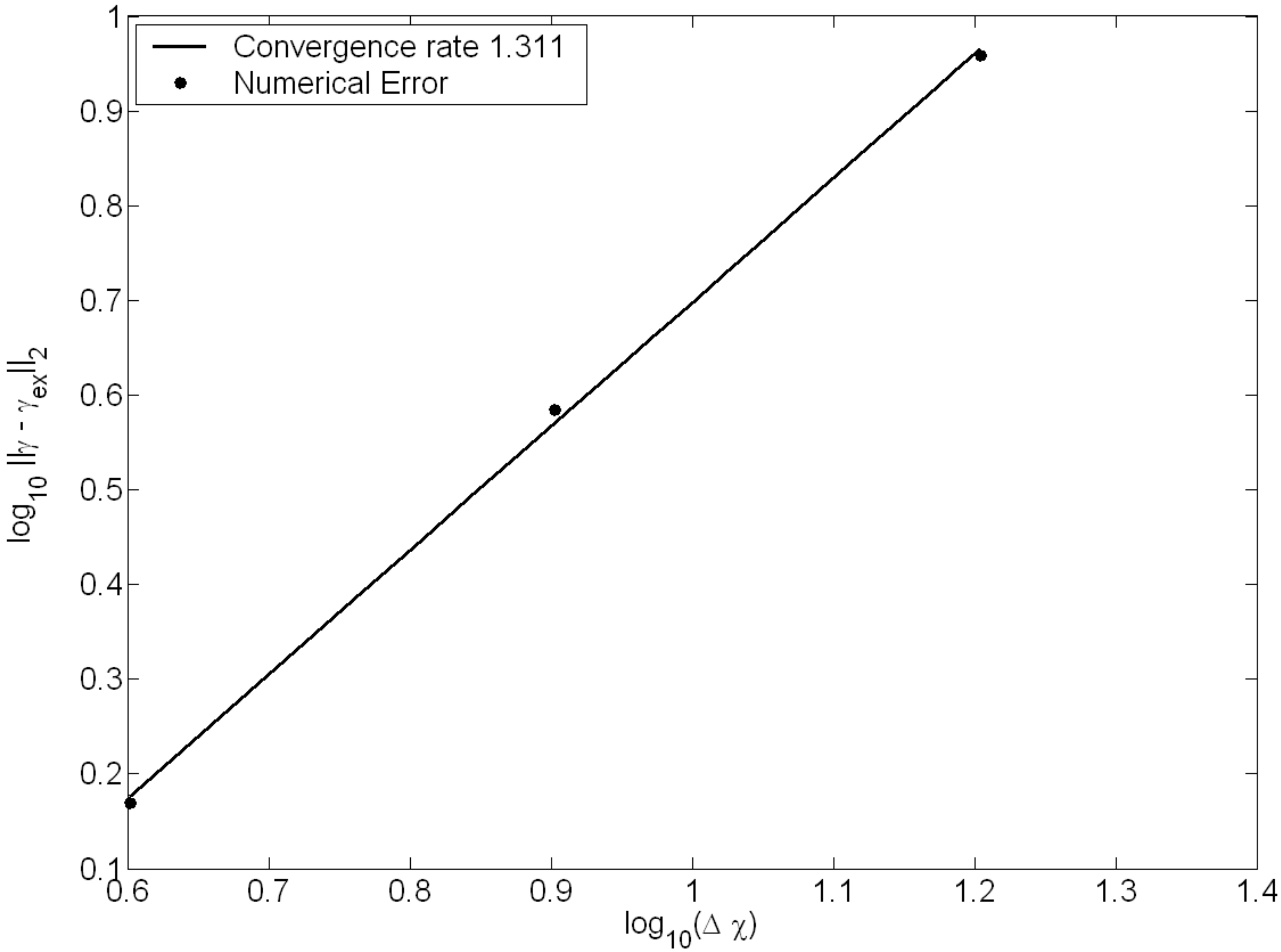
K-Vertex radial solution – front speed



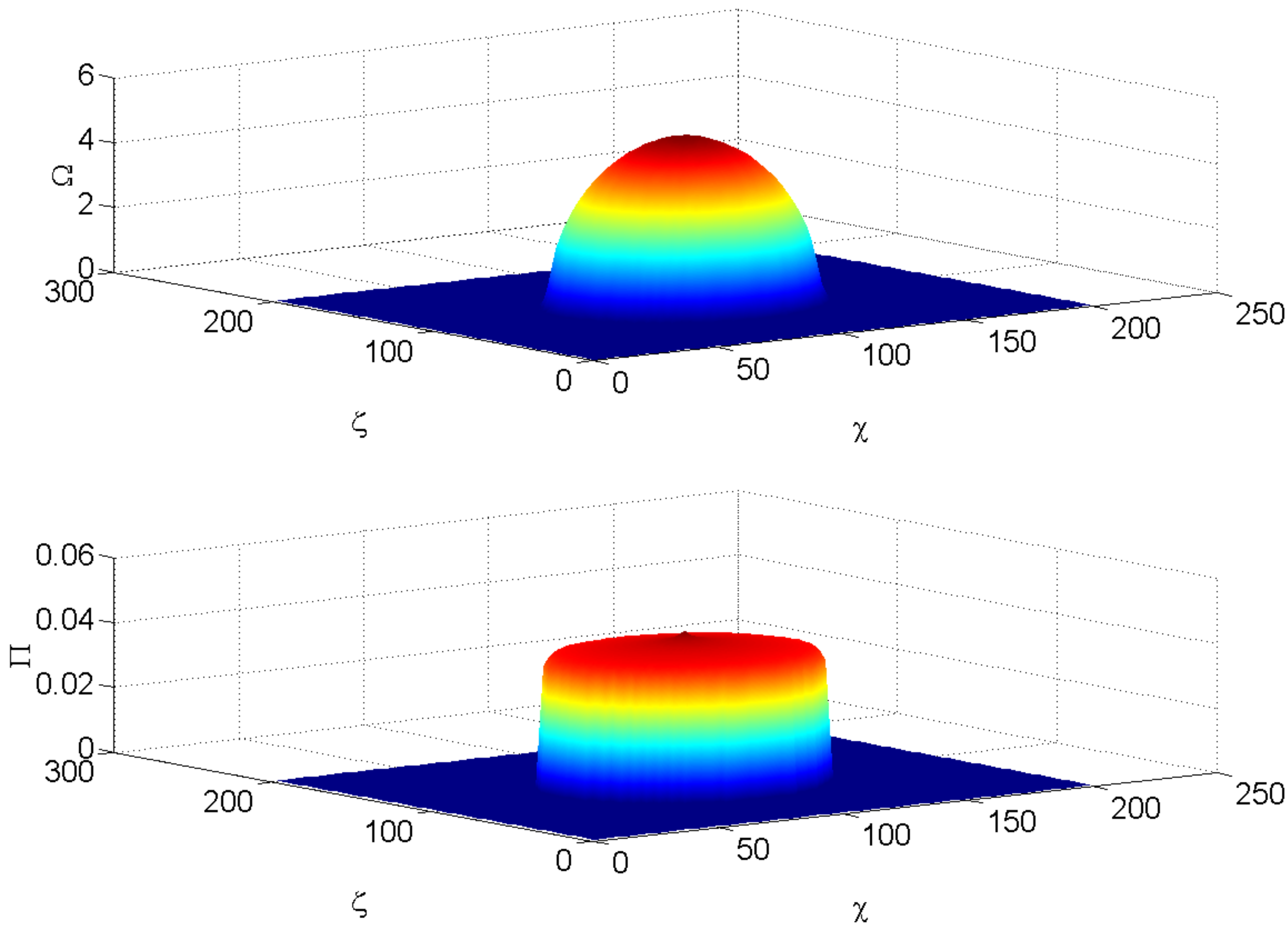
K-vertex radial solution - $\gamma(\tau)$



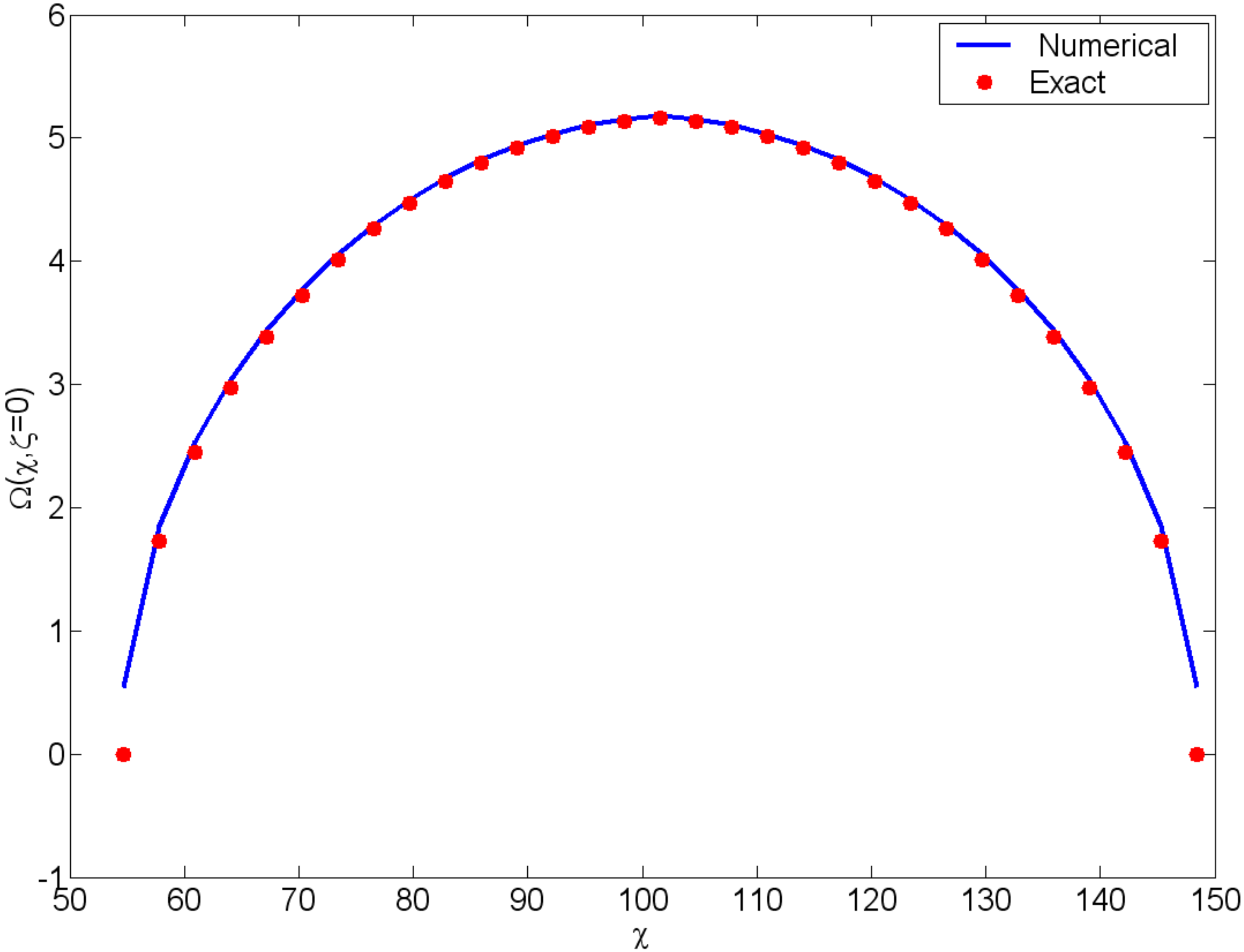
K-vertex – convergence rate for γ



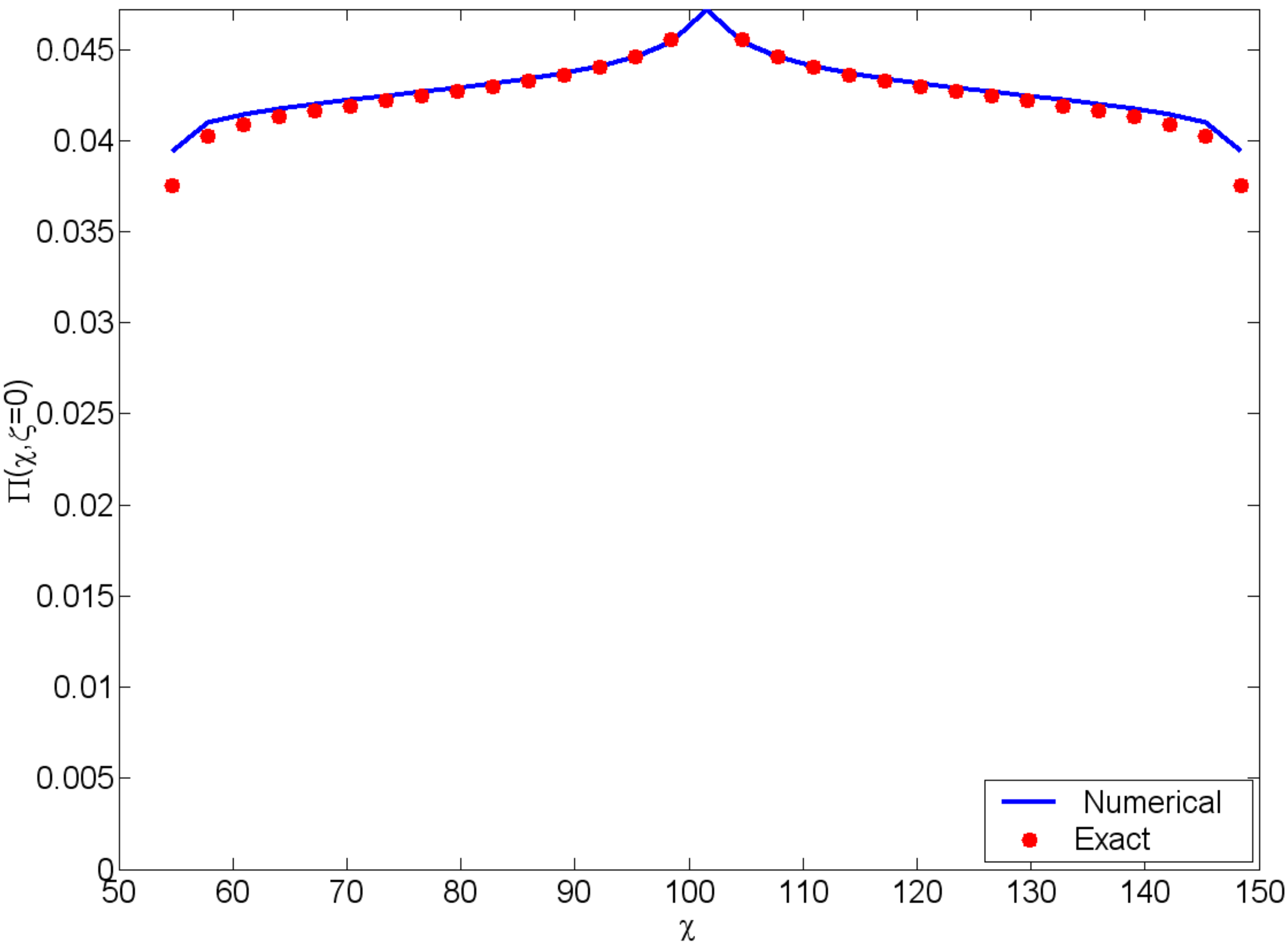
K-vertex radial soln:width & pressure



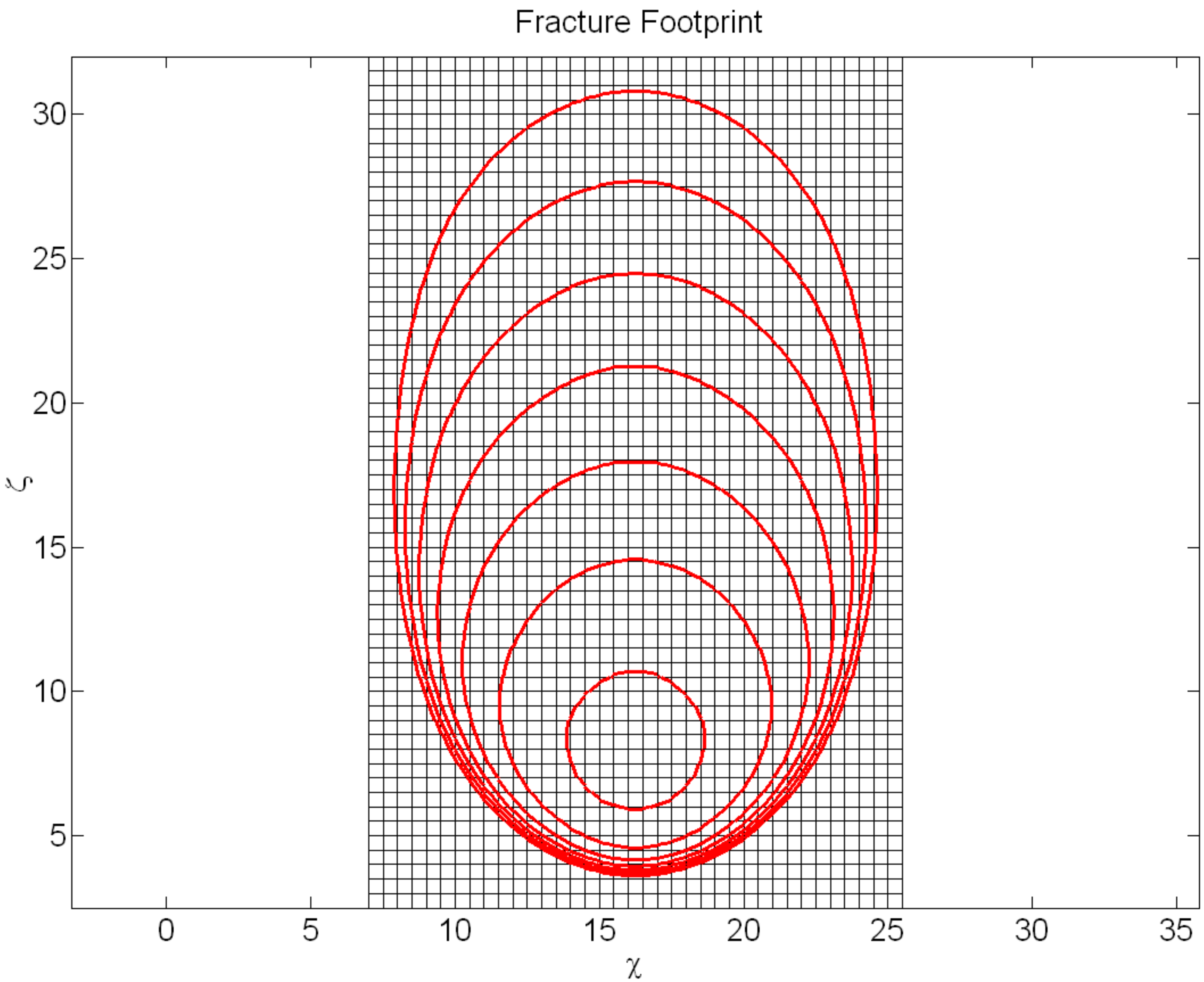
K-vertex width



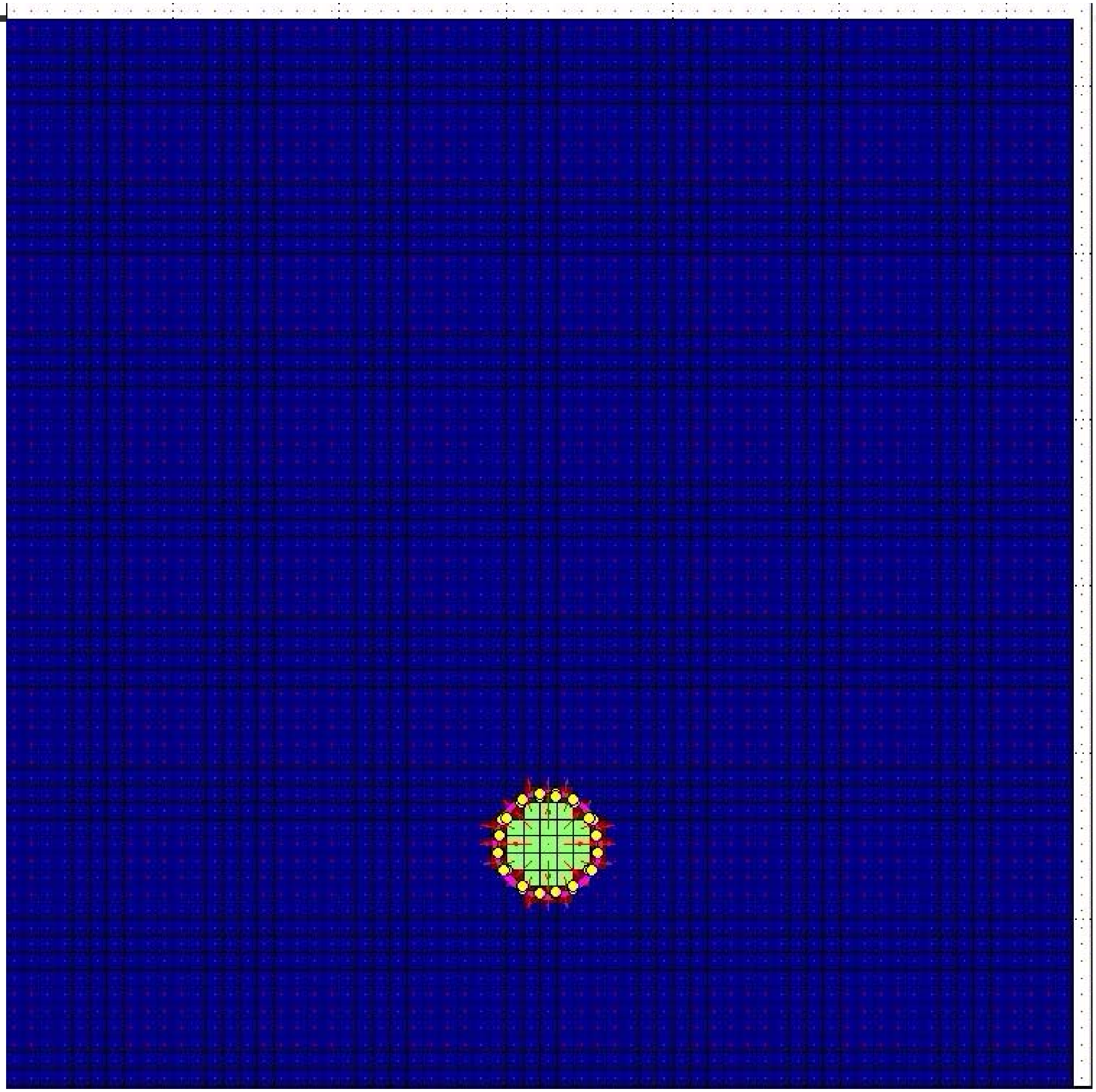
K-vertex Pressure



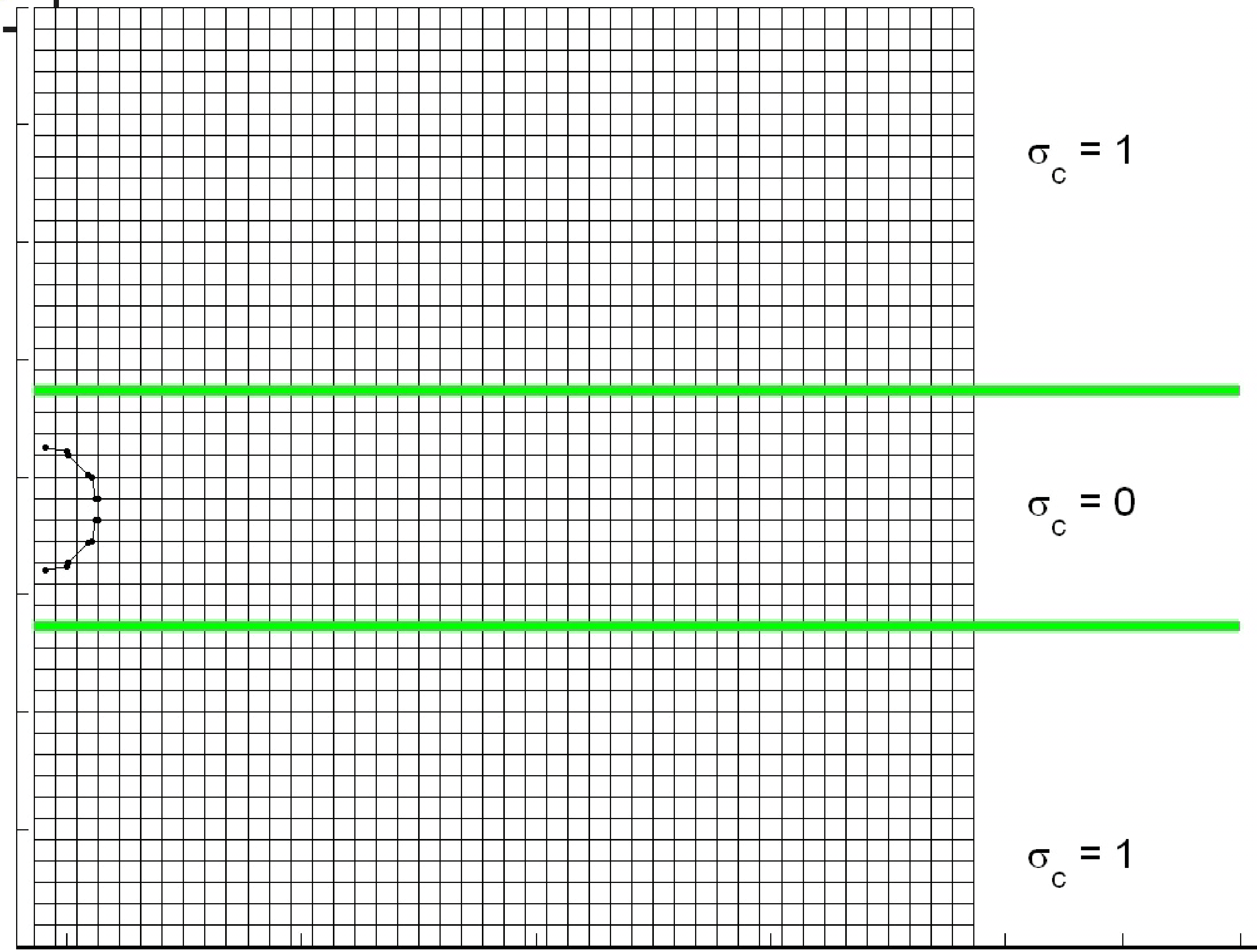
Linear *in-situ* stress field - footprints



Linear *in-situ* stress field - movie



Breakthrough: Low to High Stress



Stress contrast high->low

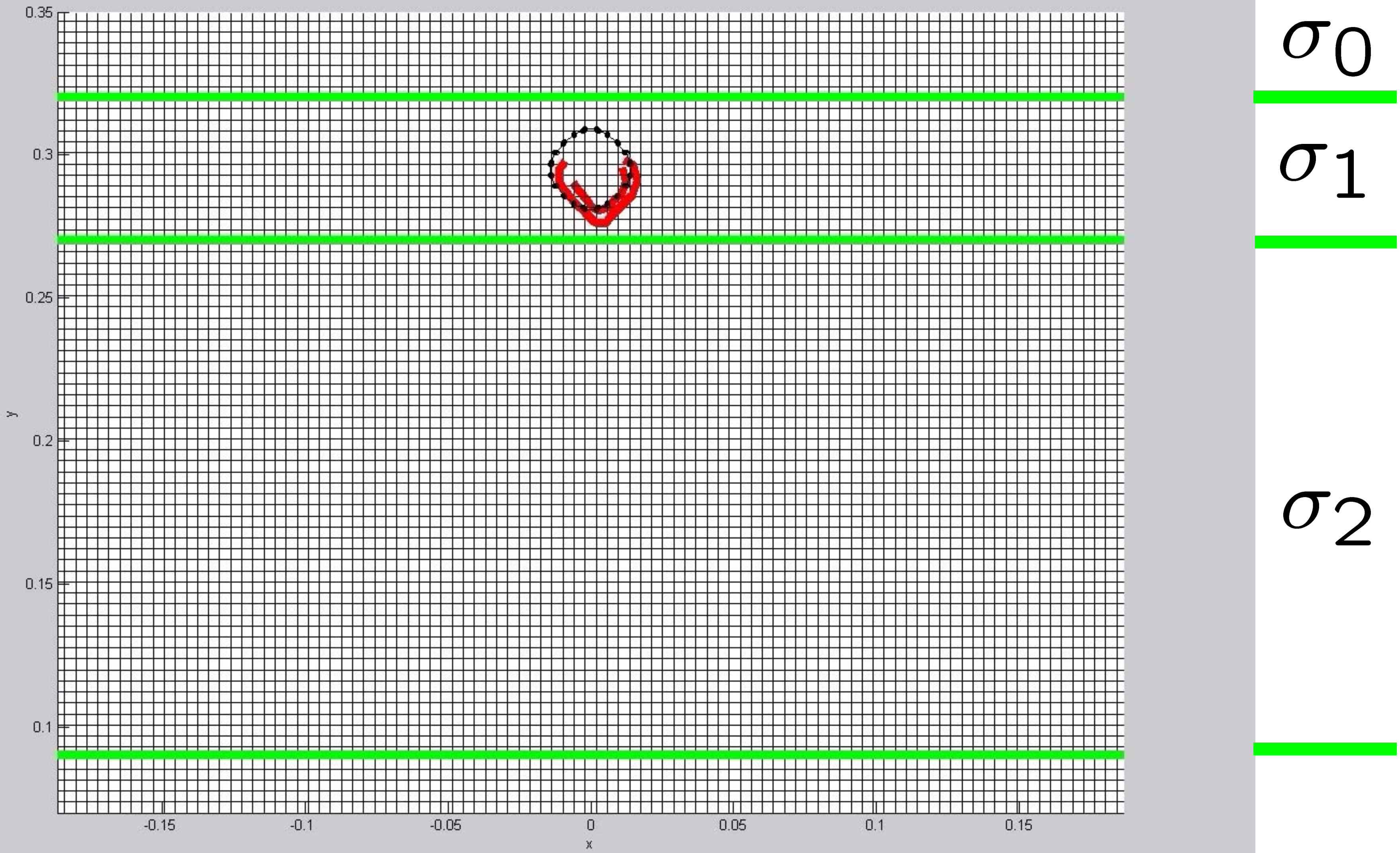
σ_0

σ_1

σ_2

$$\sigma_0 \gg \sigma_1 > \sigma_2$$

Lab Test (Bunger et al 2008)



Concluding Remarks

- Examples of hydraulic fractures
- Governing equations, scaling, asymptotic solutions 1-2D and 2-3D
- Solving The coupled Equations
  - MultiGrid Approach
  - Incomplete LU Factorization of Localized Jacobian

- Solving the free boundary problem
  - Existing methods: VOF, Level Set and K_I matching
  - Tip asymptote and Eikonal boundary value problem
  - Setting the tip volumes using the tip asymptotes
  - Coupled equations

- Numerical examples
  - M-vertex and K-vertex
  - Viscous crack propagating in a variable *in situ* stress
  - Stress jump solutions