Hydraulic Fractures: multiscale phenomena, asymptotic and numerical solutions

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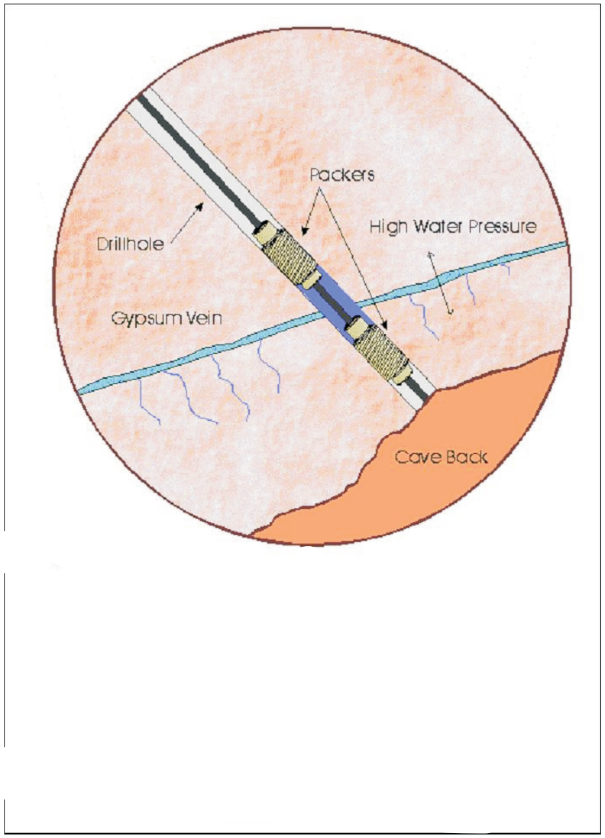
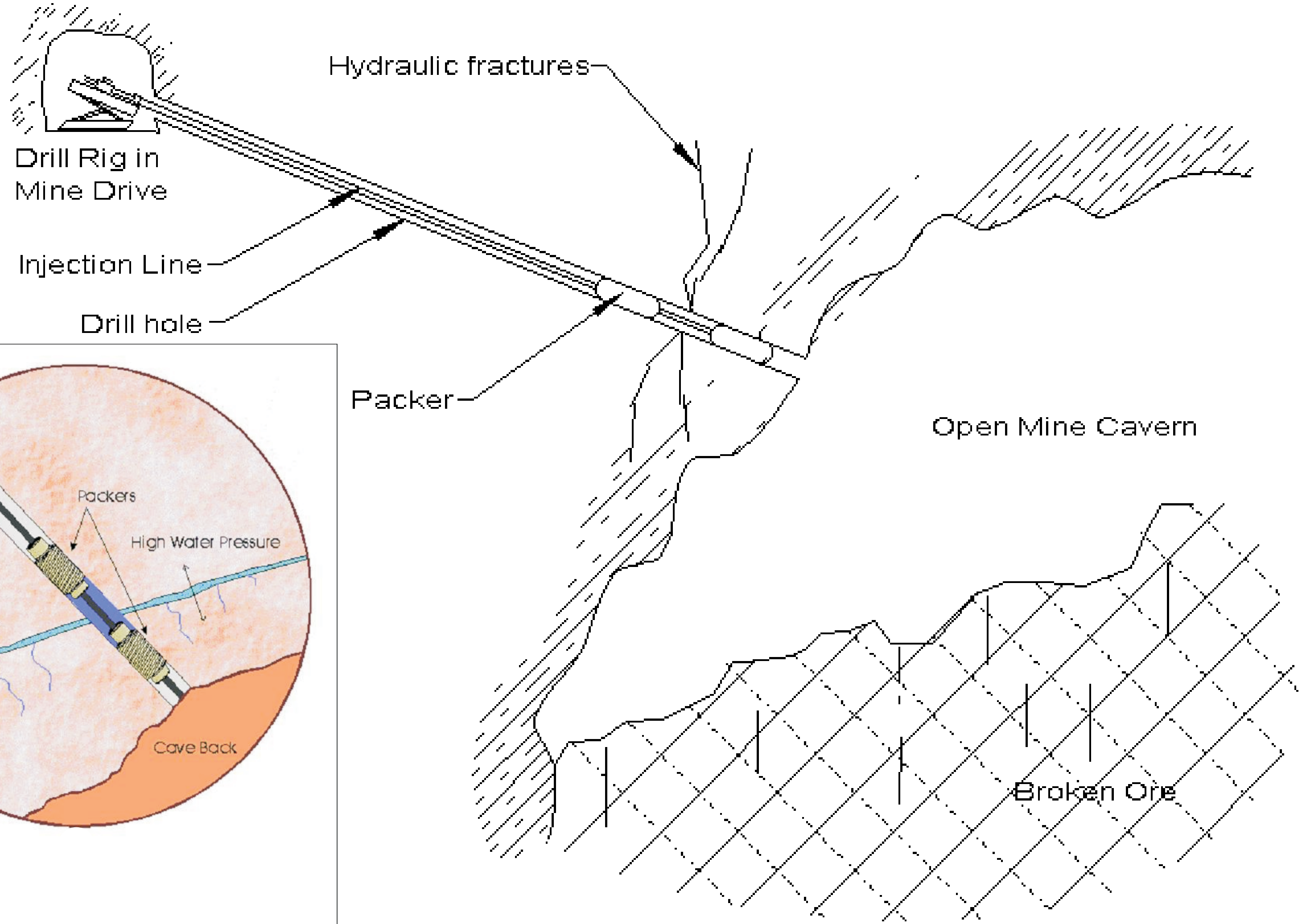
Outline

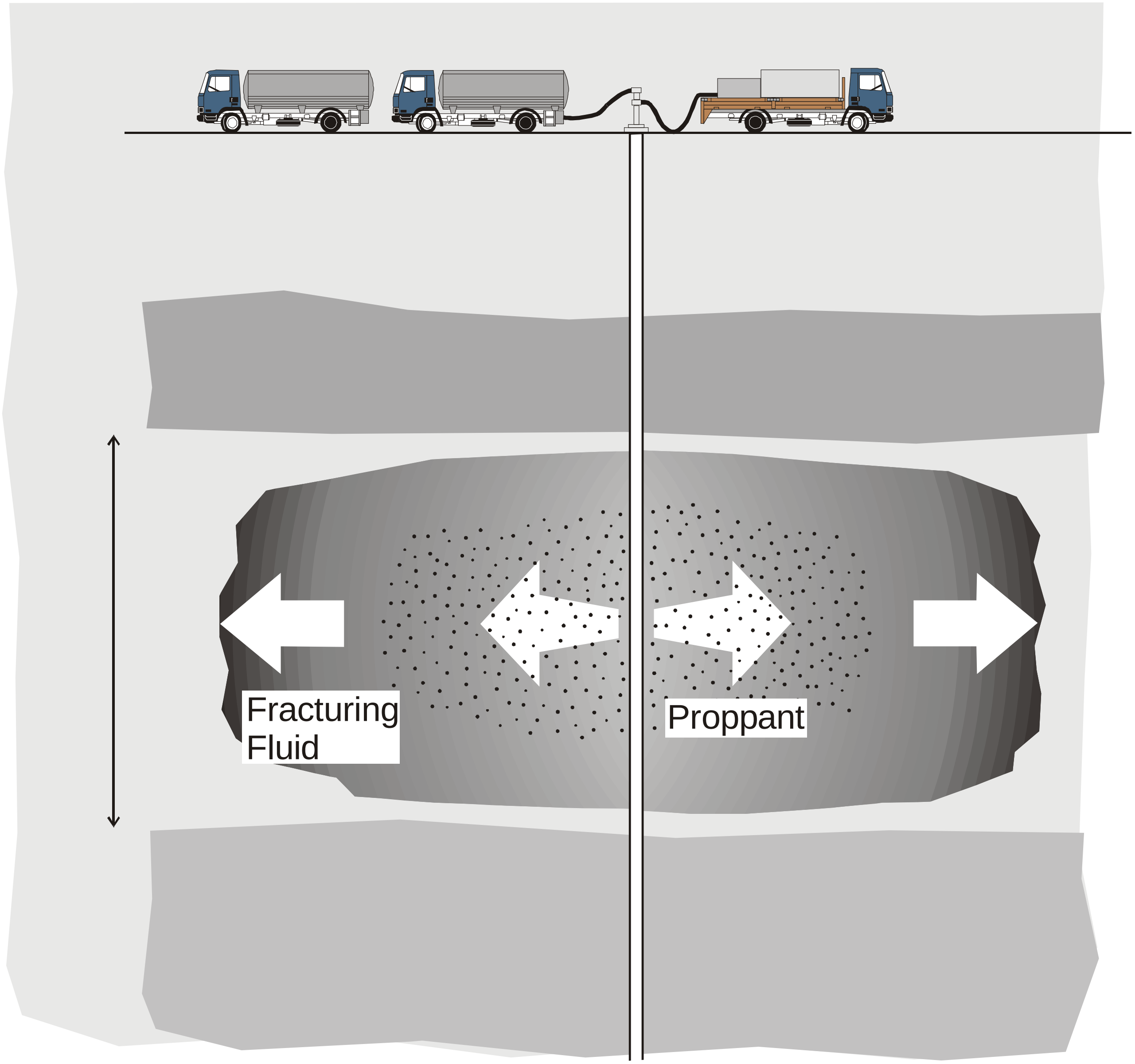
- Examples of hydraulic fractures
- Governing equations, scaling, asymptotic solutions 1-2D and 2-3D
- Solving the free boundary problem
  - Existing methods: VOF, Level Set and K_I matching
  - Tip asymptote and Eikonal boundary value problem
  - Setting the tip volumes using the tip asymptotes
  - Coupled equations
- Numerical examples
  - M-vertex and K-vertex
  - Viscous crack propagating in a variable *in situ* stress
  - Stress jump solutions

HF Examples - block caving





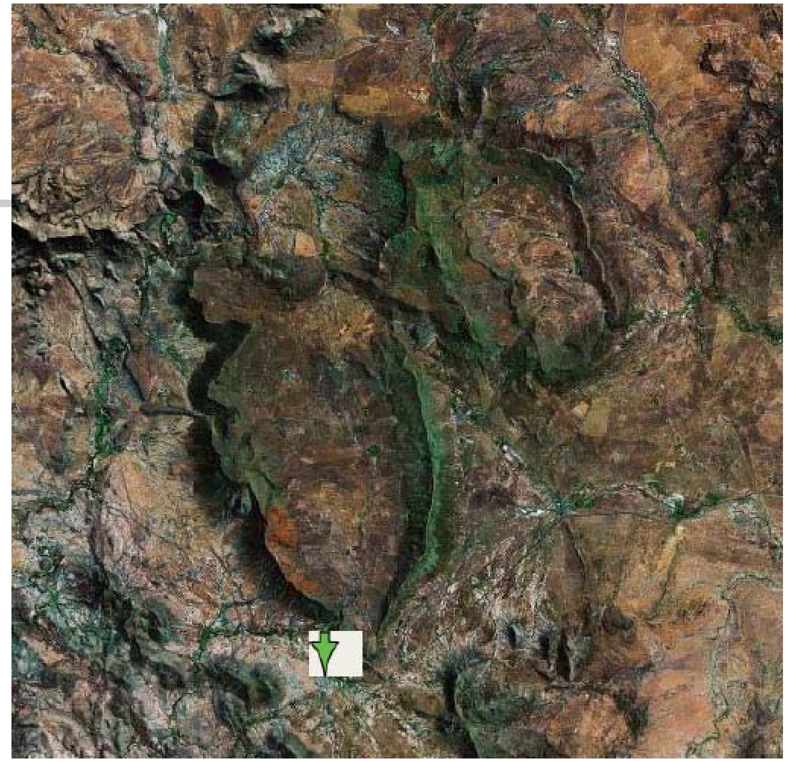
Oil well stimulation



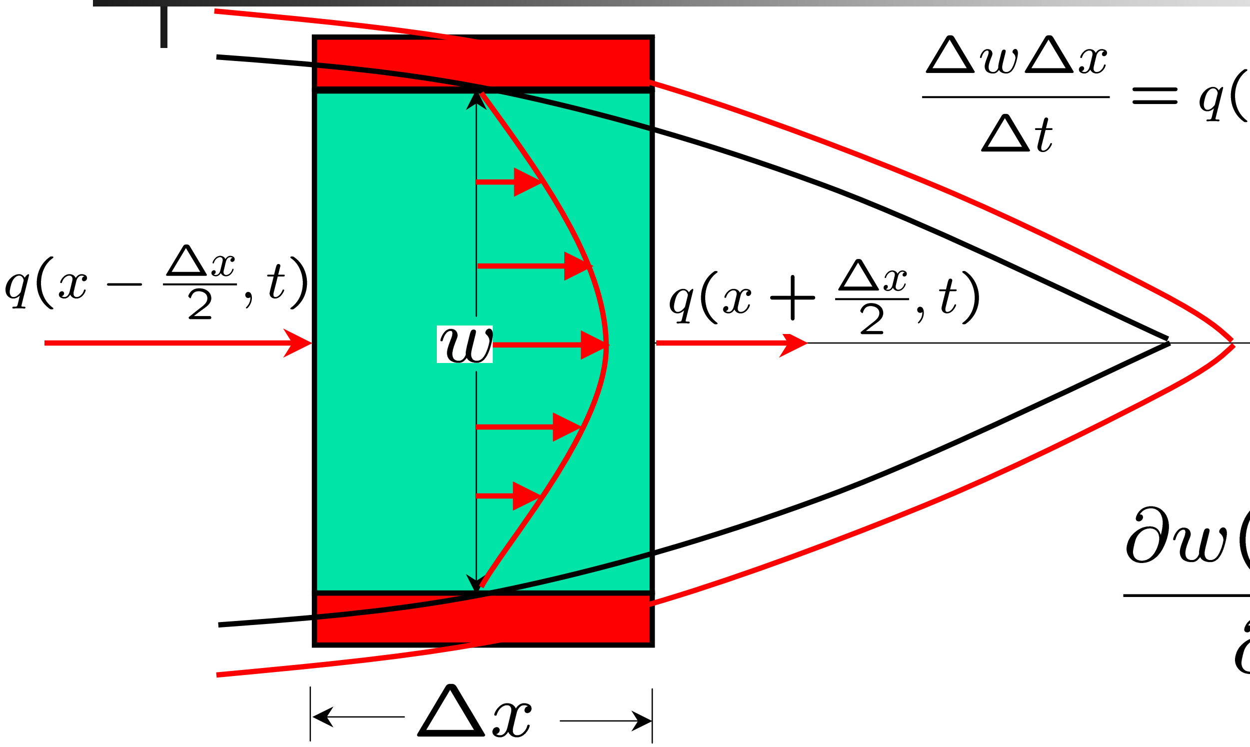
Quarries

Magma flow


Windhoek
Walvisbaai
Botswana
Messina
Pietersburg
Inhambai
Gaborone
Pretória
Witbank
Barberton
Maputo
Lüderitz
Johannesburg
Ermelo
Swaziland
Vryburg
Orkney
Sasolburg
Standerton
Kroonstad
Newcastle
Odendaalsrus
Upington
Harrismith
Dundee
Kimberley
Bloemfontein
Lesotho
Pietermaritzburg
South Africa
Middelburg
Beaufort West
Cradock
Bisho
East London
Paarl
Oudtshoorn
Grahamstown
Robertson
George
Cape Town
Port Elizabeth
Tarkastad



Model EQ 1: Conservation of mass



$$\frac{\Delta w \Delta x}{\Delta t} = q(x - \frac{\Delta x}{2}, t) - q(x + \frac{\Delta x}{2}, t)$$

$$\Downarrow \lim_{\Delta x,\ \Delta t \to 0}$$

$$\frac{\partial w(x,t)}{\partial t} = -\frac{\partial q(x,t)}{\partial x}$$

$$v_x = -\frac{1}{2\mu}\frac{\partial p}{\partial x}\left(\frac{w^2}{4} - y^2\right)$$

$$q(x,t) = -\frac{w^3}{12\mu}\frac{\partial p}{\partial x}$$

$$\boxed{\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(\frac{w^3}{\mu'}\frac{\partial p}{\partial x}\right)}$$

Model EQ 2: The elasticity equation

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2G\epsilon_{ij}$$

$$\sigma_{ij,j} + f_i = 0$$

$$[\sigma_{yy}] = 0, \ [\sigma_{xy}] = 0, \ [u_y] = \delta(x), \ [u_x] = 0$$

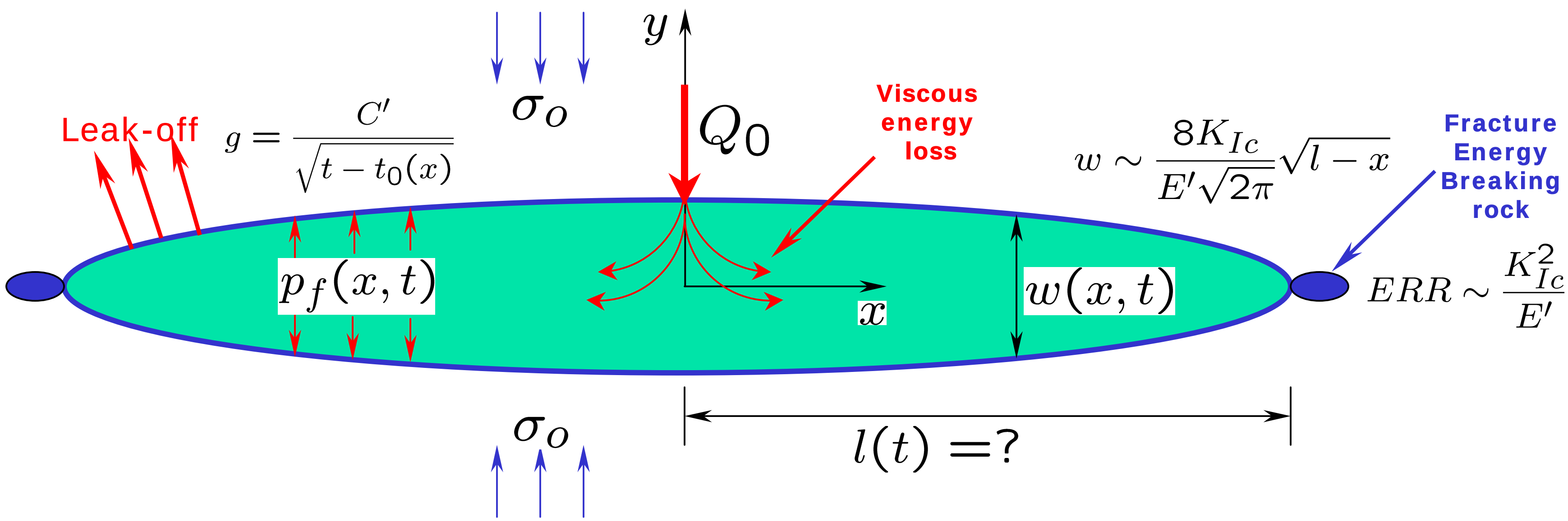
$$\overset{FT_x}{\Rightarrow} \quad \hat{\sigma}_{yy} = -\frac{E'}{4}(k^2|y| + |k|)e^{-|ky|}$$

$$\sigma_{yy} = \frac{E'}{8\pi}\left[\frac{2}{r^2} + \frac{8y^2}{r^4} - \frac{16y^4}{r^6}\right]$$

$$\lim_{y \to 0} \quad \hat{\sigma}_{yy} = -\frac{E'}{4}|k| \quad \sigma_{yy} = \frac{E'}{4\pi}\left[\frac{1}{x^2}\right]$$

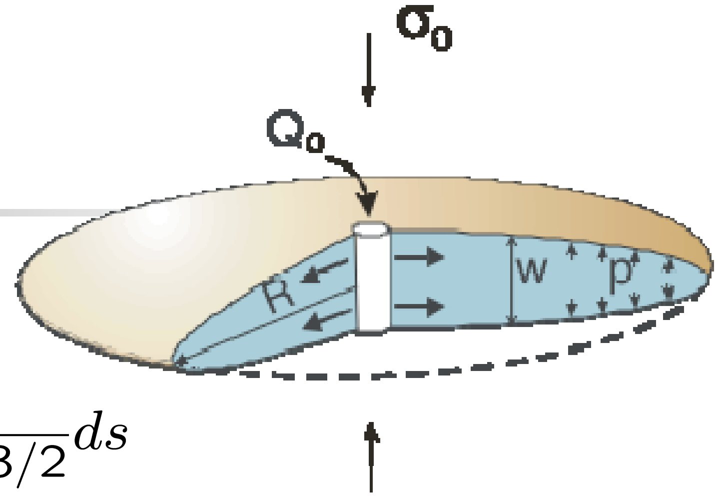
1-2D model and physical processes

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(\frac{w^3}{\mu'}\frac{\partial p_f}{\partial x}\right) + Q_0\delta(x) - g; \quad w^3\frac{\partial p_f}{\partial x}\Big]_{\pm l} = 0$$



$$p_f(x,t) - \sigma_o = -\frac{E'}{4\pi}\int_{-l(t)}^{l(t)} \frac{\partial w}{\partial \xi}\frac{d\xi}{\xi - x}; \quad w(\pm l, t) = 0$$

2-3D HF Equations



- Elasticity (non-locality)

$$p_f - \sigma_o = -\frac{E'}{8\pi}\int_{S(t)} \frac{w(x',y',t)}{\left[(x'-x)^2+(y'-y)^2\right]^{3/2}}ds$$

$$\Rightarrow \quad p_f - \sigma_o = Cw$$

- Lubrication (non-linearity)

$$\frac{\partial w}{\partial t} = \nabla \cdot \left(\frac{w^3}{\mu'}\nabla p_f\right) - \frac{C'}{\sqrt{t - t_0(x,y)}} + Q(t)\delta(x,y)$$

$$\Rightarrow \quad \frac{\Delta w}{\Delta t} = A(w)p_f + S$$

$$\frac{\partial w}{\partial t} = A(w)Cw + S$$

- Boundary conditions at moving front (free boundary)

$$\lim_{s\to 0}\frac{w}{s^{1/2}} = \frac{K'}{E'}, \qquad \lim_{s\to 0} w^3\frac{\partial p_f}{\partial s} = 0 \qquad v = \lim_{s\to 0} -\frac{1}{\mu'}w^2\nabla p$$

Scaling

- Rescale the physical quantities as follows:

$$x = L_*\chi, \qquad y = L_*\zeta, \qquad t = T_*\tau$$
$$w = W_*\Omega, \qquad p_f = P_*\Pi_f, \qquad Q = Q_0\psi(\tau), \ \sigma = \sigma_0\phi(\chi,\zeta)$$

- Dimensionless groups:

$$\mathcal{G}_c = \frac{C' L_*^2}{Q_0 T_*^{1/2}}, \quad \mathcal{G}_e = \frac{L_* P_*}{E' W_*}, \quad \mathcal{G}_m = \frac{\mu' Q_0}{P_* W_*^3}, \quad \mathcal{G}_k = \frac{K' L_*^{1/2}}{E' W_*}, \quad \mathcal{G}_v = \frac{Q_0 T_*}{L_*^2 W_*}$$

- Scaled Equations

$$\Pi(\chi,\zeta) = -\frac{1}{8\pi \boxed{\mathcal{G}_e}} \int_S \frac{\Omega(\chi',\zeta')}{[(\chi'-\chi)^2 + (\zeta'-\zeta)^2]^{\frac{3}{2}}} d\chi' d\zeta'$$

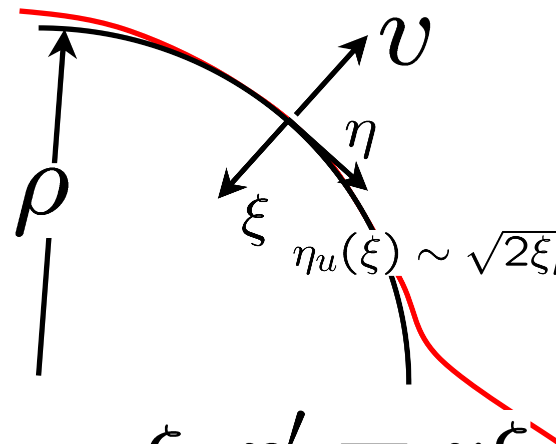
$$\frac{1}{\boxed{\mathcal{G}_v}} \frac{\partial \Omega}{\partial \tau} = \frac{1}{\boxed{\mathcal{G}_m}} \nabla \cdot \left(\Omega^3 \nabla \Pi\right) - \frac{\boxed{\mathcal{G}_c}}{\sqrt{\tau - \tau_0}} + \delta(\chi,\zeta)\psi(\tau)$$

$$\lim_{\xi \to 0} \frac{\Omega}{\xi^{1/2}} = \boxed{\mathcal{G}_k}, \qquad \lim_{\xi \to 0} \Omega^3 \frac{\partial \Pi_f}{\partial \xi} = 0, \qquad v = \lim_{\xi \to 0} \Omega^2 \frac{\partial \Pi_f}{\partial \xi}$$

Reduced eq near a smooth front

$$I(\xi,\eta) = \int_0^a i(\xi,\xi')d\xi'$$

$$(\xi-\rho)^2+\eta^2=\rho^2$$



$$i(\xi,\xi') = \int_{\eta_l(\xi')}^{\eta_u(\xi')} \frac{\Omega(\xi',\eta')}{[(\xi'-\xi)^2+(\eta'-\eta)^2]^{\frac{3}{2}}}d\eta'$$

$$\sim \xi^{-2} \int_{\eta_l(u\xi)/\xi}^{\eta_u(u\xi)/\xi} \frac{\Omega(u\xi,v\xi)}{[(u-1)^2+v^2]^{\frac{3}{2}}}dv$$

$$\xi' = u\xi \quad \eta' = v\xi$$

$$\sim \frac{\Omega_s(u\xi)}{\xi^2} \int_{-\sqrt{2u\rho/\xi}}^{\sqrt{2u\rho/\xi}} \frac{dv}{[(u-1)^2+v^2]^{\frac{3}{2}}}$$

$$\xi/\rho \ll 1 \quad \& \quad \eta/\rho \ll 1$$

$$\eta_l(\xi) \sim -\sqrt{2\xi\rho}, \; \eta_u(\xi) \sim \sqrt{2\xi\rho}$$

$$\Omega(u\xi,v\xi) \overset{\xi\to 0}{\sim} \Omega_s(u\xi)$$

$$\sim \frac{2\Omega_s(u\xi)}{\xi^2(u-1)^2}$$

$$I(\xi,0) = 2\int_0^{a/\xi} \Omega_s(u\xi)\frac{du}{\xi^2(u-1)^2} \overset{\xi\to 0}{\sim} 2\int_0^{\infty} \frac{d\Omega_s(\xi')}{d\xi'}\frac{d\xi'}{(\xi'-\xi)}$$

Tip asymptotic behaviour $\Omega = A\xi^{\alpha}$

- Elasticity:

$$\Pi = -\frac{\mathcal{G}_e}{4\pi}\int_0^\infty \frac{\mathrm{d}\Omega}{\mathrm{d}\xi'}\frac{\mathrm{d}\xi'}{(\xi'-\xi)} = -\frac{\mathcal{G}_e A}{4\pi}\int_0^\infty \frac{s^{\alpha-1}\mathrm{d}s}{(s-\xi)} = \frac{\mathcal{G}_e A}{4\pi}\pi\cot(\pi\alpha)\xi^{\alpha-1}$$

- Lubrication: $\Omega(\hat{\xi},\hat{\eta},\tau) \approx \Omega(V\tau - \hat{\xi})$ $\quad \tau - \tau_0(\xi) \approx \frac{\xi}{V}$

$$\frac{\Omega^3}{\mathcal{G}_m}\frac{\mathrm{d}\Pi}{\mathrm{d}\xi} = \frac{1}{\mathcal{G}_v}V\Omega + 2\mathcal{G}_c V^{1/2}\xi^{1/2}$$

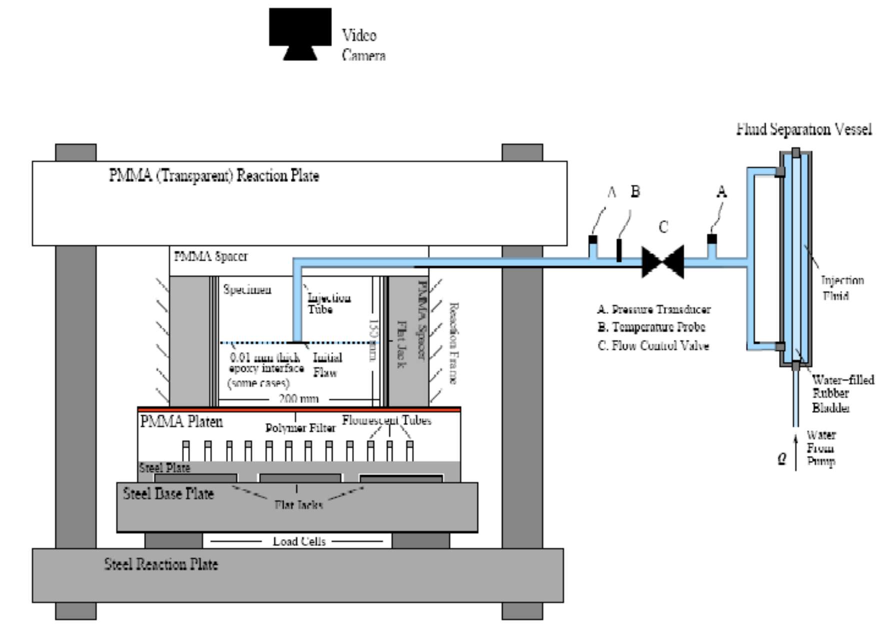
$\xi^{4\alpha-2}$ $\quad \xi^{\alpha}$ $\quad \xi^{1/2}$

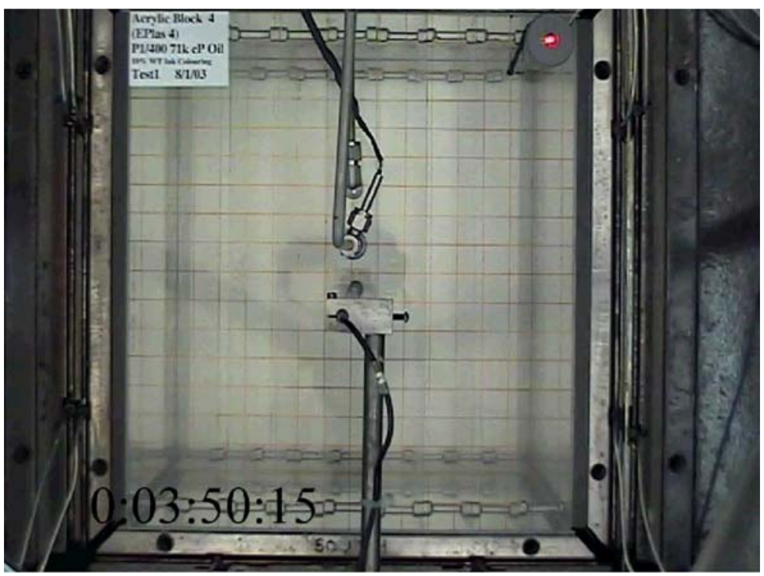
$$K\ vertex:\quad \Omega \sim \xi^{1/2} \Rightarrow \Pi \sim log\xi$$

$$M\ vertex: 4\alpha - 2 = \alpha \ \Rightarrow \Omega \sim \beta_{m0}V^{1/3}\xi^{2/3}$$

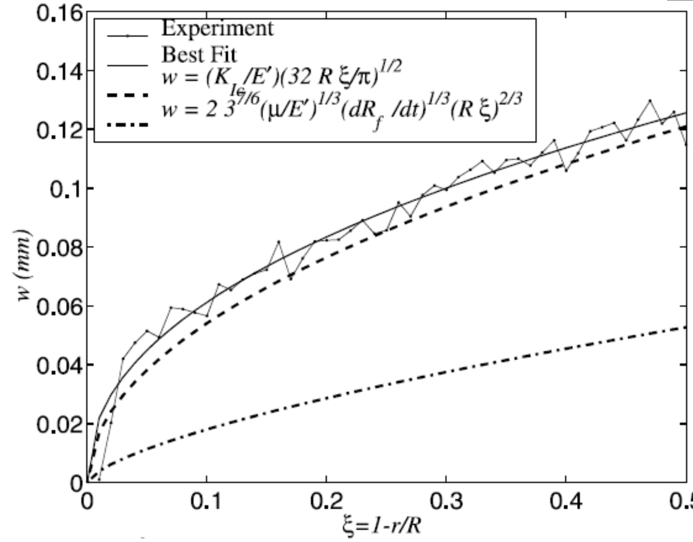
$$\tilde{M}\ vertex: 4\alpha - 2 = 1/2 \ \Rightarrow \Omega \sim \beta_{\tilde{m}0}V^{1/8}\xi^{5/8}$$

HF experiment (*Bunger & Jeffrey* CSIRO)

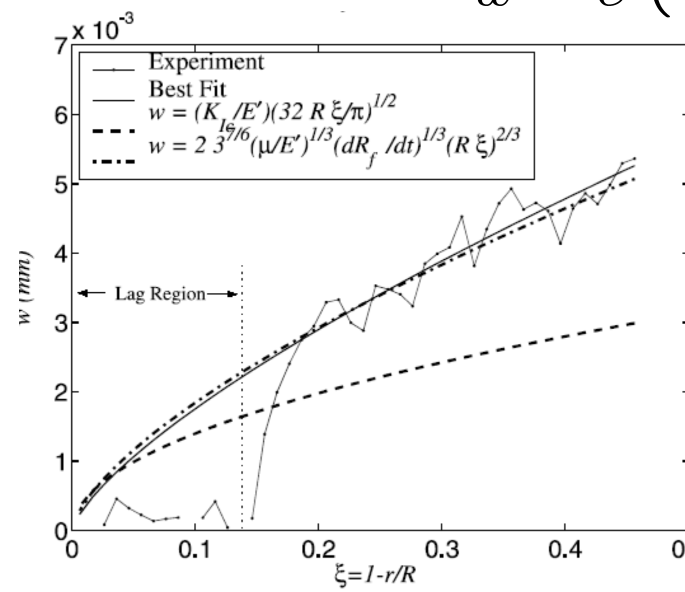




$$w \sim \frac{K'}{E'} s^{1/2}$$



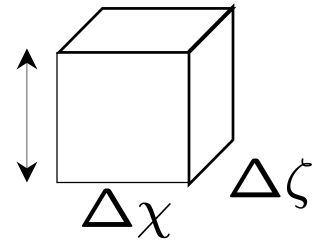
$$w \sim c \left(\frac{\mu' v}{E'}\right)^{1/3} s^{2/3}$$



Discretizing the Elasticity Equation

- Piecewise constant DD:

$$\Omega(\chi,\zeta) \approx \sum_{m,n} \Omega_{m,n} H_{m,n}(\chi,\zeta), \quad H_{m,n}(\chi,\zeta) = 1$$



- Discrete Elasticity Equation:

$$\Pi(\chi,\zeta) = -\frac{1}{8\pi}\sum_{m,n}\Omega_{m,n}\int_S \frac{H_{m,n}(\chi',\zeta')}{[(\chi'-\chi)^2+(\zeta'-\zeta)^2]^{\frac{3}{2}}}d\chi' d\zeta'$$

$$= -\frac{1}{8\pi}\sum_{m,n}\Omega_{m,n}\frac{r(\chi-\chi_m,\zeta-\zeta_n)}{(\chi-\chi_m)(\zeta-\zeta_n)}$$

- Operator form:

$$\Pi = C\Omega$$

Discretizing the Fluid Flow Equation

- The fluid flow equation: $\frac{\partial \Omega}{\partial \tau} = \nabla \bullet (\Omega^3 \nabla \Pi) + \psi(\tau)\delta$

- Integrate over cell: $\frac{\partial}{\partial \tau} \int_{S_e} \Omega dS = \int_{\partial S_e} \Omega^3 \frac{\partial \Pi}{\partial n} dC + \psi(\tau)\delta_e$

- Approximate the integrals:

$$\Delta\chi\Delta\zeta \left(\frac{\Delta\Omega_{i,j}}{\Delta\tau} \right) = \Delta\zeta(J_{i+1/2,j} - J_{i-1/2,j}) + \Delta\chi(J_{i,j+1/2} - J_{i,j-1/2}) + \Psi\delta_{i,j}$$

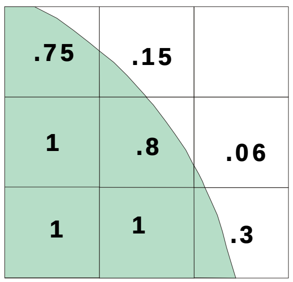
where $J_{i-1/2,j} = \Omega^3_{i-1/2,j} \left(\frac{\Pi_{i,j} - \Pi_{i-1,j}}{\Delta\chi} \right)$

- Operator form:

$$\frac{\Delta\Omega}{\Delta\tau} = A(\Omega)\Pi + \Psi\delta$$

How do we find the fracture front?

- VOF method



$$\frac{\partial \chi}{\partial t} + \mathbf{v} \cdot \nabla \chi = 0$$

$$\nabla \cdot \mathbf{v} = 0$$

$$A \frac{\partial}{\partial t}\left(\frac{1}{A} \int_A \chi dA \right) = -\int_A \nabla \cdot (\chi \mathbf{v})\, dA$$

$$\frac{\partial F}{\partial t} = -\frac{1}{A} \int_{\partial A} \chi v_n dl$$

Problem: we need an accurate

- Level set method

  -move contours of a surface

$$\phi(x, y, t) = 0$$

  - differentiate

$$\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi = 0$$

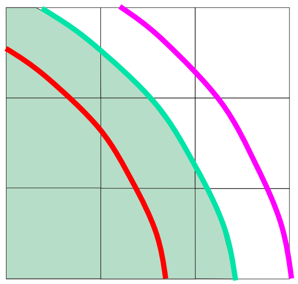
$$\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \mathbf{n} |\nabla \phi| = 0$$

$$\frac{\partial \phi}{\partial t} + v_n |\nabla \phi| = 0$$

$$v = \lim_{\xi \to 0} \Omega^2 \frac{\partial \Pi_f}{\partial \xi}$$

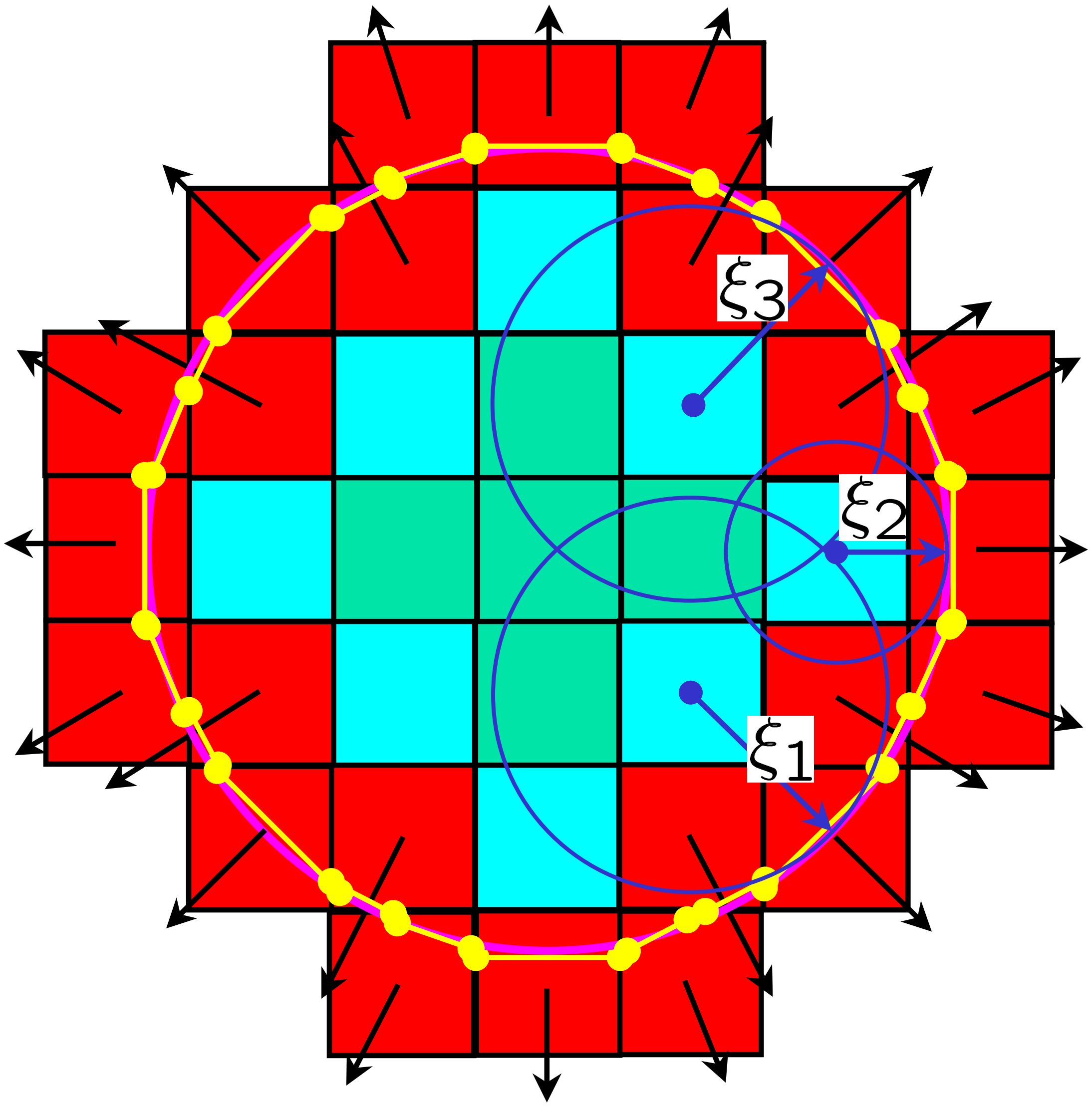
Why not use stress intensity factor?

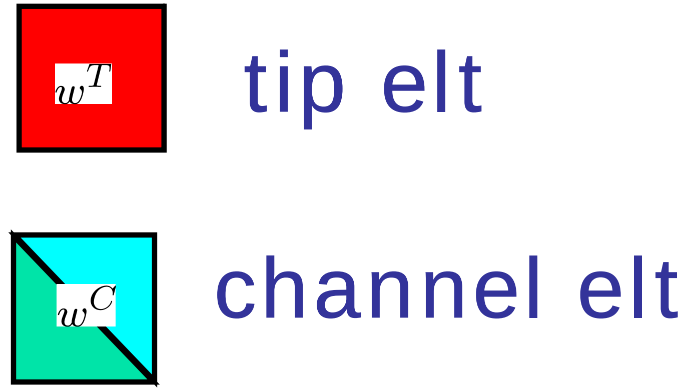
- Move front till SIF = K_{Ic}
  - Find the SIF numerically
  - if $K_I > K_{Ic}$ move forward
  - if $K_I < K_{Ic}$ move back
  - if $K_I = K_{Ic}$ stay



- Problems
  - Accurate calculation of K_I
  - How far do you move?
  - What about viscous and leak-off dominated propagation regimes

Divide elements into tip & channel





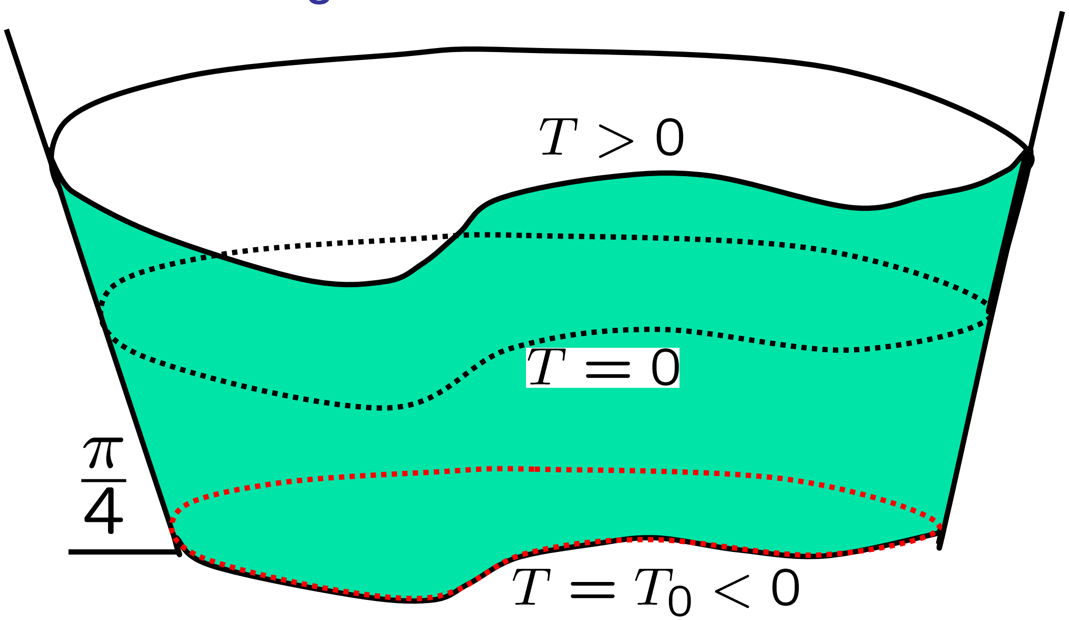
At survey points

$$\Omega = \xi^{1/2}$$

$$\xi = \Omega^2$$

Eikonal solution surfaces

The signed distance function



$$|\nabla T| = 1$$
$$T = T_0$$

Solving the Eikonal BVP

- The Eikonal Equation $|\nabla \mathcal{T}(\chi,\zeta)| = 1$
- First order discretization

$$\left[\max\left(\frac{\mathcal{T}_{ij}-\mathcal{T}_{i-1j}}{\Delta\chi}, \frac{\mathcal{T}_{ij}-\mathcal{T}_{i+1j}}{\Delta\chi}, 0\right)\right]^2 + \left[\max\left(\frac{\mathcal{T}_{ij}-\mathcal{T}_{ij-1}}{\Delta\zeta}, \frac{\mathcal{T}_{ij}-\mathcal{T}_{ij+1}}{\Delta\zeta}, 0\right)\right]^2 = 1$$

- Reduction to a quadratic

Let $\mathcal{T}_1 = \min\left(\mathcal{T}_{i-1j}, \mathcal{T}_{i+1j}\right)$ and $\mathcal{T}_2 = \min\left(\mathcal{T}_{ij-1}, \mathcal{T}_{ij+1}\right)$

$$\left[\max\left(\frac{\mathcal{T}-\mathcal{T}_1}{\Delta\chi}, 0\right)\right]^2 + \left[\max\left(\frac{\mathcal{T}-\mathcal{T}_2}{\Delta\zeta}, 0\right)\right]^2 = 1$$

- Solution to the quadratic equation

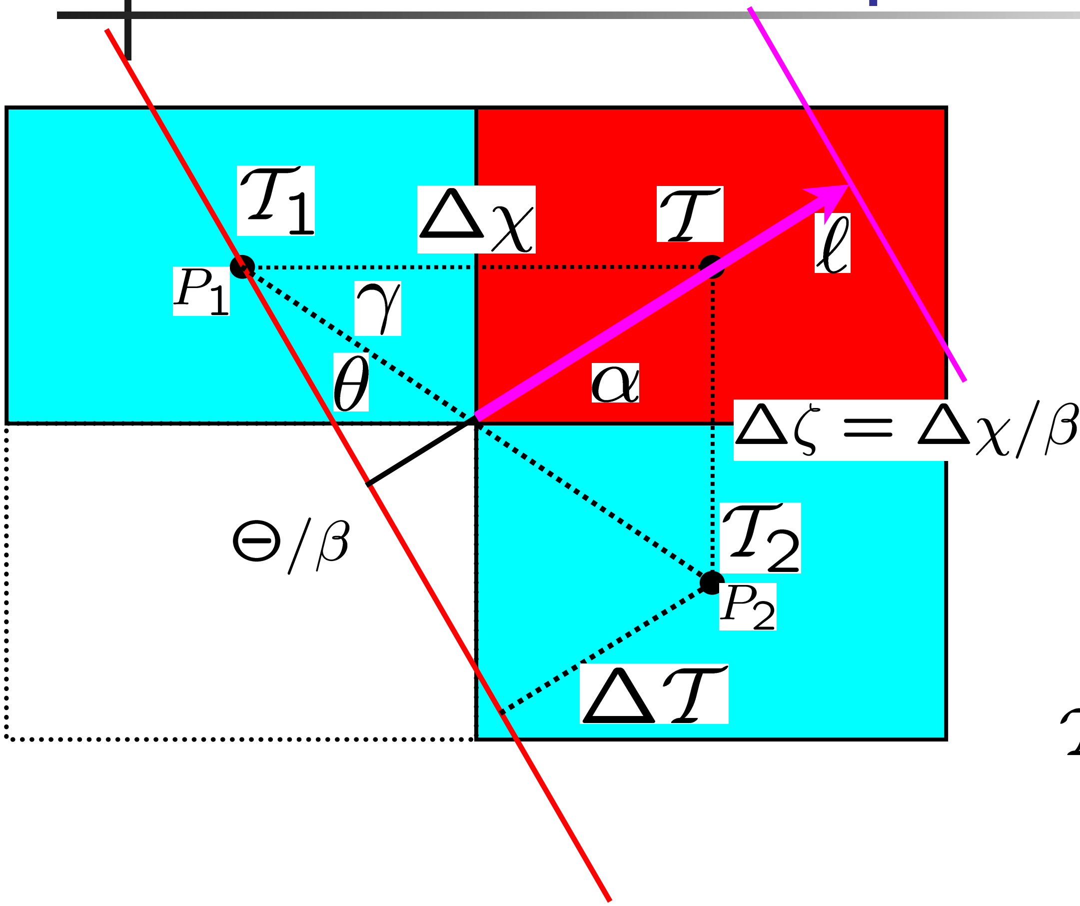
$$\mathcal{T} = \frac{\mathcal{T}_1+\beta^2\mathcal{T}_2+\Theta}{1+\beta^2}$$

$$\Theta = \sqrt{\Delta\chi^2(1+\beta^2) - \beta^2\Delta\mathcal{T}^2}$$

$$\Delta\mathcal{T} = \mathcal{T}_2 - \mathcal{T}_1$$

$$\Delta\chi = \beta\Delta\zeta$$

Geometric interpretation



$$\overline{P_1 P_2} = \frac{\Delta\chi}{\beta}\sqrt{(1+\beta^2)}$$

$$\Theta = \sqrt{\Delta\chi^2(1+\beta^2) - \beta^2\Delta\mathcal{T}^2}$$

$$\mathcal{T} = \mathcal{T}_1 + \Delta\chi \sin(\theta+\gamma)$$
$$= \frac{\mathcal{T}_1 + \beta^2\mathcal{T}_2 + \Theta}{1+\beta^2}$$

$$\ell = -\left(\frac{\mathcal{T}_1+\mathcal{T}_2}{2}\right) \qquad \tan\alpha = \frac{\beta(\Theta - \Delta\mathcal{T})}{\Theta + \beta^2\Delta\mathcal{T}}$$

Initializing $\mathcal{T}(\chi,\zeta)$ from tip asymptotes

- General asymptote: $\Omega \sim \mathcal{W}(\xi; v)$

$$\mathcal{T}(\chi,\zeta) = -\xi \sim -\mathcal{W}^{-1}(\Omega; v)$$

- K-vertex: $\Omega \sim \xi^{\frac{1}{2}}$

$$\mathcal{T}(\chi,\zeta) = -\xi \sim -\Omega^2$$

- M-vertex: $\Omega \sim \beta_{m0}\, v^{1/3}\xi^{2/3}$

$$\mathcal{T}(\chi,\zeta) = -\xi \sim -\left(\frac{\Omega}{\beta_{m0}\, v^{1/3}}\right)^{\frac{3}{2}}$$

$$v = -\frac{\mathcal{T}-\mathcal{T}_0}{\Delta\tau} \;\Rightarrow\; \mathcal{T}^3 - \mathcal{T}_0\mathcal{T}^2 + b = 0$$

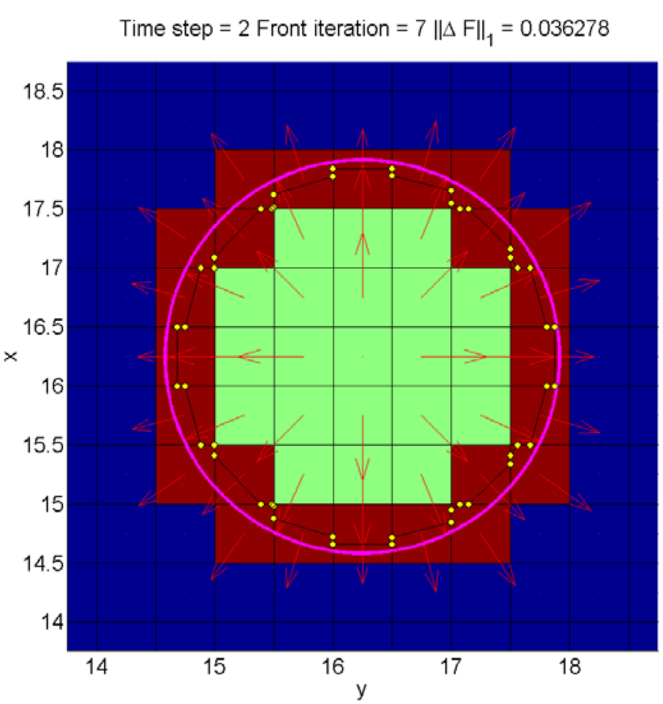
$$b = \Delta\tau\left(\frac{\Omega}{\beta_{m0}}\right)^3 > 0$$

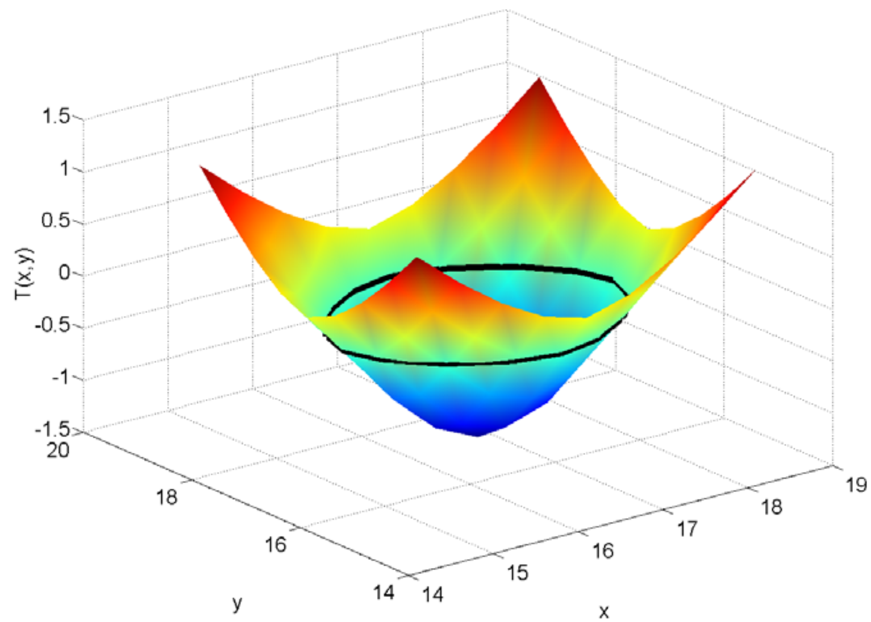
Sample signed distance function

$\mathcal{T}(\chi, \zeta)$ $\Delta\chi = 0.5 = \Delta\zeta$

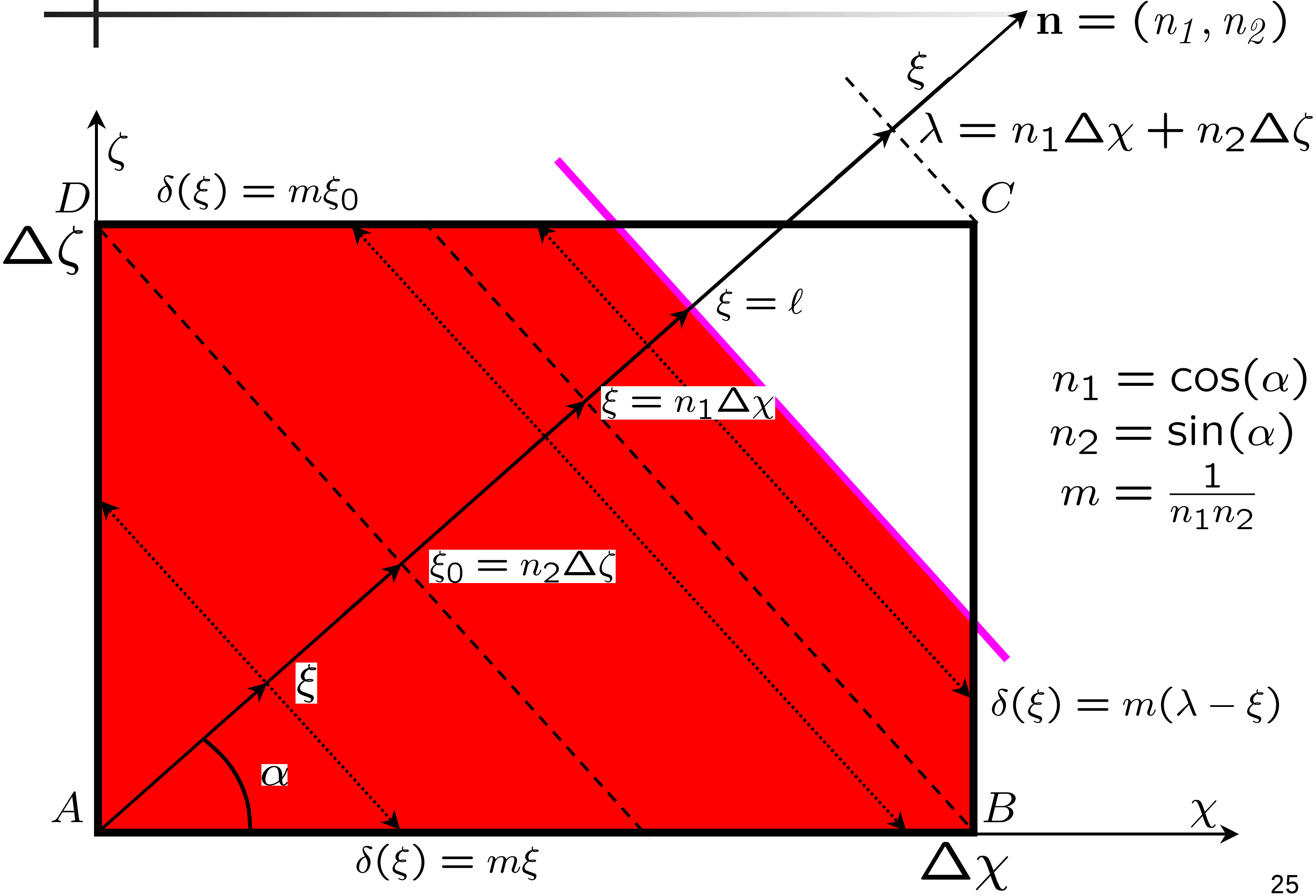
χ (rows), ζ (columns)

1.44	1.09	0.78	0.53	0.41	0.52	0.75	1.05	1.4
1.09	0.7	0.34	0.05	-0.09	0.03	0.31	0.66	1.05
0.79	0.35	-0.07	-0.43	-0.59	-0.46	-0.11	0.3	0.74
0.55	0.07	-0.42	-0.83	-0.97	-0.86	-0.47	0.01	0.5
0.43	-0.07	-0.57	-0.96	-1.16	-0.99	-0.63	-0.13	0.37
0.55	0.07	-0.42	-0.83	-0.97	-0.86	-0.47	0.01	0.5
0.79	0.35	-0.07	-0.43	-0.59	-0.46	-0.11	0.3	0.74
1.09	0.7	0.34	0.05	-0.09	0.03	0.31	0.66	1.05
1.44	1.09	0.78	0.53	0.41	0.52	0.75	1.05	1.4





Calculating the tip volume



Tip volume & width as a function of ℓ

- Width $\delta(\xi)$ of the tip element:

$$\delta(\xi) = \begin{cases} m\xi, & 0 < \xi < \xi_o \\ m\xi_0, & \xi_o < \xi < \lambda - \xi_o \\ m(\lambda - \xi), & \lambda - \xi_o < \xi < \lambda \end{cases}$$

- Volume of the tip element:

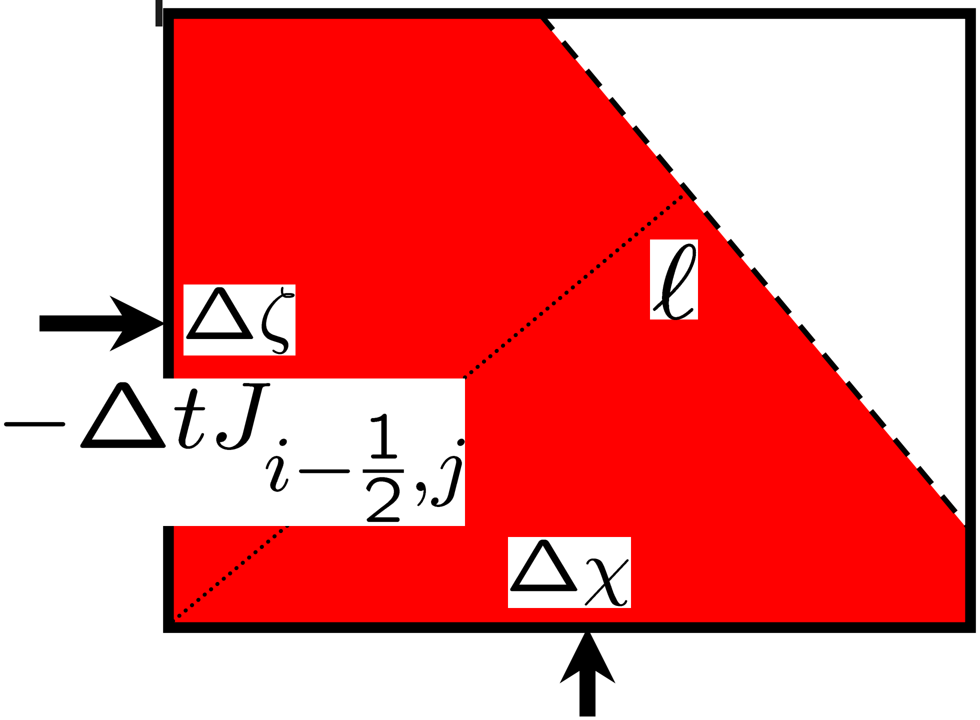
$$V(\ell) = cA_\alpha(\ell) = c\int_0^\ell (\ell - \xi)^\alpha \delta(\xi) d\xi$$

$$A_\alpha(\ell) = \begin{cases} \frac{m}{(\alpha+1)(\alpha+2)}\ell^{\alpha+2}, & 0 < \ell < \xi_o \\ \frac{m}{(\alpha+1)(\alpha+2)}\left[\ell^{\alpha+2} - (\ell - \xi_o)^{\alpha+2}\right], & \xi_o < \ell < \lambda - \xi_o \\ \frac{m}{(\alpha+1)(\alpha+2)}\left[\ell^{\alpha+2} - (\ell - \xi_o)^{\alpha+2} - (\ell - \lambda + \xi_o)^{\alpha+2}\right], & \lambda - \xi_o < \ell < \lambda \end{cases}$$

- Tip widths

$$\Omega_n^t = \frac{V_n^{\tau+\Delta\tau}}{\Delta A}$$

Tip widths and Pressures



$$V(\ell) = c\int_0^{\ell} (\ell - \xi)^{\alpha}\delta(\xi)\,d\xi$$

$\checkmark$

$$\Omega^t = \frac{V(\ell)}{\Delta A}$$

?

$$-\Delta t J_{i,j-\frac{1}{2}} = -\Delta t\,\Omega^3_{i,j-1/2}\left(\frac{\Pi_{i,j}-\Pi_{i,j-1}}{\Delta\chi}\right)$$

$$V_{\tau+\Delta\tau} = V_{\tau} - \Delta\tau\left(\Delta\chi J_{i,j-\frac{1}{2}} + \Delta\zeta J_{i-\frac{1}{2},j}\right)$$
$$= V(\ell) = C A_{\alpha}(\ell)$$

The coupled equations

- Channel lubrication equation

$$\Delta\Omega^c = \Omega^c - \Omega_0^c = \Delta\tau\left(A^{cc}\Pi^c + A^{ct}\Pi^t\right) + \Delta\tau\Gamma^c$$

- Tip lubrication equation

$$\Delta\Omega^t = \Omega^t - \Omega_0^t = \Delta\tau\left(A^{tc}\Pi^c + A^{tt}\Pi^t\right) + \Delta\tau\Gamma^t$$

- Elasticity Equation (eliminate channel pressure)

$$\Pi^c = C^{cc}\Omega^c + C^{ct}\Omega^t$$

- Coupled system

$$\begin{bmatrix} I - \Delta\tau A^{cc}C^{cc} & -\Delta\tau A^{ct} \\ -\Delta\tau A^{tc}C^{cc} & -\Delta\tau A^{tt} \end{bmatrix} \begin{bmatrix} \Delta\Omega^c \\ \Pi^t \end{bmatrix} = \begin{bmatrix} \Delta\tau A^{cc}\left(C_0^{cc}\Omega^c + C^{ct}\Omega^t\right) + \Delta\tau\Gamma^c \\ -\Delta\Omega^t - \Delta\tau A^{tc}\left(C_0^{cc}\Omega^c + C^{ct}\Omega^t\right) + \Delta\tau\Gamma^t \end{bmatrix}$$

Time stepping and front evolution

Time step loop: $\tau \leftarrow \tau + \Delta\tau$

Front iteration loop:

Coupled Solution

$$\begin{bmatrix} I - \Delta\tau A^{cc}C^{cc} & -\Delta\tau A^{ct} \\ -\Delta\tau A^{tc}C^{cc} & -\Delta\tau A^{tt} \end{bmatrix} \begin{bmatrix} \Delta\Omega^{c} \\ \Pi^{t} \end{bmatrix} = \begin{bmatrix} r^{c} \\ r^{t} \end{bmatrix}$$

end

set $\mathcal{T}(\chi,\zeta) = -\xi \sim -\mathcal{W}^{-1}(\Omega; v)$ in

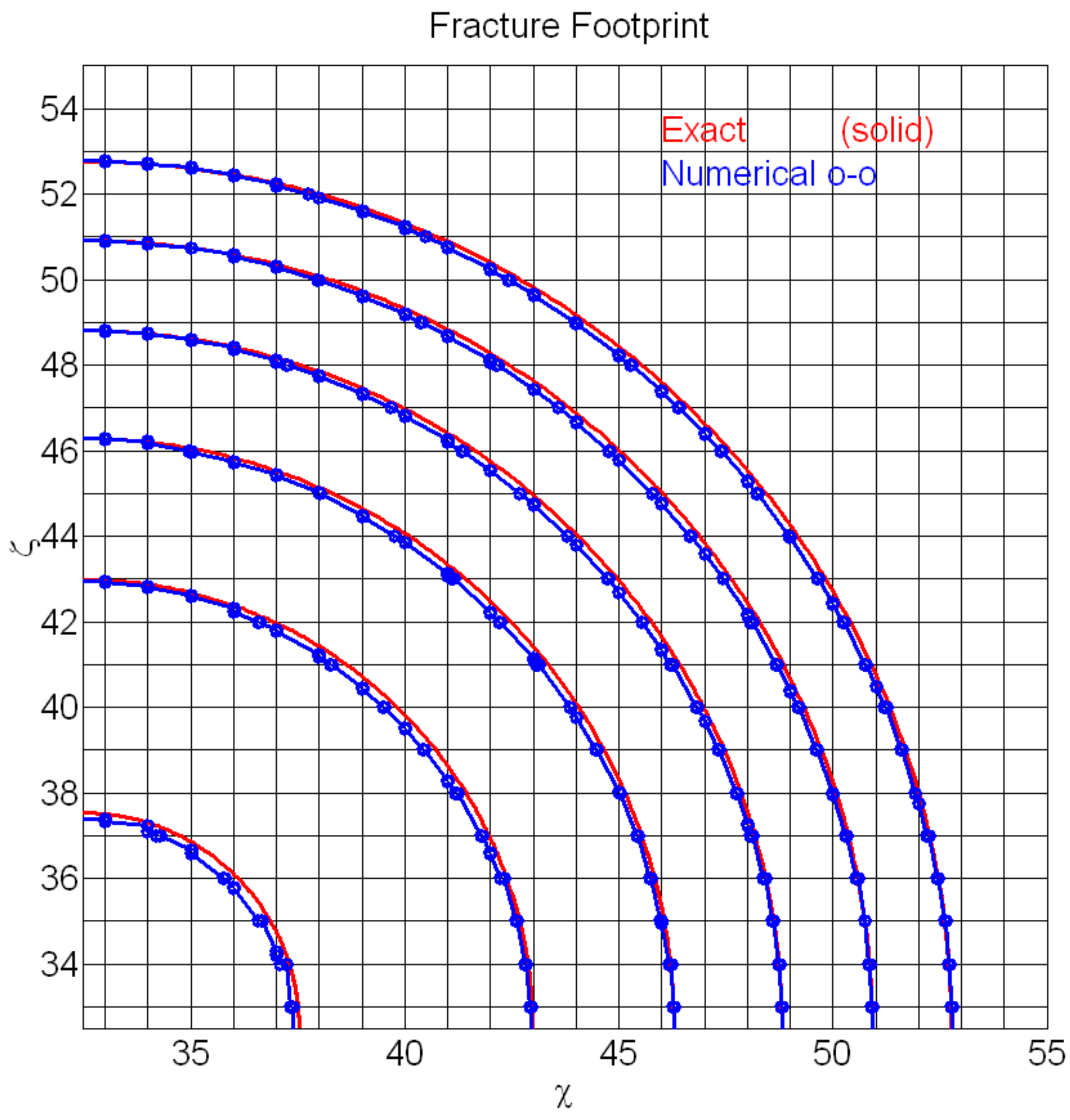
use FMM to solve $|\nabla\mathcal{T}(\chi,\zeta)| = 1$

Locate front $\ell, \alpha, \Omega^{t} = V^{t}(\ell)/\Delta A$

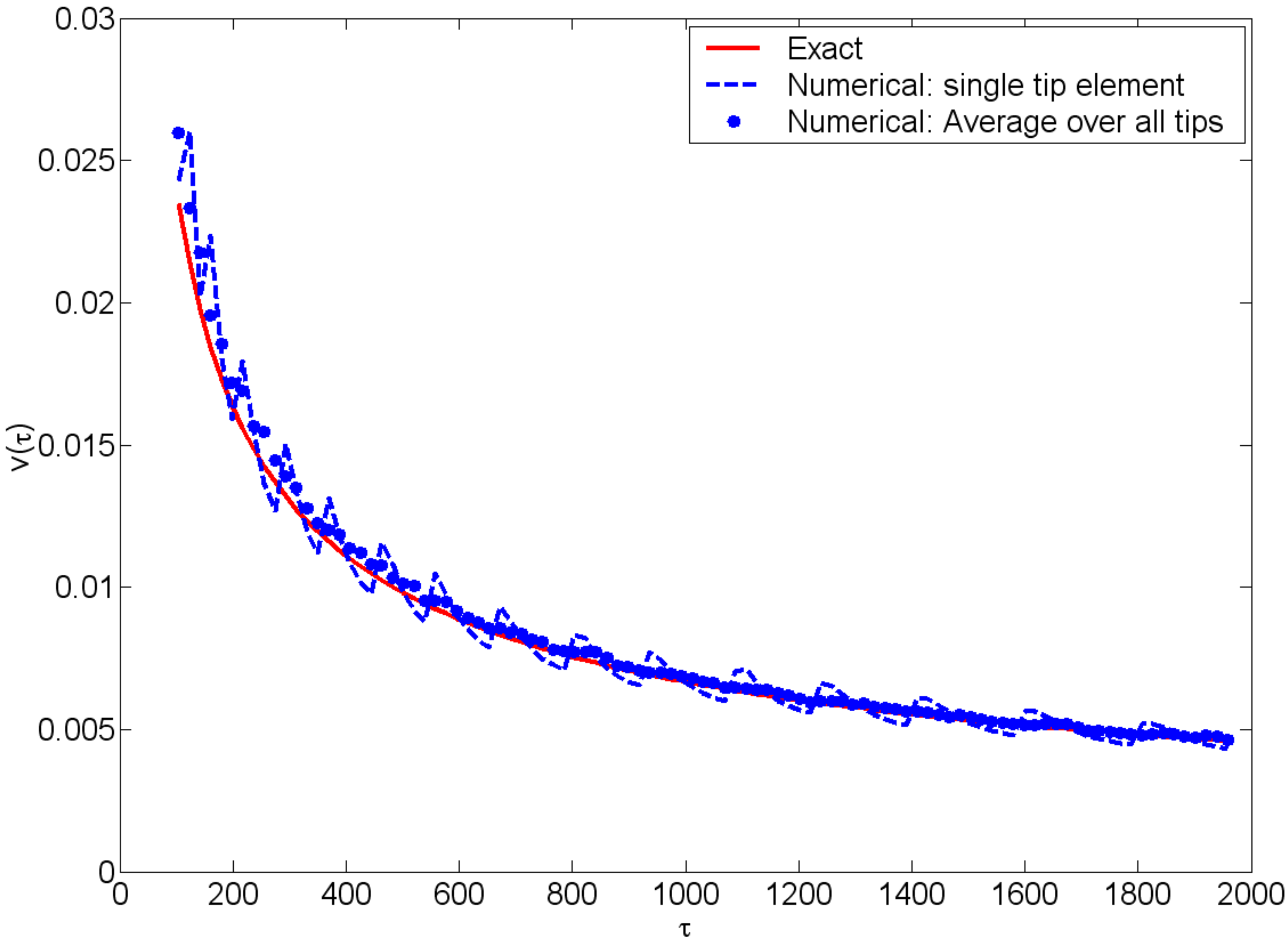
next front iteration

next time step

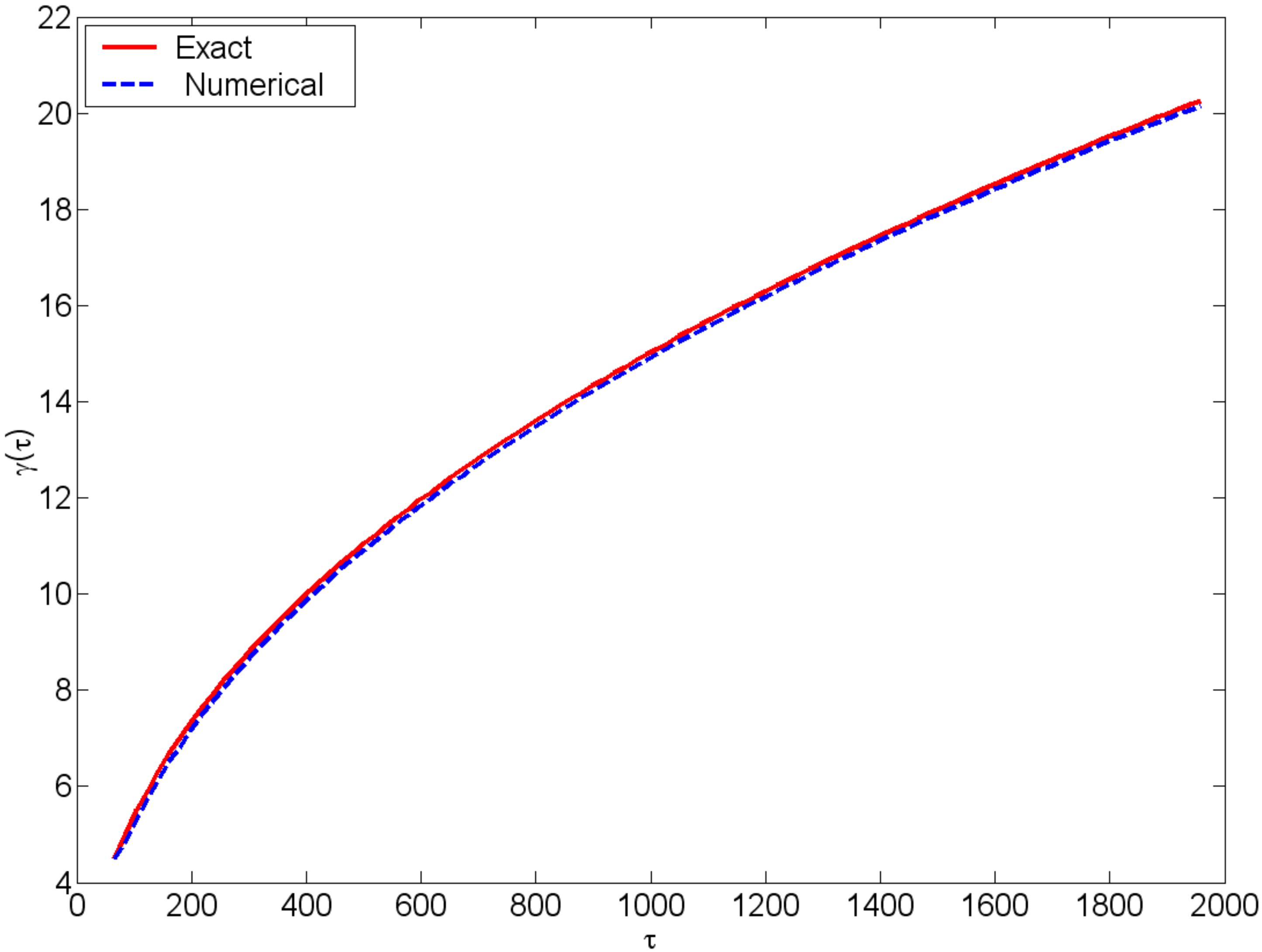
M-vertex radial sol'n – footprints



M-Vertex radial solution – front speed

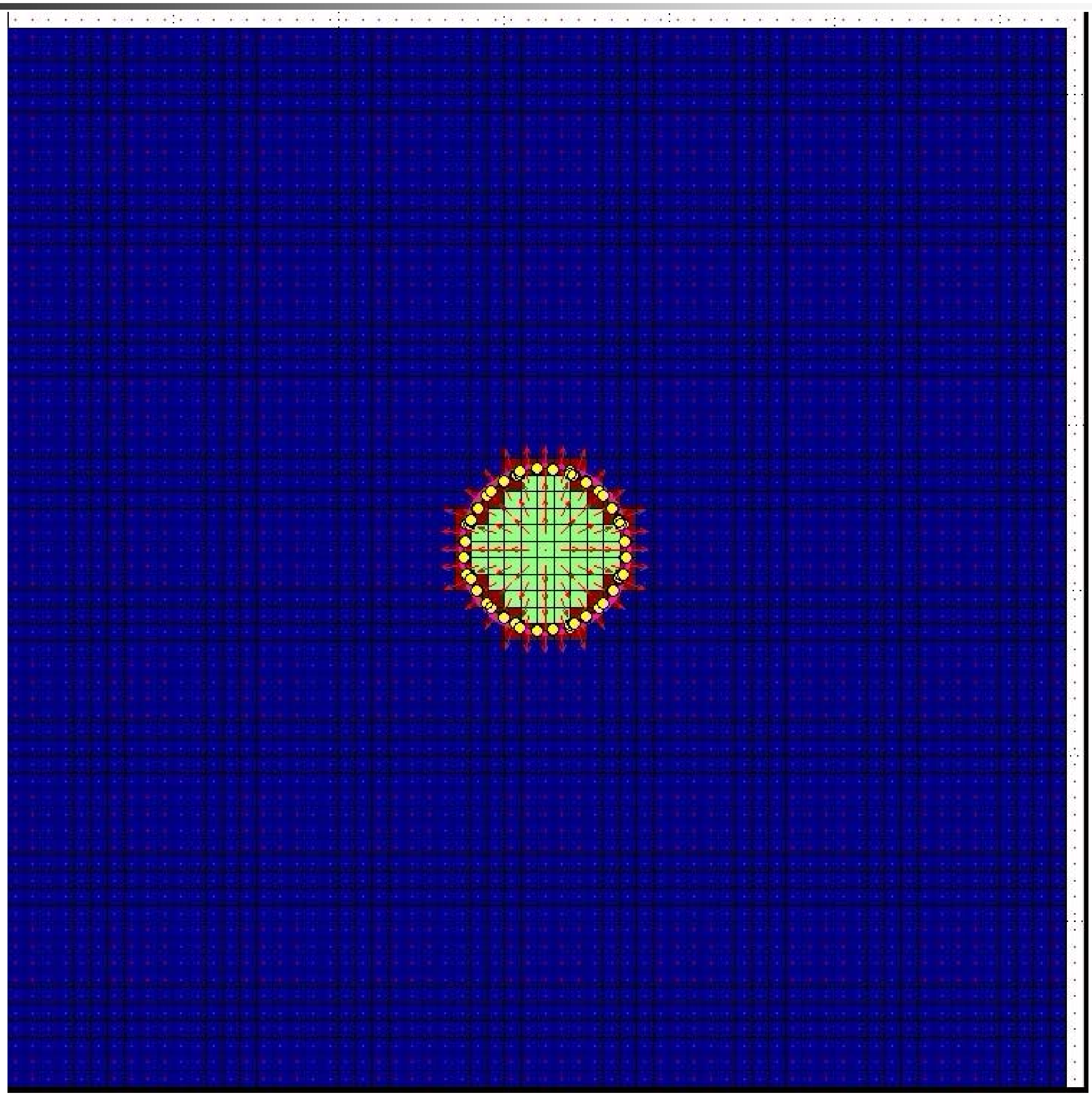


M-vertex radial solution - $\gamma(\tau)$

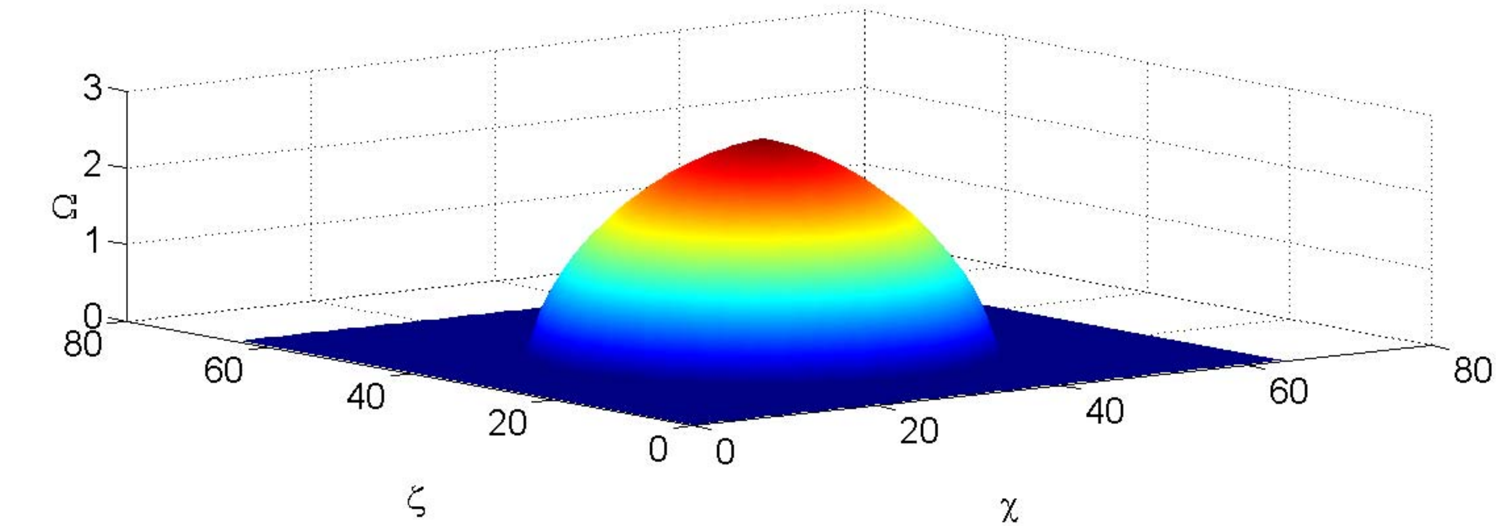


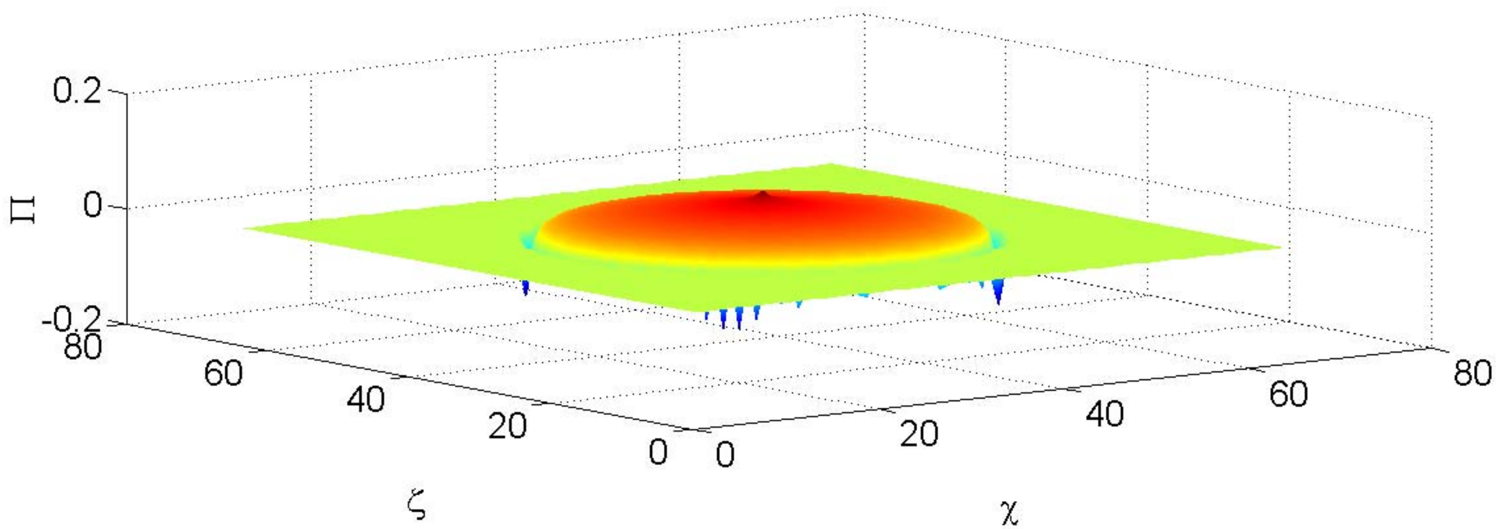
M-Vertex Footprint evolution

ILSA
EXACT

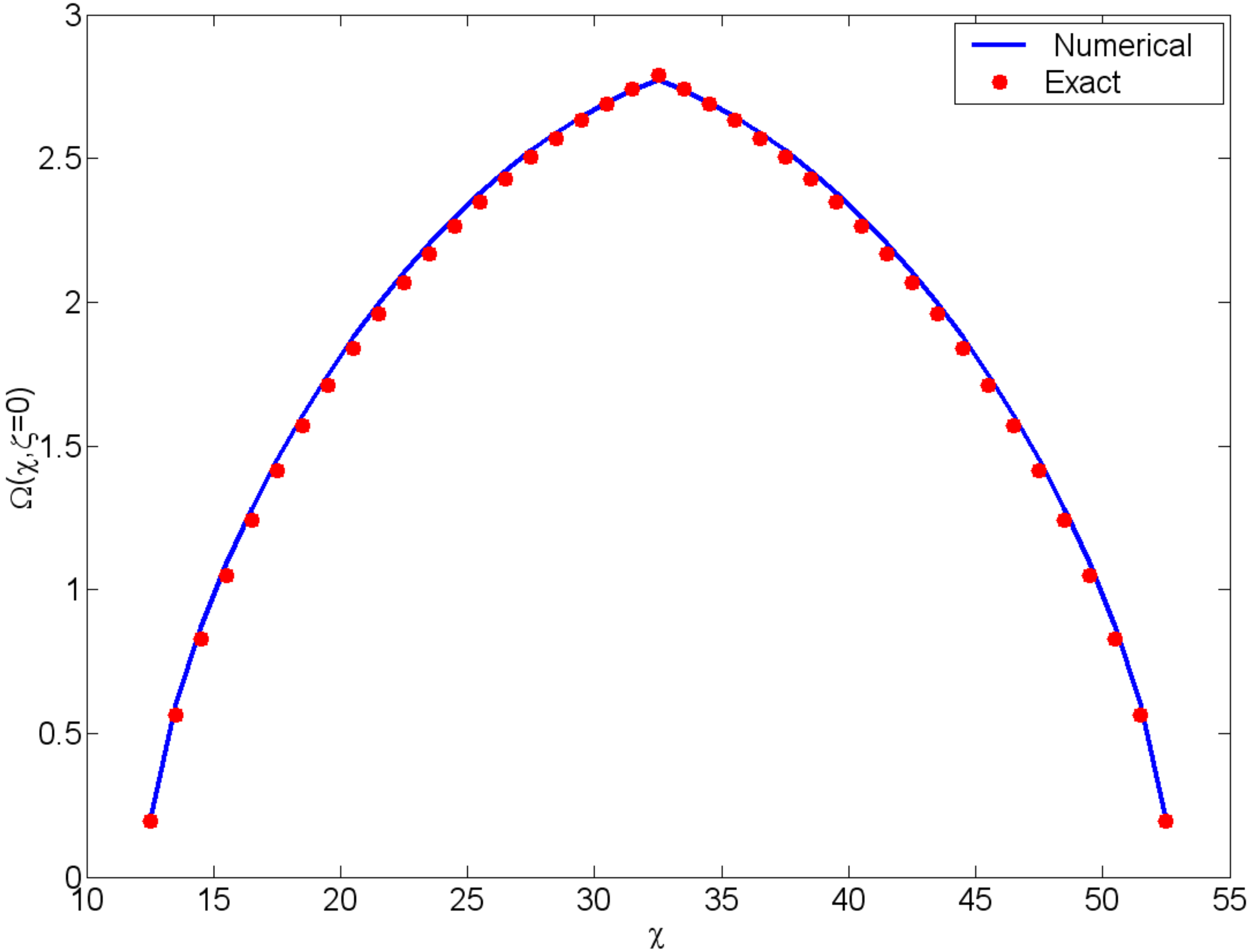


M-vertex radial soln:width & pressure

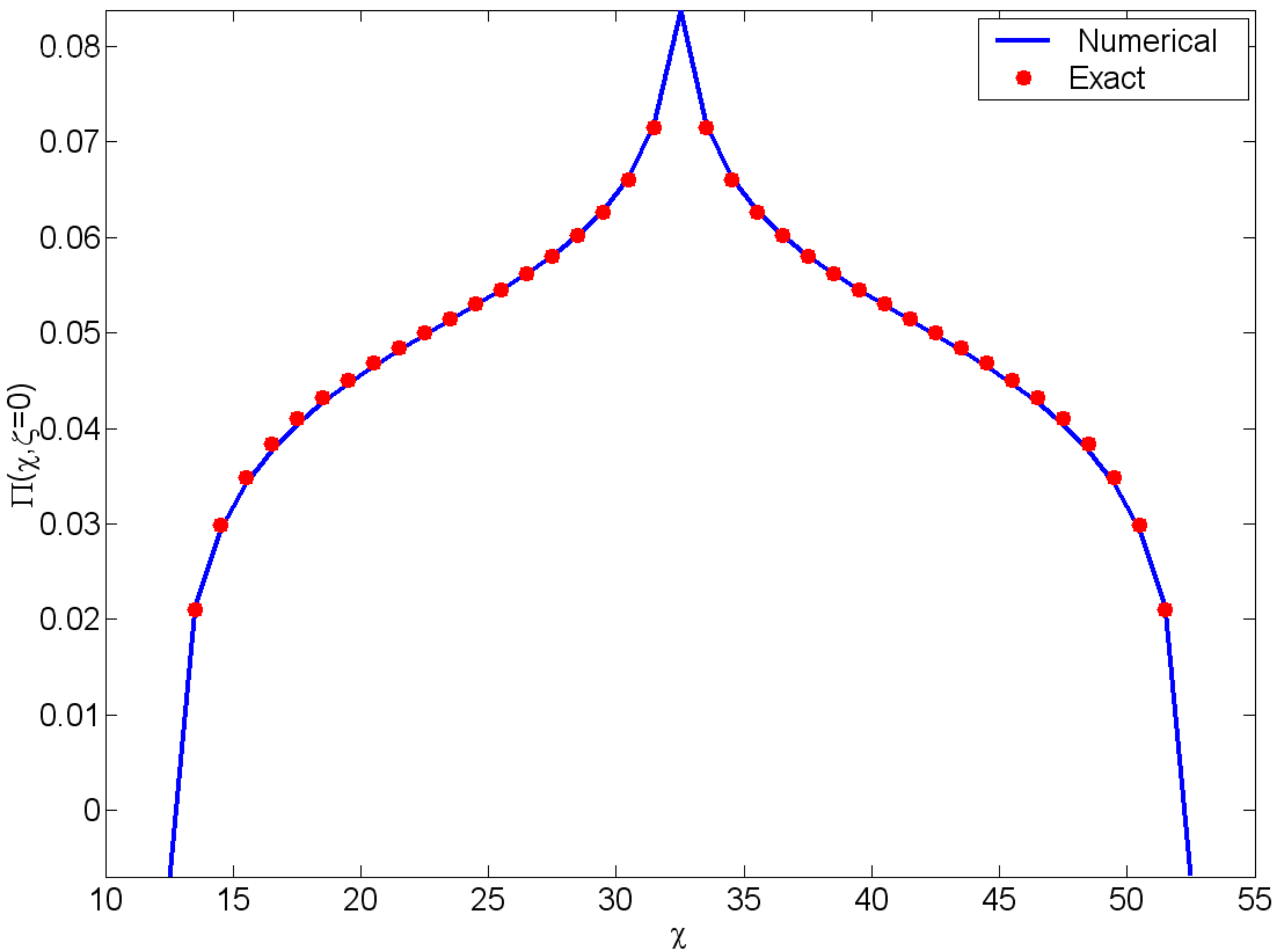




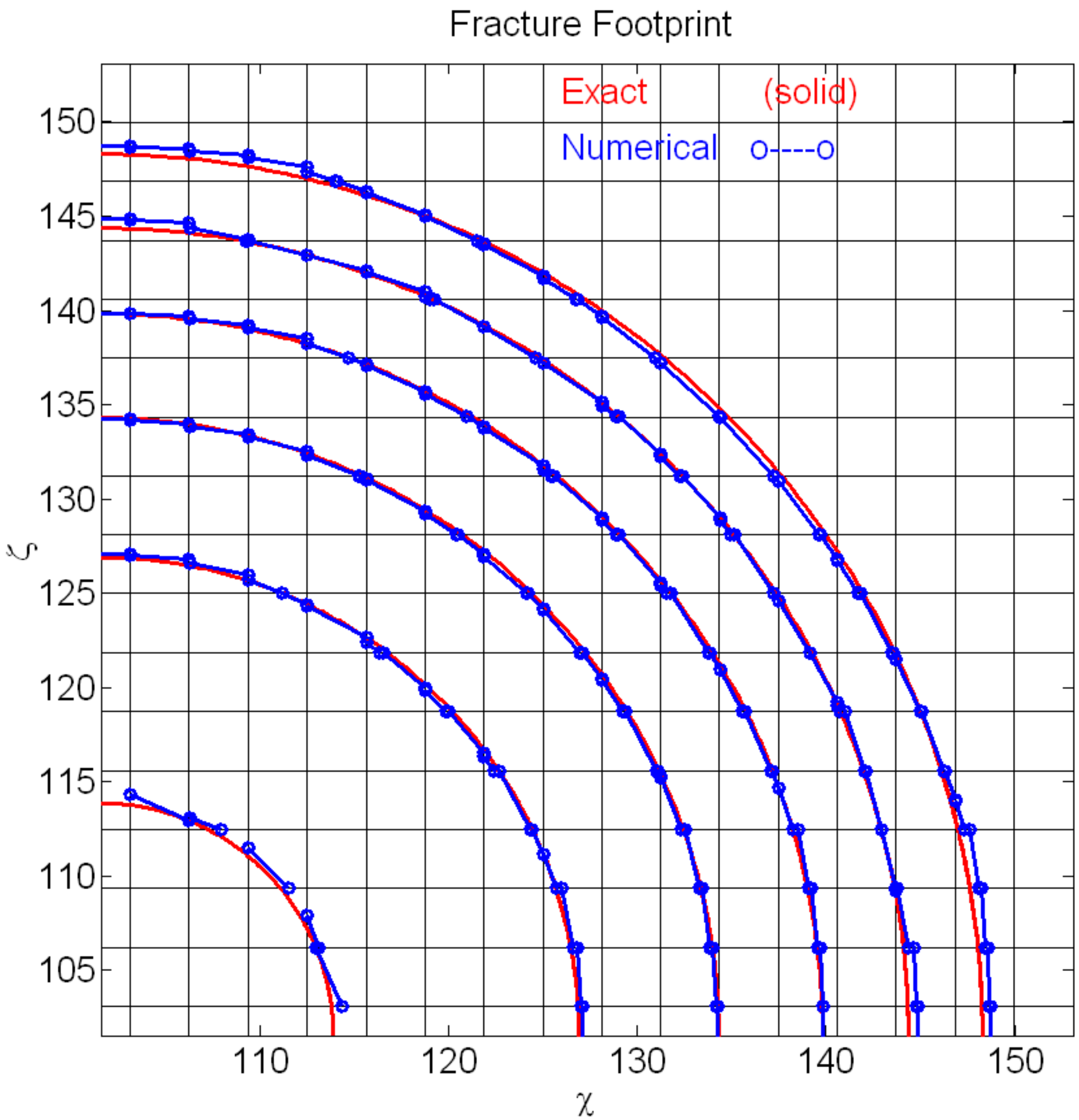
M-vertex width



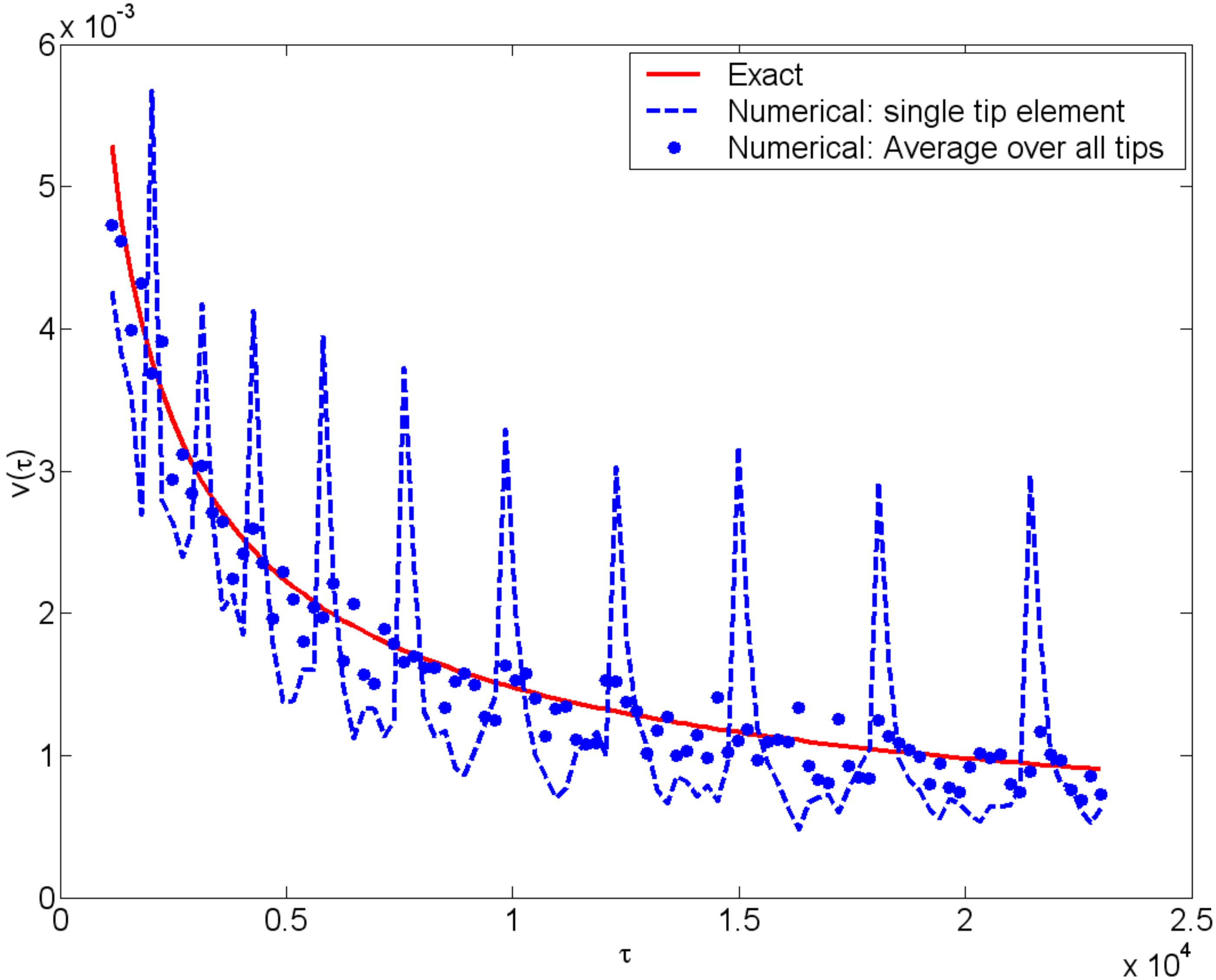
M-vertex pressure



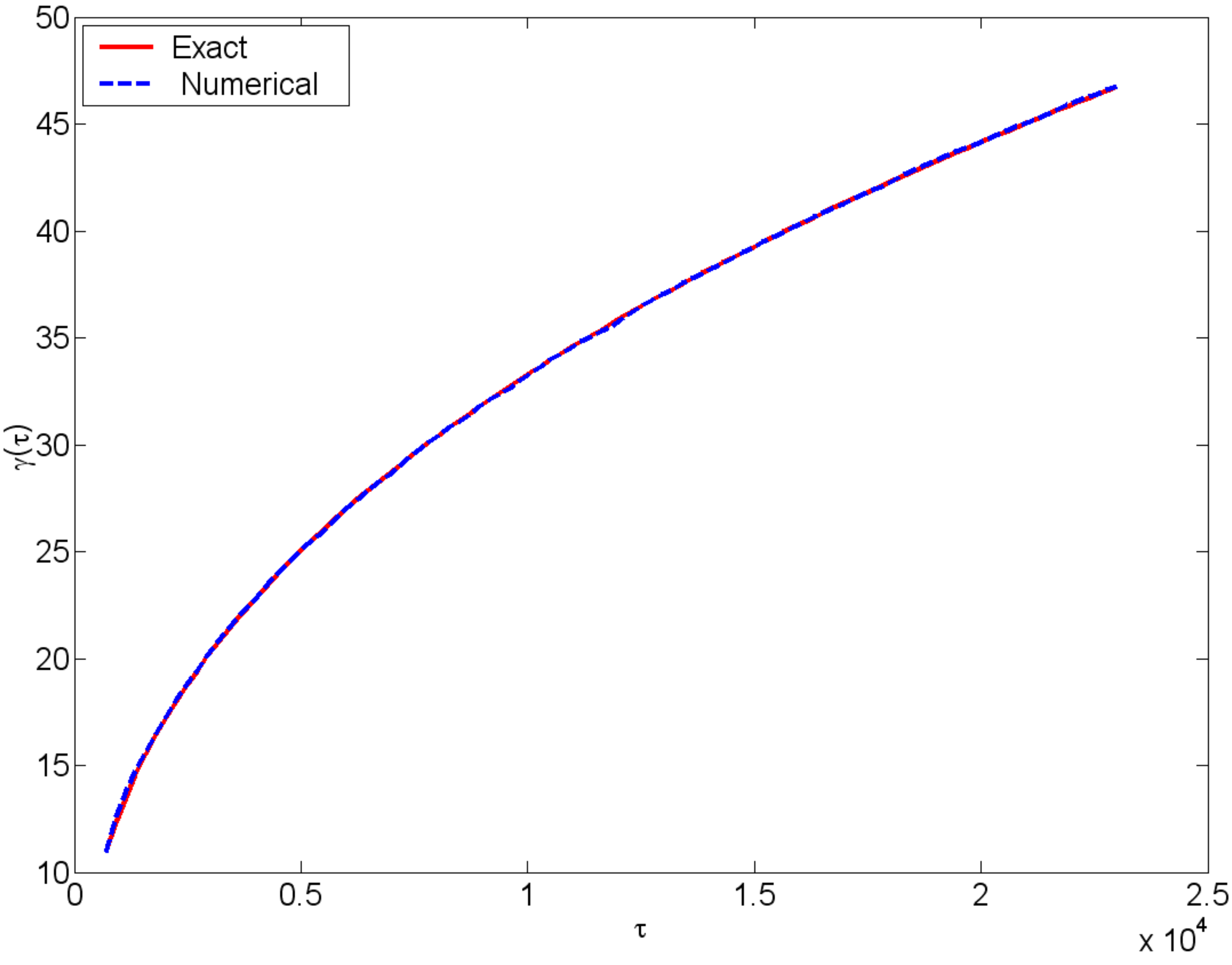
K-vertex radial sol'n – footprints



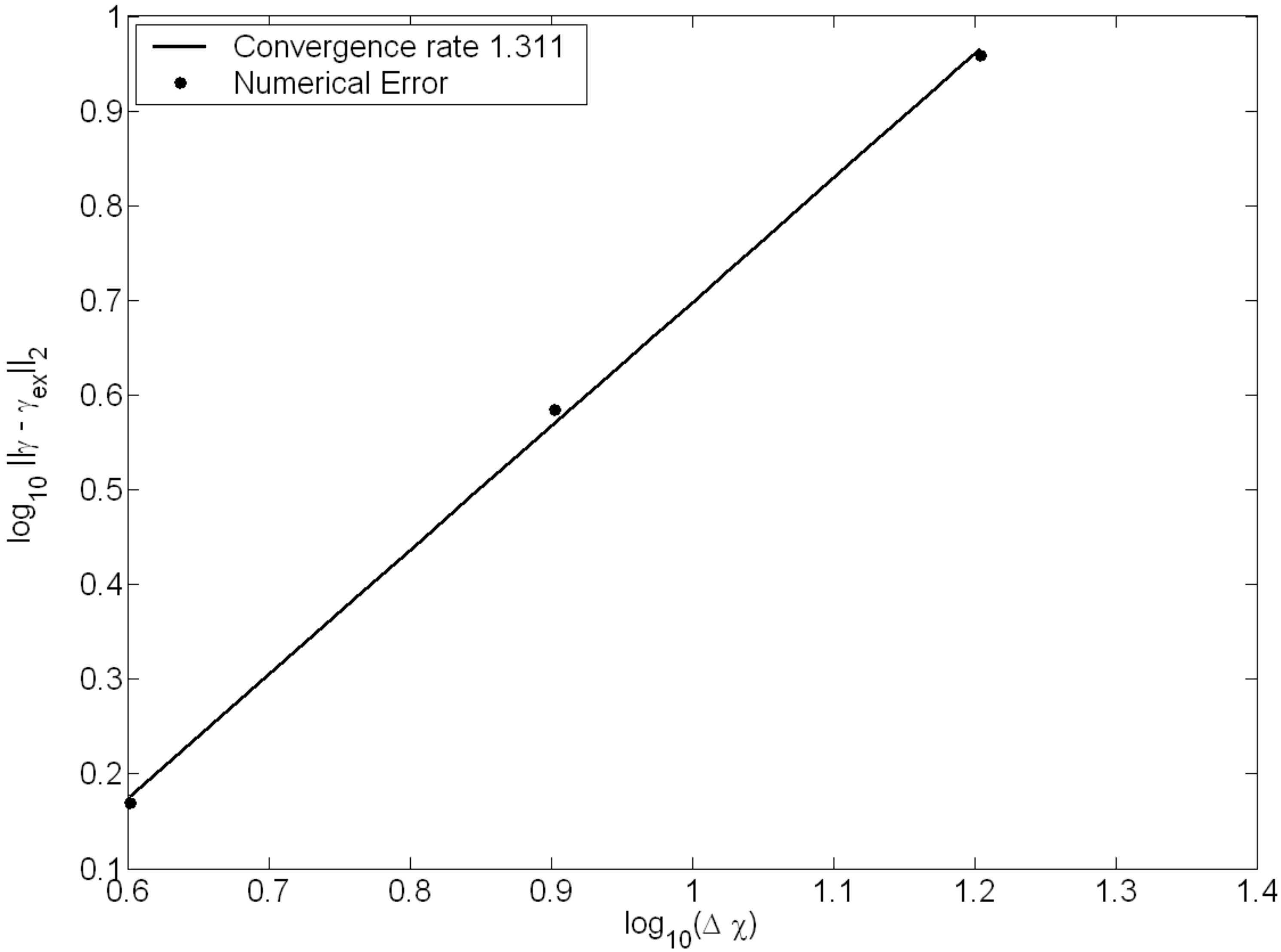
K-Vertex radial solution – front speed



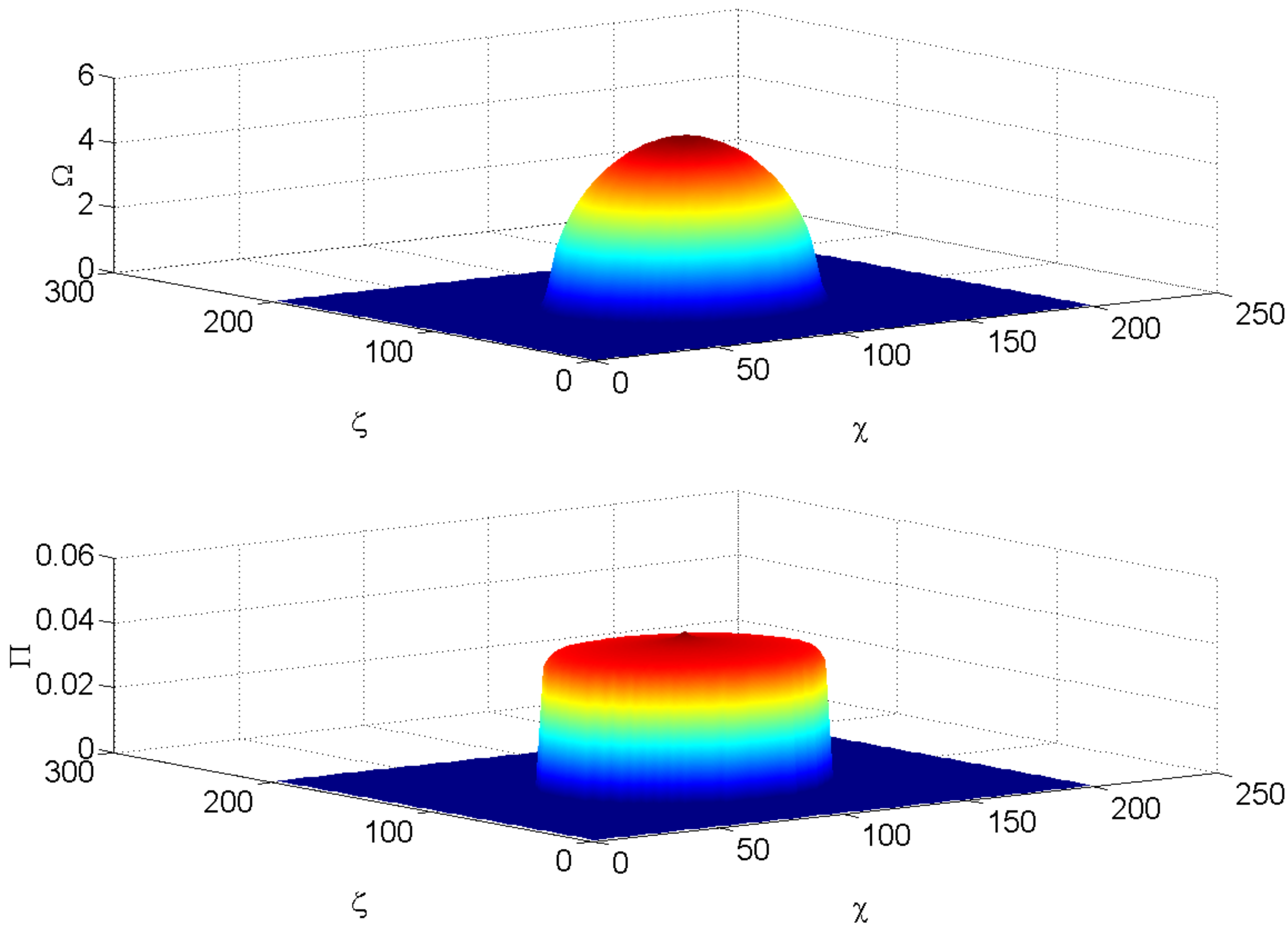
K-vertex radial solution - $\gamma(\tau)$



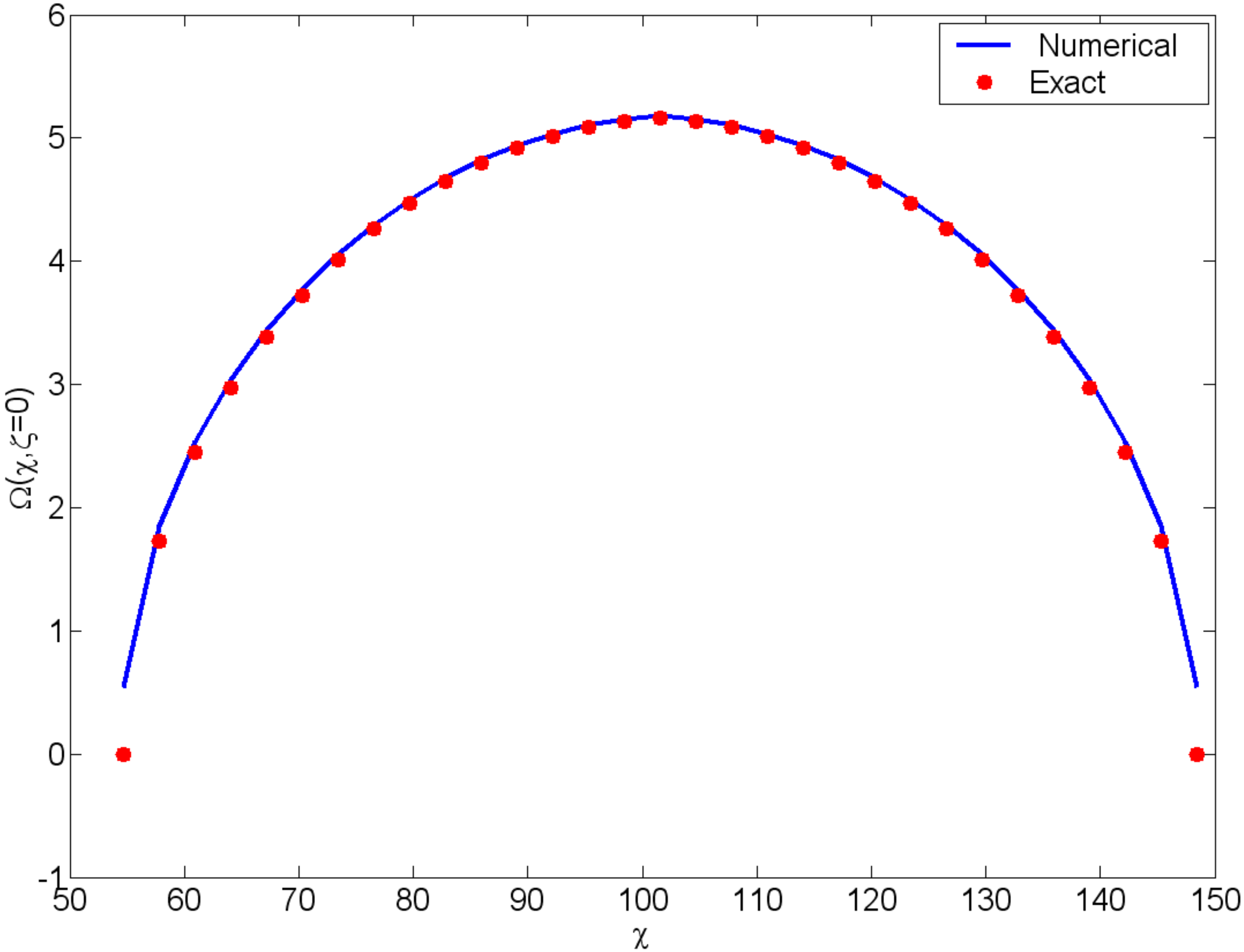
K-vertex – convergence rate for γ



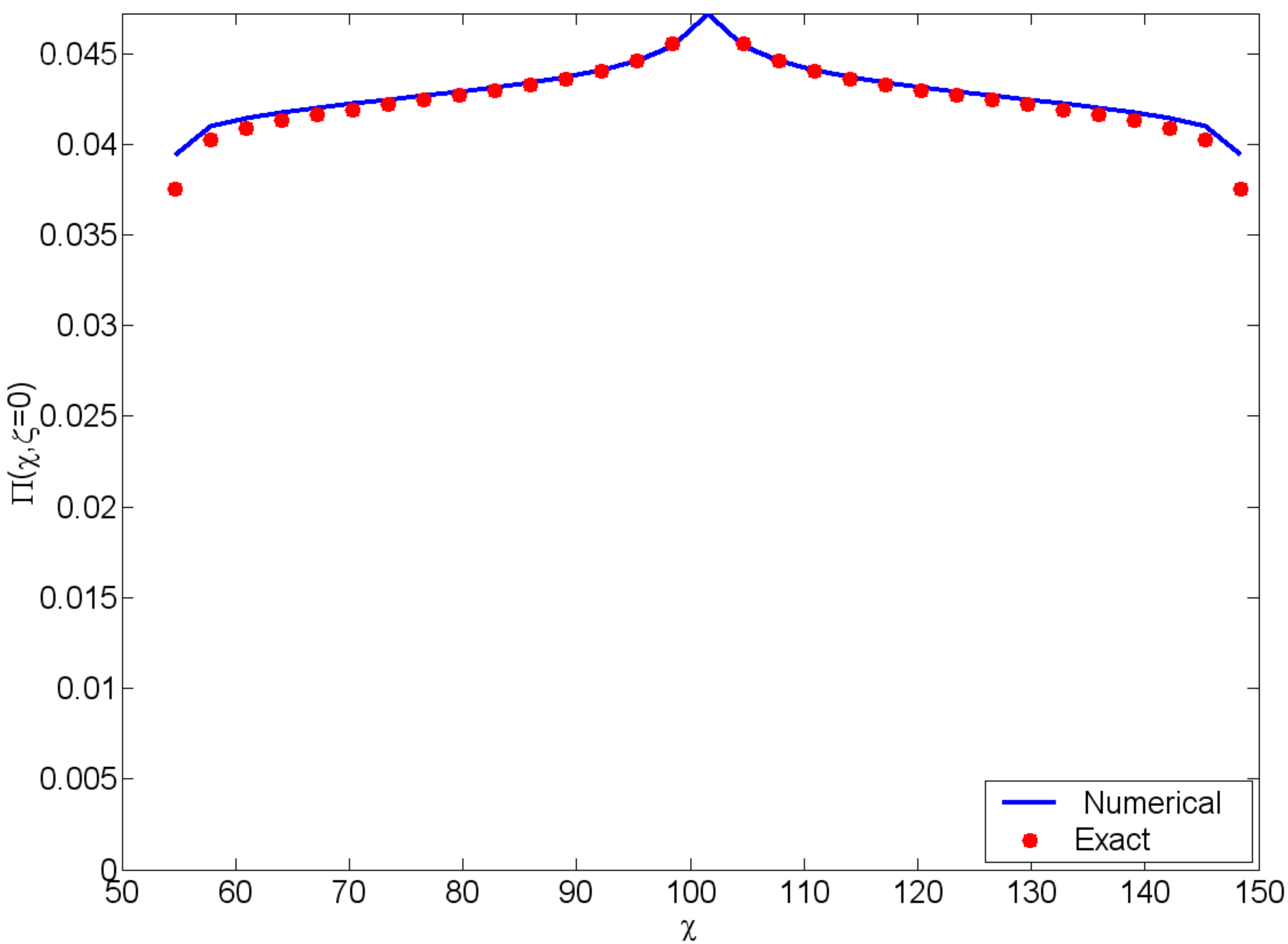
K-vertex radial soln:width & pressure



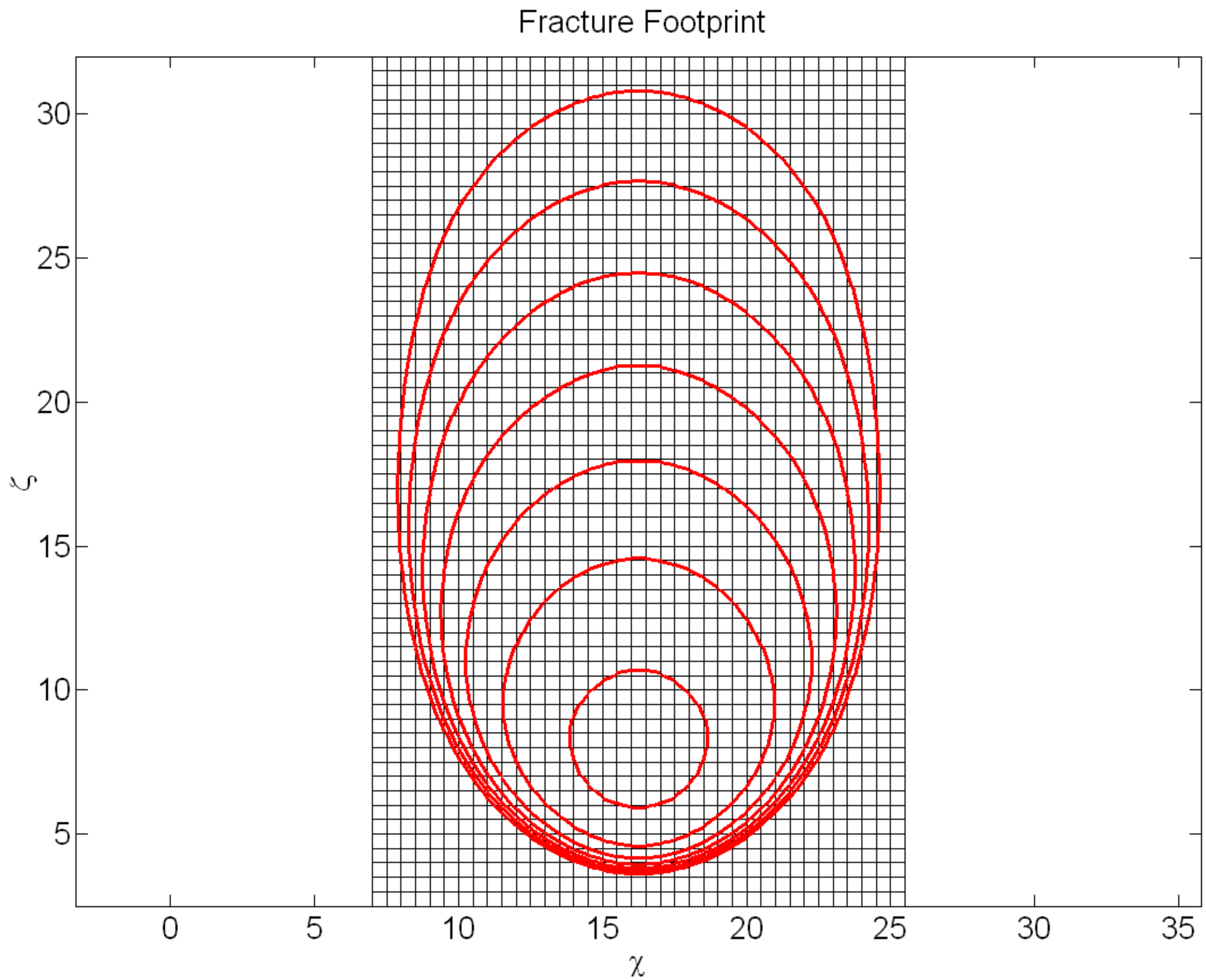
K-vertex width



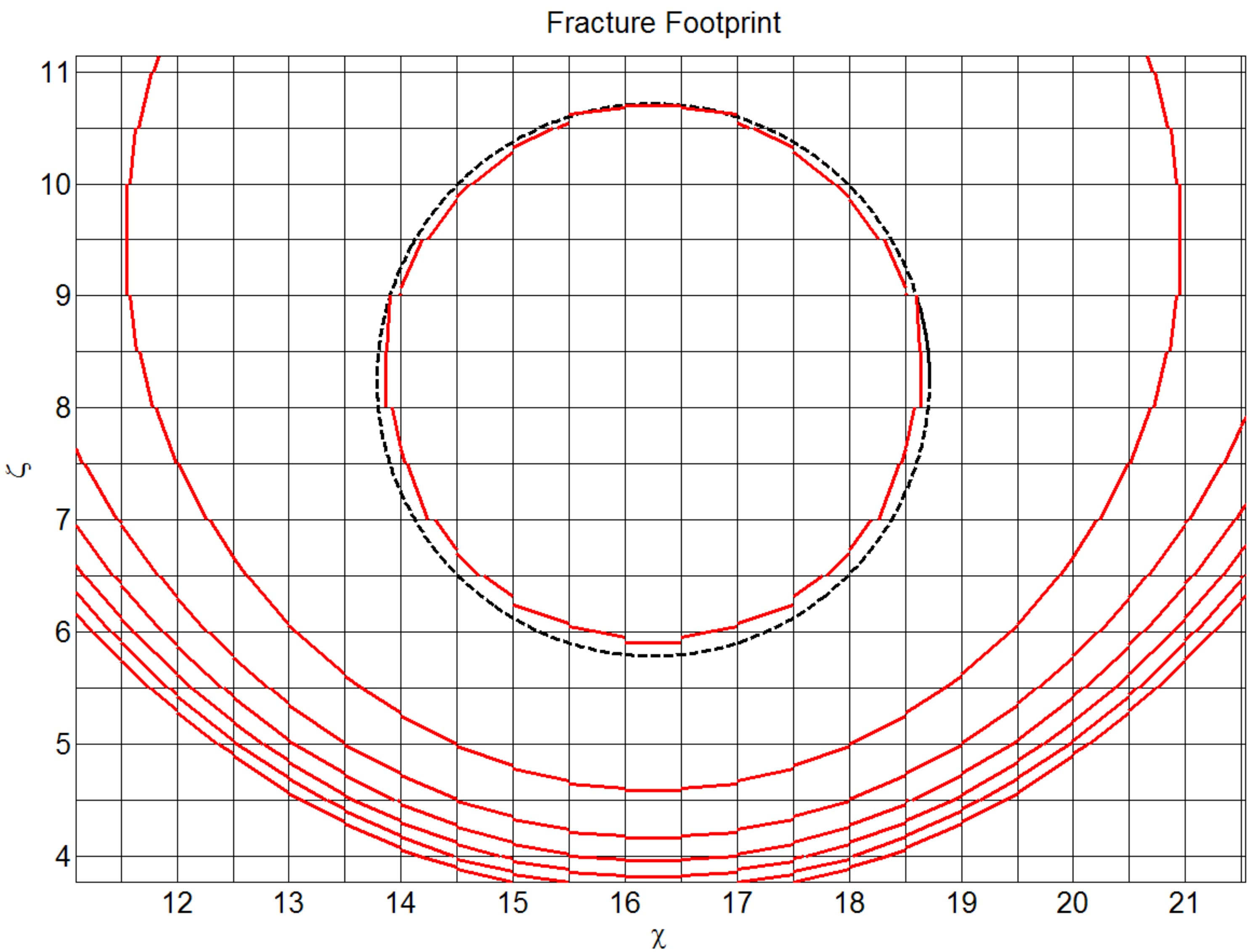
K-vertex Pressure



Linear *in-situ* stress field - footprints

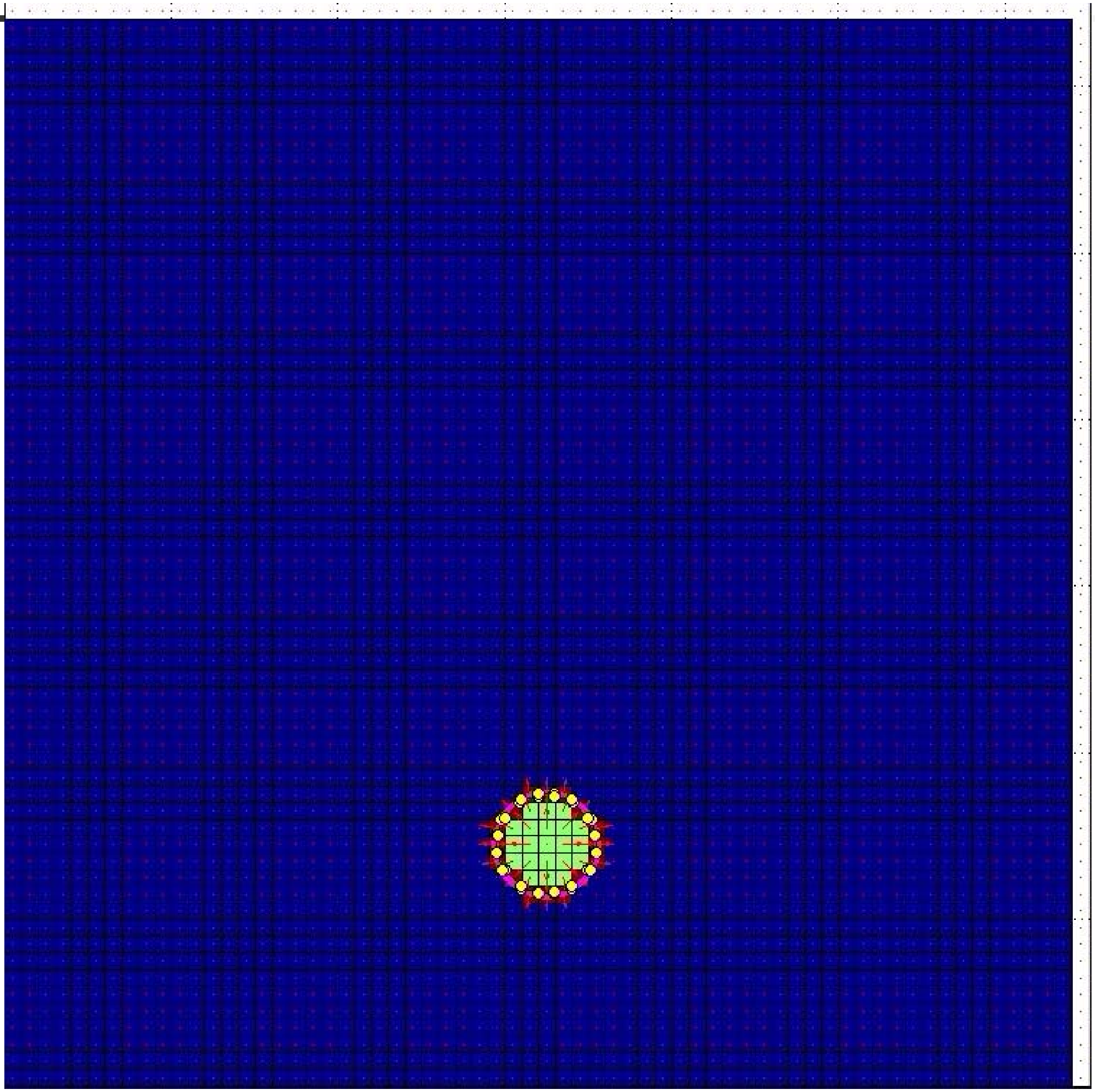


Zoom in

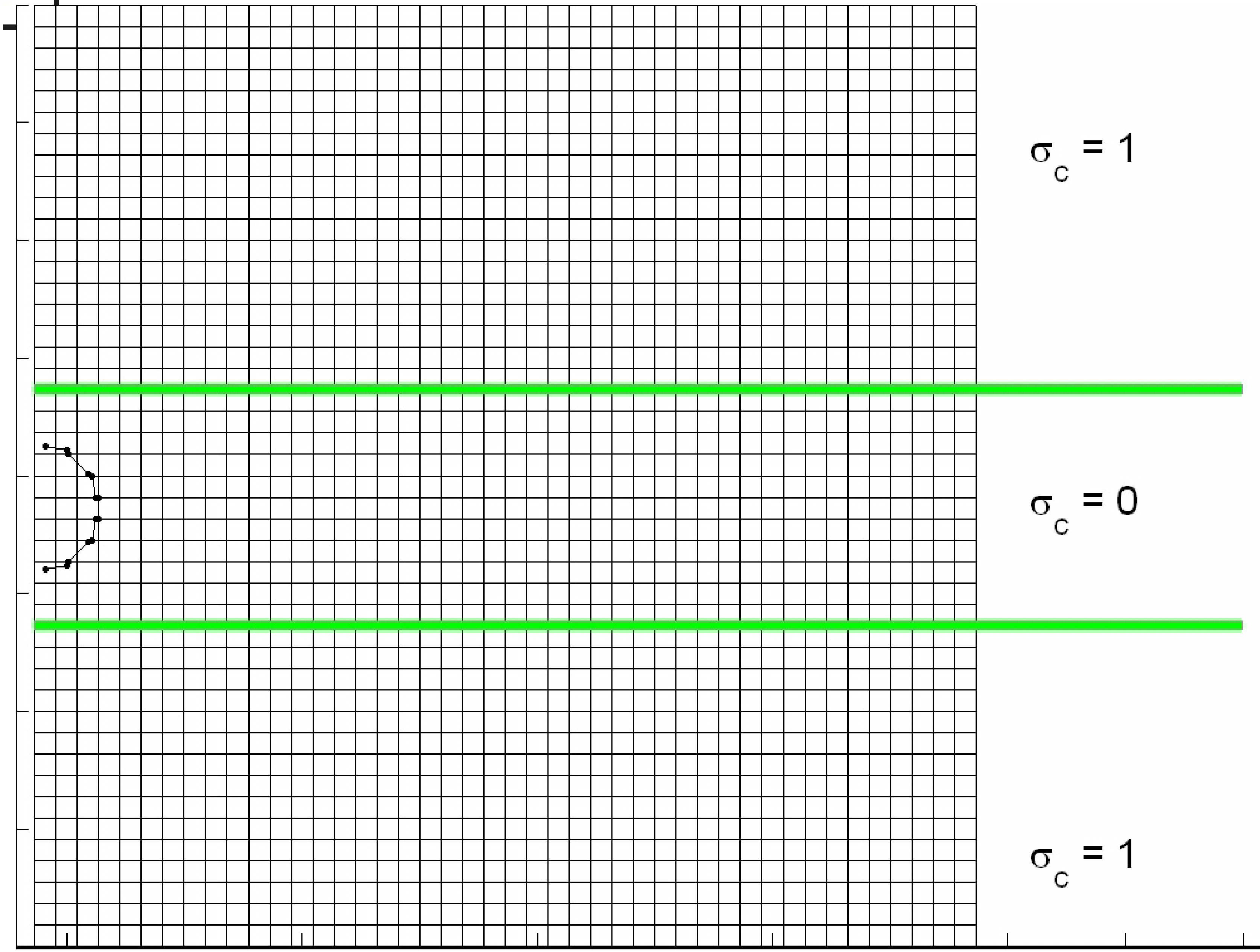


ILSA Linear — Uniform field

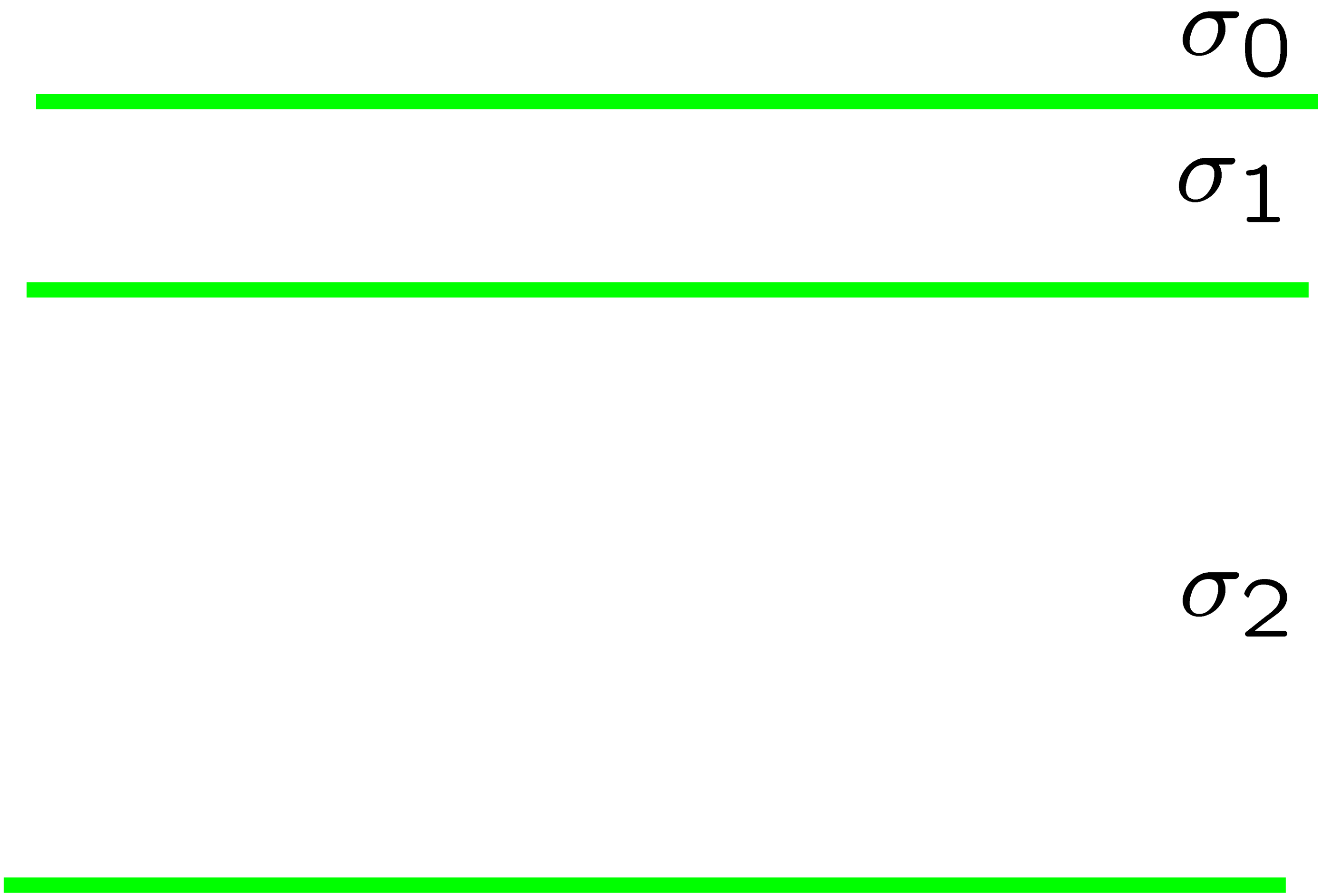
Linear *in-situ* stress field - movie



Breakthrough: Low to High Stress

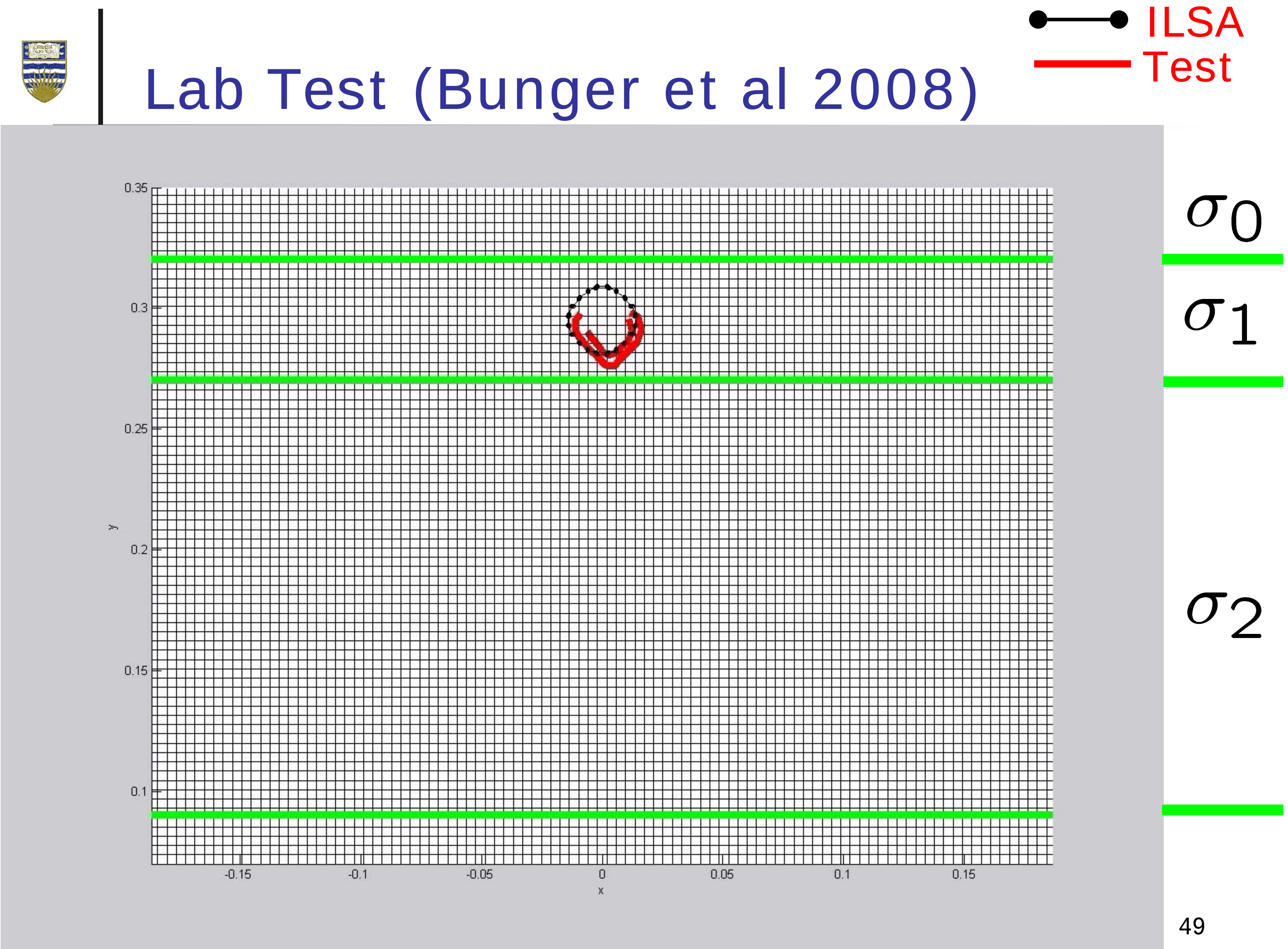


Stress contrast high->low



$$\sigma_0 \gg \sigma_1 > \sigma_2$$

Lab Test (Bunger et al 2008)



Conclusions

- Governing equations, scaling, asymptotic solutions 1-2D and 2-3D

- Solving the free boundary problem
  - Existing methods: VOF, Level Set and K_I matching
  - Tip asymptote and Eikonal boundary value problem
  - Setting the tip volumes using the tip asymptotes
  - Coupled equations

- Numerical examples
  - M-vertex and K-vertex
  - Viscous crack propagating in a variable *in situ* stress
  - Stress jump solutions