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Comput. Methods Appl. Mech. Engrg. 190 (2001) 5935–5956

Computer methods in applied mechanics and engineering

www.elsevier.com/locate/cma

The scaled flexibility matrix method for the efficient solution of boundary value problems in 2D and 3D layered elastic media ☆

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Received 1 January 2000; received in revised form 17 September 2000

Abstract

We present a method that extends the flexibility matrix method for multilayer elasticity problems to include problems with very thin layers. This method is particularly important for solving problems in which one or a number of very thin layers are juxtaposed with very thick layers. The standard flexibility matrix method suffers from round-off errors and poor scaling of the flexibility equations which occur when one of the layers in the multilayered medium becomes much smaller than the others. The method proposed in this paper makes use of power series expansions of the various components of the flexibility matrix in order to arrive at a system of equations that is appropriately scaled. The effectiveness of the scaled flexibility matrix method is demonstrated on a number of test problems.

Keywords: The flexibility matrix method; Layered elastic media; Fourier transforms

1. Introduction

The solution of boundary value problems in layered elastic materials is important in a number of applications of the theory of elasticity. For example, it is important to determine the stresses around underground excavations in rock which comprises distinct layers due to the sedimentary deposition of materials over time. Another example includes the analysis of the effect of layering in natural or man-made deposits on foundations and pavements. A third important example, which motivates the work in this paper, involves modeling the hydraulic fracturing process in the oil and gas recovery industry. In the hydraulic fracturing process, a fracture is forced to propagate through a layered reservoir by means of a fluid which is injected under high pressure at a well-bore. A complete model of this process involves the solution of the elasto-hydrodynamic equations, which couple the fluid flow equations expressing the conservation of mass to the elasticity equations expressing a balance of forces between the fluid pressure and the elastic response of the rockmass adjacent to the fracture. Poro-elastic effects, which are important in reservoir simulations, occur on a much longer time scale than the hydraulic fracturing process and can be ignored. Local fluid leak-off into the reservoir, however, is typically accounted for by means of loss terms in the fluid equations.

A crucial component in the hydraulic fracturing simulation process is the ability to model the elastic response of a pressurized crack which may intersect a number of layers. It is not unusual for the elastic moduli in these layers to differ by at least an order of magnitude and for the thickness of the layers to change from 0.2 to 200–1000 m. In order to be able to model such problems in an efficient manner, we have developed the scaled flexibility matrix method described in this paper.

The method we describe can be used to determine the appropriate Green's function for a multilayered elastic material which comprises a number of layers with different material moduli, each layer containing uniform properties, which are bonded together at interfaces that are all parallel (see Fig. 1). The Green's function can then be used to determine the kernel functions for a boundary integral equation that can be used to solve boundary value problems for cavities or cracks that exist within the layered elastic material.

In order to determine the Green's function, we will apply the Fourier transform (FT) in one (in 2D) or two (in 3D) directions parallel to the layers in order to reduce the system of partial differential equations to a system of coupled ordinary differential equations. The general solution of the homogeneous ordinary differential equations for a given layer can be determined since any given layer is uniform and therefore has constant coefficients. The general solution in each layer can then be expressed in terms of a small number (4 in 2D and 6 in 3D) free constants, which we shall refer to in this paper as spectral coefficients. The appropriate the Green's function can then be constructed by stitching together the solutions within each of the layers by applying the conditions that the stresses and displacements are continuous across the layers, while the source term for Green's function can be derived by specifying the appropriate jump conditions across the pseudo-interface at the horizon at which the desired source falls. In order to obtain the spatial form of Green's function, the FTs are inverted. For 3D problems it is perhaps more convenient to use the Hankel transform (or an expansion in terms of Hansen potentials). The method we describe will apply equally to a formulation in terms of Fourier or Hankel transforms, but for the clarity of exposition and to allow the possibility of presenting the method for both 2D and 3D problems within the same framework, we will restrict our discussion to the FT formulation. The notation we use in this paper will follow that presented by Wardle [20].

The FT method described above for the construction of the Green's functions was essentially pioneered by Sneddon [18]. Thompson [19] introduced the first systematic approach to layered materials. The method described above leads naturally to the so-called stiffness matrix method (see [20]), which leads to a system of algebraic equations for the spectral coefficients. The expression for the stiffness matrix will involve terms of the form $e^{k\Delta y}$ and $e^{-k\Delta y}$ where k is the wavenumber and Δy is the distance

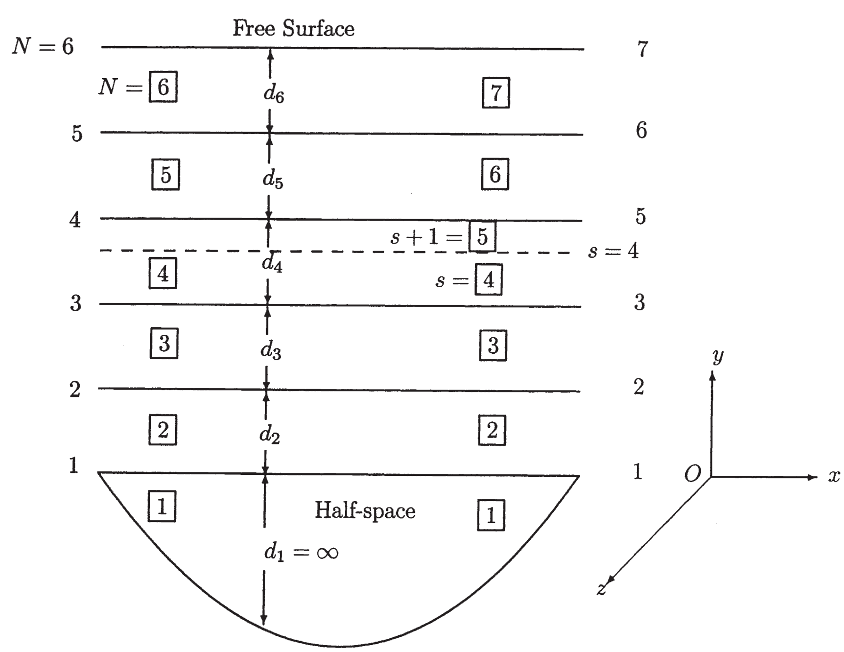

Fig. 1. Geometry and labeling of a horizontally layered body.

from the source point to the layer interface. If the layers are even moderately thick, then the terms $e^{k\Delta y}$ that need to be evaluated in order to calculate the terms in the stiffness matrix become excessively large.

Gilbert and Backus [5] introduced the so-called propagator matrix method which was developed further in the geophysical literature (see for example [6,16,17]). In order to deal with the problem of excessively large exponentials for large kd products (where k is the wavenumber and d is the layer thickness) a special rescaling of the propagator matrix is necessary (see [14]). A variant of this approach, which is known as the transfer matrix method, has also been considered (see [10,11,13]) for the solution of the systems of chain-like equations associated with layered media. Maier and Novati [11] have reported serious ill-conditioning problems associated with this method when the ratio of the layer thickness d to the mesh size h becomes large. This is analogous to the problems with large kd products associated with the propagator matrix method.

A third method known as the flexibility matrix method was developed by Buffler [1–3] and has been developed further by a number of authors (see for example [9]). A variant of this technique, called the successive stiffness method, was introduced by Maier and Novati [12] in the context of the solution of the boundary element equations of layered structures to remedy the above-mentioned ill-conditioning problems associated with the transfer matrix method. The flexibility matrix method has the distinct advantage over the previous two methods in that it can deal with extremely thick layers and large wavenumbers k – both of which manifest themselves in large products of the form kd. One drawback of this method, however, is that it cannot treat extremely thin layers and small wavenumbers – which manifest themselves in extremely small products of the form kd (see [9] for a discussion and analysis of this drawback). This latter situation of extremely small kd products will typically arise when the pack of layers being considered contains a combination of very thin layers and very thick layers in the same system. Effectively this situation is equivalent to solving a coupled system of ODEs in which the length scales of the problem are very different. Such a system of ODEs is referred to as a stiff system, which is well known to be difficult to solve numerically, owing to the very different length scales in the problem.

In this paper we describe the scaled flexibility matrix method that is useful for solving boundary value problems in layered elastic materials in which there are very thin layers and very thick layers in the same problem. A satisfactory solution to such problems using a Fourier technique involves flexibility matrix components for very small kd products that result from the layers with very small thicknesses d as well as very small wavenumbers k which are needed to capture the influence of the very thick layers. The technique involves using power series expansions to express the terms of the flexibility matrices for the layers in which the product kd is very small. The dominant singular terms in the power series expansions are extracted and factored out leaving only terms which are regular. All the remaining terms of the singular flexibility matrices are rescaled using these dominant terms in the power series expansion, which yields a well-conditioned system of equations. The technique we describe can be used to solve problems involving layers that are 10 times thinner than can be treated by means of the classic flexibility matrix method. Although the method is quite general, we apply it here to multilayered problems containing cracks which are oriented at 90° to the layer interfaces.

In Section 2 we summarize the governing equations and describe the use of the FT for boundary value problems in layered materials. In Section 3 we briefly review the classic stiffness matrix method, the propagator matrix method, and derive the flexibility matrix method. In Section 4 we provide the details of the scaled flexibility matrix method, including the power series expansions and the rescaled flexibility equations. In Section 5 we provide some numerical evidence of the performance of the method. The first problem we consider involves a layered material with thin layers in which there are no very thick layers so that the solution using the traditional flexibility matrix method and the solution using the rescaled flexibility matrix method can be compared to the solution obtained using the classic stiffness matrix method. The stiffness matrix method can be used to give a good reference solution to this problem because there are no very thick layers or large wavenumbers being used in the comparison. In the second illustrative example, we use the rescaled flexibility matrix method to solve a pressurized crack problem in which there is a very thin and much stiffer layer that is sandwiched between a moderately thick and a very thick layer. In Section 6 we provide some concluding remarks.

2. Governing equations and solution by the FT

2.1. The equilibrium equations for a layered elastic medium

Consider a linear elastic material that occupies a region in 3D space and which is in a state of equilibrium. In this case the stresses σ_{ij} and the strains $\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, which are defined in terms of the displacement gradients $u_{i,j} = \partial u_i / \partial x_j$ at any point within the body, are related by

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2G\epsilon_{ij}, \tag{2.1}$$

where λ and G are Lamé's constants that can be expressed in terms of Young's modulus E and Poisson's ratio v of the material by the formulae

$$\lambda = \frac{Ev}{(1+v)(1-2v)} \quad \text{and} \quad G = \frac{E}{2(1+v)}.$$

It is convenient to introduce the constants a, b, and f that are defined by: $a = \lambda + 2G$, $b = \lambda$, and $f = 2G$.

We assume that the elastic medium is in equilibrium so that the stresses satisfy the equilibrium equations

$$\sigma_{ij,j} + f_i = 0, \tag{2.2}$$

where f_i are the applied body forces.

It is also useful in this context, in which the layer properties do not change in the x and z directions but do vary in the y direction (see Fig. 1), to rewrite the system Eqs. (2.1) and (2.2) in the form of a system in which the y derivatives have been separated from the x and z derivatives:

$$\partial_y T = \mathscr{A} T + F, \tag{2.3}$$

where T represents the vector of stresses and displacements defined by

$$T = [\,\sigma_{yy} \quad \sigma_{xy} \quad \sigma_{yz} \quad u_y \quad u_x \quad u_z\,]^{\mathrm{T}},$$

the body force vector is given by

$$F = [\,-f_y \quad -f_x \quad -f_z \quad 0 \quad 0 \quad 0\,]^{\mathrm{T}},$$

and $\mathscr{A}$ is the differential operator, involving only x and z derivatives, defined by

$$\mathscr{A} = \begin{bmatrix} 0 & -\partial_x & -\partial_z & 0 & 0 & 0 \\ -\frac{b}{a}\partial_x & 0 & 0 & 0 & \frac{(b^2-a^2)}{a}\partial_{xx} - \frac{f}{2}\partial_{zz} & \left(\frac{(b^2-ab)}{a} - \frac{f}{2}\right)\partial_{xz} \\ -\frac{b}{a}\partial_z & 0 & 0 & 0 & \left(\frac{(b^2-ab)}{a} - \frac{f}{2}\right)\partial_{xz} & \frac{(b^2-a^2)}{a}\partial_{zz} - \frac{f}{2}\partial_{xx} \\ \frac{1}{a} & 0 & 0 & 0 & -\frac{b}{a}\partial_x & -\frac{b}{a}\partial_z \\ 0 & \frac{2}{f} & 0 & -\partial_x & 0 & 0 \\ 0 & 0 & \frac{2}{f} & -\partial_z & 0 & 0 \end{bmatrix} \tag{2.4}$$

We assume that the elastic body is piecewise homogeneous and that the body can be divided into N layers in which the moduli can be different (see Fig. 1). Depending on the problem being considered, the pack of N layers can either extend to ∞ in both directions or there can be a free surface on the top of the pack of layers which rests on an elastic half-space (as shown in Fig. 1). In this paper we assume the latter situation. We assign numbers to the layers starting from layer 1 for the bottom half-space and ending with layer number N for the top layer adjacent to the free surface. These layer indices are represented by the boxed sequence of numbers on the left-hand side of Fig. 1. The layer interfaces are numbered in a similar way and the sequence of interface indices for this problem are shown on the extreme left-hand side of Fig. 1. Observe that the interface at the top of a layer has the same index as the layer itself. The thicknesses of the layers, d_i, which may all be distinct, are also shown in the figure. Similarly, the symbols E_i and v_i are used to denote the elastic moduli of the ith layer. We introduce a Cartesian coordinate system $Oxyz$ in which the x

and z axes are aligned with the horizontal layers and in which the y coordinates are measured upwards from the interface between the pack of layers and the bottom half-space (see Fig. 1).

Point displacement or force discontinuities can be introduced into the N-layer elastic medium by specifying appropriate jump conditions in the stress and displacement fields across a horizontal layer having the same y coordinate as the desired source point. This is achieved by introducing a pseudo-interface, which is represented by the dashed line through layer 4 in Fig. 1. This process divides layer 4 into two layers for the purpose of this source computation and increases the number of layers by one. For the purpose of the computation the layers are renumbered using the same procedure as before and the layer numbers and interface indices are shown on the right-hand side of Fig. 1. The symbol s will be reserved for the sth layer immediately below the pseudo "source" interface. In the algorithm that is developed in this paper, it is necessary to be able to deal with multiple sources. To avoid having to re-sort all the layer properties from the configuration on the left of the figure to that on the right of the figure, a permutation vector is used to enable one to access the appropriate material properties (whose indices are shown on the left-hand side of the figure) while running through the indices which include the pseudo-source layer.

2.2. The FT solution

There exist reports in the literature on the application of the FT to singular solutions for elastic media [18] and to layered isotropic [1,2,7,16], and even layered transversely isotropic media [8,14,17,20]. The FT is the fundamental device that we will use in this paper to exploit the horizontal layering of the elastic medium being considered. The fact that the material properties do not vary in the x and z directions implies that the FT can be applied to the system of partial differential Eqs. (2.1) and (2.2) to reduce them to a system of ordinary differential equations in the independent variable y for each of the stress and displacement components in each of the layers (see Appendix A for the definition of the FT used in this paper). Interpreting the spatial wavenumber as a parameter, it is possible to obtain the general solution to the system of ordinary differential equations in each of the layers which involves six arbitrary constants that need to be determined for each layer. The stresses and displacements in the whole pack of N layers are obtained by setting up and solving a system of algebraic equations for the undetermined constants that express the appropriate conditions of continuity of tractions and displacements across the layer interfaces as well as the appropriate jump conditions across the pseudo-interface representing the source. Once these constants have been determined, the FTs of the stress and displacement fields are known and the spatial stress and displacement fields can be obtained by applying the inverse FT.

2.2.1. Reduction of the layer PDEs to a system of ODEs

By taking the FT of the equilibrium equations (2.2) in the absence of body forces as well as the stress–strain relations (2.1), eliminating the stresses from the resulting system of equations, introducing the change of variables (see [20]):

$$\begin{aligned}\widehat{u}_s &= -i(m\widehat{u}_x + n\widehat{u}_z)/k,\\ \widehat{u}_t &= -i(n\widehat{u}_x - m\widehat{u}_z)/k,\end{aligned} \tag{2.5}$$

where $k = \sqrt{m^2 + n^2}$, and making use of the relations $\frac{1}{2}(a+b) = \lambda + G = b + (f/2)$ and $\frac{1}{2}(a-b) = G = f/2$, we obtain the following system of ordinary differential equations for the FT of the displacement components in a typical layer:

$$\begin{aligned}&a\widehat{u}_y'' + \frac{1}{2}(a+b)k\widehat{u}_s' - \frac{1}{2}(a-b)k^2\widehat{u}_y = 0,\\ &\frac{1}{2}(a-b)\widehat{u}_s'' - \frac{1}{2}(a+b)k\widehat{u}_y' - ak^2\widehat{u}_s = 0,\\ &\widehat{u}_t'' - k^2\widehat{u}_t = 0.\end{aligned} \tag{2.6}$$

Alternatively by taking the FT of the system of equations (2.3) we obtain

$$\partial_y \widehat{T} = \widehat{\mathscr{A}}\widehat{T} + \widehat{F}, \tag{2.7}$$

where

$$\widehat{\mathscr{A}} = \begin{bmatrix} 0 & -k & 0 & 0 & 0 & 0 \\ \frac{b}{a}k & 0 & 0 & \frac{(a^2-b^2)}{a}k^2 & 0 & 0 \\ \frac{1}{a} & 0 & 0 & -\frac{b}{a}k & 0 & 0 \\ 0 & \frac{2}{f} & k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{f}{2}k \\ 0 & 0 & 0 & 0 & \frac{2}{f} & 0 \end{bmatrix} \tag{2.8}$$

where the elements of $\widehat{T}$ and $\widehat{F}$ have been arranged as follows:

$$\widehat{T} = \left[\widehat{\sigma}_{yy}, \widehat{\tau}_s, \widehat{u}_y, \widehat{u}_s, \widehat{\tau}_t, \widehat{u}_t\right]^{\mathrm{T}}$$

and

$$\widehat{F} = [-\widehat{f}_y, -\widehat{f}_s, 0, 0, -\widehat{f}_t, 0]^{\mathrm{T}},$$

where

$$\begin{aligned} \widehat{\tau}_s &= -i(m\widehat{\sigma}_{xy} + n\widehat{\sigma}_{yz})/k, \\ \widehat{\tau}_t &= -i(n\widehat{\sigma}_{xy} - m\widehat{\sigma}_{yz})/k. \end{aligned} \tag{2.9}$$

We observe that unknowns involving $\widehat{\sigma}_{yy}$, $\widehat{\tau}_s$, $\widehat{u}_y$ and $\widehat{u}_s$ (the s-subsystem) are completely decoupled from the unknowns involving $\widehat{\tau}_t$ and $\widehat{u}_t$ (the t-subsystem). The s-subsystem is sufficient to determine boundary value problems for 2D plane strain, while the autonomous t-subsystem is the only additional part that needs to be added to the plane strain equations in order to determine boundary value problems in 3D. A similar decoupling of the spectral ODEs also occurs if the Hankel transformation is applied to the layered elasticity problem (see for example [6,16]).

2.2.2. Exact solution to the layer ODEs and spectral coefficients

Considering the wavenumber k as a parameter, we can now determine the homogeneous solution to the system of ODEs (2.6) (see [20]), which can be expressed in terms of solutions to the s-subsystem and the t-subsystem as follows:

$$\begin{bmatrix} T_s \\ T_t \end{bmatrix} = \begin{bmatrix} Z_s & 0 \\ 0 & Z_t \end{bmatrix} \begin{bmatrix} A_s \\ A_t \end{bmatrix}, \tag{2.10}$$

where

$$T_s = [\,\widehat{\sigma}^l_{yy}/k \quad \widehat{\tau}^l_s/k \quad \widehat{u}^l_y \quad \widehat{u}^l_s\,]^{\mathrm{T}} \quad \text{and} \quad T_t = [\,\widehat{\tau}^l_t/k \quad \widehat{u}^l_t\,]^{\mathrm{T}}$$

$$A_s = [\,A_1 \quad A_2 \quad A_3 \quad A_4\,]^{\mathrm{T}} \quad \text{and} \quad A_t = [\,A_5 \quad A_6\,]^{\mathrm{T}}$$

and

$$Z_s = \begin{bmatrix} -fe^{-ky} & (l_4 - fky)e^{-ky} & fe^{ky} & (l_4 + fky)e^{ky} \\ -fe^{-ky} & (l_5 - fky)e^{-ky} & -fe^{ky} & -(l_5 + fky)e^{ky} \\ e^{-ky} & kye^{-ky} & e^{ky} & kye^{ky} \\ e^{-ky} & (ky - l_2)e^{-ky} & -e^{ky} & -(ky + l_2)e^{ky} \end{bmatrix}$$

and

$$Z_t = \begin{bmatrix} -\frac{f}{2}e^{-ky} & \frac{f}{2}e^{ky} \\ e^{-ky} & e^{ky} \end{bmatrix}.$$

The constants l_j in (2.10) are defined as follows:

$$l_2 = \frac{\lambda + 3G}{\lambda + G}, \quad l_4 = \frac{2G^2}{\lambda + G}, \quad l_5 = \frac{2G(\lambda + 2G)}{\lambda + G}. \tag{2.11}$$

The following identities between the above constants are useful:

$$fl_2 = l_5 + l_4, \quad f = l_5 - l_4. \tag{2.12}$$

It is important to note that the spectral coefficients required to define the primary variables can be expressed entirely in terms of the single wavenumber parameter $k = \sqrt{m^2 + n^2}$. This property can be exploited to reduce the FT inversion problem from one which involves sampling the integrand at points throughout the (m, n) plane to what amounts to a 1D sampling of the wavenumber parameter k. We observe that the system of equations (2.6) for a typical layer remains invariant if a new length scale $Y = y/D$ is introduced while a new wavenumber variable $K = kD$ is defined. This invariance manifests itself in the layer solution (2.10) in that the independent variable y is always multiplied by the wavenumber k in the final solution.

The unknown coefficients $A_j(k)$ depend on the parameter k and we will refer to them as the spectral coefficients throughout this paper. It will be seen that the spectral coefficients provide a useful representation of the solution as they separate the exponentially decaying part of the solution from the exponentially growing part of the solution. Once the spectral coefficients in any one layer are known, it is then possible using (2.10) to determine the stresses and displacements at any desired point within that layer. Other representations of the solution that arise, depending on the technique used to solve the system of algebraic equations, involve the displacements and stresses at the interfaces between layers.

In order to model vertical fractures that run perpendicular to the layers, it is necessary to have an expression for the stress normal to the fracture surface. In the coordinate systems defined in Fig. 1 it is necessary to determine the stress component $\widehat{\sigma}_{zz}$, which can be defined in terms of the spectral coefficients as follows:

$$\begin{aligned} k\widehat{\sigma}_{zz} = {} & fn^2A_1e^{-ky} + (-l_6n^2 - l_7m^2 + fn^2ky)A_2e^{-ky} - fmnA_5e^{-ky} \\ & - fn^2A_3e^{ky} + (-l_6n^2 - l_7m^2 - fn^2ky)A_4e^{ky} - fmnA_6e^{ky}, \end{aligned} \tag{2.13}$$

where we have defined the new constants

$$l_6 = \frac{2G(2\lambda + 3G)}{\lambda + G} \quad \text{and} \quad l_7 = \frac{2\lambda G}{\lambda + G}.$$

Once the values of the spectral coefficients $A_j(k)$ in each layer have been determined (this will be dealt with in the following section) and the FT of the displacements $\widehat{u}_i(k)$ and stresses $\widehat{\sigma}_{ij}(k)$ within each layer have been determined, then the displacements and stresses within each layer can be determined by applying formula (A.2) for the inversion of the FT.

3. Methods to solve the layer ODEs

In this section we describe three methods that can be used to determine the solution to the coupled system of algebraic equations that need to be solved in order to determine the spectral coefficients $A_j(k)$ or equivalently the FTs of the displacements and stresses on the layer interfaces. These methods have been described in the literature [3,5,9,20] and will only be summarized here for the sake of completeness. We will be exploiting various aspects of all three methods of solution so it will be necessary to introduce them in a uniform notation and to briefly discuss their relative advantages and disadvantages.

All the methods that we discuss rely on a fairly simple idea that is common to all techniques for solving problems for layered elastic media. We first establish the equations that determine the stiffness and compliance properties of each of the layers in terms of the degrees of freedom of the model. For example the degrees of freedom for a finite element, finite difference, or boundary element model will be unknown nodal displacements at the mesh points of the numerical model. For the spectral methods that we use in this paper, the degrees of freedom in the model are represented by the unknown constants, which we call spectral coefficients, that are parameterized by the wavenumber k. Once we have established equations for the stresses and displacements within each of the layers in terms of the internal degrees of freedom, we bond all the layers together at their common interfaces by imposing conditions of continuity in displacements and tractions across the interfaces. Discontinuous sources (such as force discontinuities or displacement discontinuities) can be represented by introducing the appropriate jump conditions across pseudo-interfaces introduced for this purpose. Finally, the whole mechanical problem is completed into a well-posed system of equations by introducing the appropriate conditions at the boundaries of the pack of layers, e.g., specified tractions, specified displacements, or a complementary combination of tractions and displacements.

3.1. The stiffness matrix method

The first of the three methods we describe in this section involves setting up a system of algebraic equations that express the continuity or jumps in displacements and stresses across layer interfaces. This method, described by Wardle [20], essentially involves using (2.10) to set up a system of equations for the jump conditions across the various interfaces in the layered medium. Two materials that are bonded are represented in these equations by tractions and displacements that are continuous across their shared interface, while a source with the desired properties can be represented by the appropriate jump conditions across an interface. If the source is not actually located at an interface between two materials, then a pseudo-interface is introduced at the desired y coordinate and appropriate jump conditions are prescribed across such an interface.

Consider the normal and shear components of the stresses and displacements from (2.10) for the lth layer, which we can rewrite in the following form:

$$\begin{bmatrix} \widehat{\sigma}^l_{yy}/k \\ \widehat{\tau}^l_s/k \\ \widehat{u}^l_y \\ \widehat{u}^l_s \\ \widehat{\tau}^l_t/k \\ \widehat{u}^l_t \end{bmatrix} = \begin{bmatrix} -f^l e^{-ky} & (l^l_4 - f^l ky)e^{-ky} & f^l e^{ky} & (l^l_4 + f^l ky)e^{ky} & 0 & 0 \\ -f^l e^{-ky} & (l^l_5 - f^l ky)e^{-ky} & -f^l e^{ky} & -(l^l_5 + f^l ky)e^{ky} & 0 & 0 \\ e^{-ky} & kye^{-ky} & e^{ky} & kye^{ky} & 0 & 0 \\ e^{-ky} & (ky - l^l_2)e^{-ky} & -e^{ky} & -(ky + l^l_2)e^{ky} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{f^l}{2}e^{-ky} & \frac{f^l}{2}e^{ky} \\ 0 & 0 & 0 & 0 & e^{-ky} & e^{ky} \end{bmatrix} \begin{bmatrix} A^l_1(k) \\ A^l_2(k) \\ A^l_3(k) \\ A^l_4(k) \\ A^l_5(k) \\ A^l_6(k) \end{bmatrix} \tag{3.1}$$

Here the superscripts l indicate that the material constants l^l_j, the spectral coefficients $A^l_j(k)$, and the FTs of the stresses and displacements are located in the lth layer. We can express (3.1) in the following compact forms:

$$T^l_s(k,y) = Z^l_s(k,y)A^l_s(k) \quad \text{and} \quad T^l_t(k,y) = Z^l_t(k,y)A^l_t(k), \tag{3.2}$$

where

$$T^l_t(k,y) = [\,\widehat{\sigma}^l_{yy}/k \quad \widehat{\tau}^l_s/k \quad \widehat{u}^l_y \quad \widehat{u}^l_s\,]^{\mathrm{T}}, \quad T^l_t(k,y) = [\,\widehat{\tau}^l_t/k \quad \widehat{u}^l_s\,]^{\mathrm{T}},$$

$$A^l_s(k) = [\,A^l_1(k) \quad A^l_2(k) \quad A^l_3(k) \quad A^l_4(k)\,]^{\mathrm{T}}, \quad A^l_t(k) = [\,A^l_5(k) \quad A^l_6(k)\,]^{\mathrm{T}},$$

and $Z^l_s(k,y)$ and $Z^l_t(k,y)$ from (2.10) represent the s and t submatrices evident in (3.1).

We observe that for very thick layers some of the y_j will be very large so that even for relatively moderate values of k, the system matrix in (3.1) will become poorly conditioned due to the exponentially large and exponentially small terms that appear simultaneously in the matrix. For this reason, the stiffness matrix approach is not widely used in the computations of such layered spectra.

3.2. The propagator matrix method

In this section we briefly describe the propagator matrix method introduced by Gilbert and Backus [5]. Consider the stresses and displacements $T^l_s(k,y)$ and $T^l_t(k,y)$ at any point y within the lth layer of the layered medium, which can be expressed in terms of the lth layer spectral coefficients according to (3.2). It is possible, by inverting the s-stiffness matrix, to rewrite the spectral coefficients in terms of the stress and displacement vector $T_s^l(k,y)$ as follows:

$$\begin{bmatrix} A_1^l(k) \\ A_2^l(k) \\ A_3^l(k) \\ A_4^l(k) \end{bmatrix} = \frac{1}{2l_5^l} \begin{bmatrix} (ky - l_2^l)e^{ky} & -kye^{ky} & (l_5^l - f^l ky)e^{ky} & (-l_4^l + f^l ky)e^{ky} \\ -e^{ky} & e^{ky} & f^l e^{ky} & -f^l e^{ky} \\ (l_2^l + ky)e^{-ky} & kye^{-ky} & (l_5^l + f^l ky)e^{-ky} & (l_4^l + f^l ky)e^{-ky} \\ -e^{-ky} & -e^{-ky} & -f^l e^{-ky} & -f^l e^{-ky} \end{bmatrix} \begin{bmatrix} \widehat{\sigma}_{yy}^l/k \\ \widehat{\tau}_s^l/k \\ \widehat{u}_y^l \\ \widehat{u}_s^l \end{bmatrix} \tag{3.3}$$

Similarly for the t-subsystem we obtain the following expression for the spectral coefficients in terms of the stress–displacement vector:

$$\begin{bmatrix} A_5^l(k) \\ A_6^l(k) \end{bmatrix} \begin{bmatrix} -e^{ky}/f^l & e^{ky}/2 \\ e^{-ky}/f^l & e^{-ky}/2 \end{bmatrix} \begin{bmatrix} \widehat{\tau}_t^l/k \\ \widehat{u}_t^l \end{bmatrix}. \tag{3.4}$$

These equations can be rewritten in the more compact form:

$$A_s^l(k) = Z_s^l(k,y)^{-1} T_s^l(k,y) \quad \text{and} \quad A_t^l(k) = Z_t^l(k,y)^{-1} T_t^l(k,y). \tag{3.5}$$

If we combine (3.2) and (3.5) evaluated, respectively, at y_l (the coordinate of the interface at the top of the lth layer) and y_{l-1} (the coordinate of the interface at the bottom of the lth layer), then eliminating the $A^l(k)$ we obtain the following relationship between the stresses and displacements at the bottom and at the top of the lth layer:

$$T^l(k,y_l) = Z^l(k,y_l)\left\{Z^l(k,y_{l-1})^{-1} T^l(k,y_{l-1})\right\} \tag{3.6}$$

$$= \left[Z^l(k,y_l) Z^l(k,y_{l-1})^{-1}\right] T^l(k,y_{l-1}) \tag{3.7}$$

$$= P(y_l,y_{l-1}) T^l(k,y_{l-1}). \tag{3.8}$$

This procedure can be followed for both the s- and t-subsystems. From (3.5) we observe that the expression in the curly brackets in (3.6) represents the spectral coefficient $A^l(k)$ that has been eliminated. The expression in the square brackets in (3.7) represents the propagator matrix $P(y_l,y_{l-1})$ that transfers the stresses and displacements from the bottom interface of the lth layer to the top interface of the lth layer. The explicit expression for the propagator matrix is most efficiently derived by choosing the coordinate system to coincide with the bottom interface of the lth layer so that $y_{l-1} = 0$ and $y_l = d_l$. In this case $P(y_l,y_{l-1})$ for the s-subsystem can be written in the form:

$$\begin{aligned} P_{l,l-1} &= P(y_l = d_l, y_{l-1} = 0) \\ &= \frac{1}{2l_5} \begin{bmatrix} l_5 ch - fkdsh & -l_4 sh - fkdch & f^2(sh - kdch) & -f^2 kdsh \\ -l_4 sh + fkdch & l_5 ch + fkdsh & f^2 kdsh & f^2(sh + kdch) \\ l_2 sh - kdch & -kdsh & l_5 ch - fkdsh & l_4 sh - fkdch \\ kdsh & l_2 sh + kdch & l_4 sh + fkdch & l_5 ch + fkdsh \end{bmatrix} \\ &= \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}_{l,l-1}. \end{aligned} \tag{3.9}$$

Here for the sake of brevity, the superscripts for identifying the lth layer have been suppressed, and the symbols $sh = \sinh(kd_l)$ and $ch = \cosh(kd_l)$ have been introduced. The propagator $P(y_l,y_{l-1})$ for the t-subsystem can be written in the form

$$P_{l,l-1} = P(y_l = d_l, y_{l-1} = 0) = \begin{bmatrix} ch & \frac{f}{2} sh \\ \frac{2}{f} sh & ch \end{bmatrix}.$$

In order to illustrate how the propagator matrix method can be used, assume that in our N-layer problem the displacements at the bottom interface (interface number 1 in Fig. 1) of a pack of layers are known and that the top interface (interface number $N+1$ using the right-hand side numbering scheme in Fig. 1) is stress free. Assume that the source can be represented by the jump Δb_s in the tractions and displacements across the sth interface, i.e.,

$$T_s^{s+1} = T^{s+1}(k, y_s) = T^s(k, y_s) + \Delta b_s.$$

We also note that we can apply (3.8) recursively starting at the top layer in the following way:

$$\begin{aligned}
T_{N+1}^{N+1} &= T^{N+1}(k, y_{N+1}) \\
&= P_{N+1,N} T_N^{N+1} \\
&= P_{N+1,N} T_N^N \quad \text{since } T \text{ is continuous} \\
&= P_{N+1,N} P_{N,N-1} T_{N-1}^N \\
&= \cdots \\
&= \prod_{r=s}^{N} P_{r+1,r} T_s^{s+1} \\
&= \prod_{r=s}^{N} P_{r+1,r} (T_s^s + \Delta b_s) && (3.10) \\
&= \prod_{r=1}^{N} P_{r+1,r} T_1^1 + \prod_{r=s}^{N} P_{r+1,r} \Delta b_s && (3.11) \\
&= \mathscr{P}_{N+1,1} T_1^1 + \mathscr{P}_{N+1,s} \Delta b_s. && (3.12)
\end{aligned}$$

Now since Δb_s is known, Eq. (3.11) can be used to write a single system of four equations in the two unknown stress components (a system of two equations in the one unknown stress component) at the bottom layer and the two unknown displacement components (the single unknown displacement component) at the free surface in the case of the s-subsystem (in the case of the t-subsystem). This assumes that the products $\mathscr{P}_{N+1,1}$ and $\mathscr{P}_{N+1,s}$ indicated in (3.12) have been evaluated to determine the operators that are applied to T_1^1 and Δb_s. Once the stresses and displacements are known in the bottom layer, it is possible to determine the stresses and displacements in any other layer by applying the appropriate product of propagator matrices.

One of the major drawbacks of the propagator matrix method is encountered in the presence of very thick layers or large wavenumbers k – either of which lead to large products of the form kd. In this case the successive application of the matrices that involve exponentially large terms leads to operators $\mathscr{P}_{N+1,1}$ and $\mathscr{P}_{N+1,s}$ that become unbounded when they are evaluated directly. However, the result of the application of the operators $\mathscr{P}_{N+1,1}$ and $\mathscr{P}_{N+1,s}$, namely T_{N+1}^{N+1}, should be bounded. This results from the cancellation of the exponentially growing and decaying terms in the sequence of component operators, which can lead to numerical cancellation. Various authors (see for example [14]) have proposed schemes which involve scaling out the exponentially growing and decaying terms explicitly so that they can be canceled analytically thus avoiding the undesirable numerical cancellation. However, the re-scaled propagator matrices still suffer from ill-conditioning problems. Indeed, a number of papers (see for example [9–11]) report serious ill-conditioning for the propagator matrix method if more than five layers of moderate thickness are used. We shall, therefore, not pursue the propagator matrix method as a computational device, but will utilize it in the theoretical developments that follow.

3.3. The flexibility matrix method

The flexibility matrix method was developed by Buffler [1–3] and has been developed further by a number of authors (see for example [9,10]). The flexibility matrix method has the distinct advantage over the previous two methods outlined in this section in that it can deal with extremely thick layers and large wavenumbers k – both of which manifest themselves in large products of the form kd. One drawback of this method is that it cannot treat extremely thin layers and small wavenumbers – which manifest themselves in extremely small products of the form kd. This latter situation of extremely small kd products will typically arise when the pack of layers being considered contains a combination of very thin layers and very thick layers in the same system.

3.3.1. The flexibility matrix

A convenient starting point for the flexibility matrix method is the expression for the propagator matrix given in (3.9). Rather than determining the relationship between the stresses and displacements (as represented by the vectors $T = [\,\widehat{\sigma}^l_{yy}/k \quad \widehat{\tau}^l_s/k \quad \widehat{u}^l_y \quad \widehat{u}^l_s\,]^{\mathrm{T}}$ for the s-subsystem, and $T = [\,\widehat{\tau}^l_t/k \quad \widehat{u}^l_t\,]^{\mathrm{T}}$ for the t-subsystem) at the top and the bottom of the l-th layer, the strategy in this case, is to separate the stresses and displacements and to represent the displacements in terms of the stresses (or vice versa) at the top and the bottom of the layers. To formulate this technique it is useful to separate the combined stress–displacement vectors T into stress components, which we represent by the symbol p, and displacement components which we represent by the symbol u. In the case of the s-subsystem we have the following decomposition:

$$T_s = \begin{bmatrix} \widehat{\sigma}_{yy}/k \\ \widehat{\tau}_s/k \\ \widehat{u}_y \\ \widehat{u}_s \end{bmatrix} = \begin{bmatrix} p \\ u \end{bmatrix} \quad \text{for the } s\text{-subsystem,}$$

where p and u are the stress and displacement vectors defined by

$$p_s = \begin{bmatrix} \widehat{\sigma}_{yy}/k \\ \widehat{\tau}_s/k \end{bmatrix} \quad \text{and} \quad u_s = \begin{bmatrix} \widehat{u}_y \\ \widehat{u}_s \end{bmatrix}.$$

While for the t-subsystem we have the following decomposition:

$$T_t = \begin{bmatrix} \widehat{\tau}_t/k \\ \widehat{u}_t \end{bmatrix} = \begin{bmatrix} p \\ u \end{bmatrix} \quad \text{for the } t\text{-subsystem,}$$

where in this case p and u are the stress and displacement components defined by: $p = \widehat{\tau}_t/k$ and $u = \widehat{u}_t$.

In what follows we will, for the sake of brevity, not use the "hats" to denote the fact that u and p are the FTs of the displacements and stresses. In addition to avoid confusion, we will not introduce notation to distinguish between the s- and t-subsystems explicitly, since their flexibility equations have precisely the same form but with different flexibility matrices in each case. Using this notation Eq. (3.8) can be rewritten in the following form:

$$\begin{bmatrix} p^l_t \\ u^l_t \end{bmatrix} = P(y_l, y_{l-1}) \begin{bmatrix} p^l_b \\ u^l_b \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}_{l,l-1} \begin{bmatrix} p^l_b \\ u^l_b \end{bmatrix}. \tag{3.13}$$

Here the index l refers to the layer number while the subscripts t and b refer to the quantities defined at the top and bottom of the layer. Eq. (3.13) can be used to express the displacements in terms of stresses using the flexibility matrix R^l as follows:

$$\begin{bmatrix} u^l_t \\ u^l_b \end{bmatrix} = \begin{bmatrix} R^l_{tt} & R^l_{tb} \\ R^l_{bt} & R^l_{bb} \end{bmatrix} \begin{bmatrix} p^l_t \\ p^l_b \end{bmatrix}, \tag{3.14}$$

where the flexibility submatrices are determined by: $R^l_{tt} = P_{22}P_{12}^{-1}$, $R^l_{tb} = P_{21} - P_{22}P_{12}^{-1}P_{11}$, $R^l_{bt} = P_{12}^{-1}$, and $R^l_{bb} = -P_{12}^{-1}P_{11}$. The explicit expressions for these flexibility submatrices for the s-subsystem are as follows:

$$R_{tt} = \frac{1}{D} \begin{bmatrix} -l_5(th + kd.se^2) & -(l_4 th^2 + fk^2d^2se^2) \\ -(l_4 th^2 + fk^2d^2se^2) & -l_5(th - kd.se^2) \end{bmatrix}, \tag{3.15}$$

$$R_{bb} = \frac{1}{D}\begin{bmatrix} l_5(th + kd.se^2) & -(l_4 th^2 + fk^2d^2se^2) \\ -(l_4 th^2 + fk^2d^2se^2) & l_5(th - kd.se^2) \end{bmatrix}, \tag{3.16}$$

$$R_{bt} = \frac{l_5}{D}\begin{bmatrix} -(th + kd)se & -k.d.th.se \\ k.d.th.se & -(th - kd)se \end{bmatrix}, \tag{3.17}$$

$$R_{tb} = \frac{l_5}{D}\begin{bmatrix} (th + kd)se & -k.d.th.se \\ k.d.th.se & (th - kd)se \end{bmatrix}, \tag{3.18}$$

where, for the sake of brevity the superscript l identifying the layer number has been omitted, we have used the notation $th = \tanh(kd)$, and $se = \operatorname{sech}(kd)$, and $D = f^2[(1 + k^2d^2)se^2 - 1]$. The explicit terms for the flexibility coefficients for the t-subsystem (for the t-system the $R_{tt}, R_{bb}, \ldots$ are numbers and not matrices) are as follows:

$$R_{tt} = \frac{2}{f}\coth(kd),$$

$$R_{bb} = -\frac{2}{f}\coth(kd),$$

$$R_{bt} = \frac{2}{f}\operatorname{cosech}(kd),$$

$$R_{tb} = -\frac{2}{f}\operatorname{cosech}(kd).$$

It is also possible to express the tractions in terms of the displacements by taking the inverse of the flexibility matrix defined in (3.14).

3.3.2. Second-order difference equations for tractions

If we assume that the ith interface (see Fig. 2) is fully bonded then the jump in the stresses $\Delta p^i = p_b^{i+1} - p_t^i$ or the displacements $\Delta u^i = u_b^{i+1} - u_t^i$ is 0. If there are prescribed jump conditions they can be expressed in the following form:

$$p_b^{i+1} = p_t^i + \Delta p^i, \quad u_b^{i+1} = u_t^i + \Delta u^i. \tag{3.19}$$

Now using the conditions (3.19) in (3.14) we obtain the following equations for the displacements either side of the ith interface that lies at the top of the ith layer:

$$\otimes: \quad u_t^i = R_{tt}^i p_t^i + R_{tb}^i p_b^i = R_{tt}^i p_t^i + R_{tb}^i (p_t^{i-1} + \Delta p^{i-1}), \tag{3.20}$$

$$\odot: \quad u_b^{i+1} = R_{bt}^{i+1} p_t^{i+1} + R_{bb}^{i+1} p_b^{i+1} = R_{bt}^{i+1} p_t^{i+1} + R_{bb}^{i+1} (p_t^i + \Delta p^i). \tag{3.21}$$

Subtracting (3.20) from (3.21) and using the jump condition for the displacements given in (3.19) we obtain

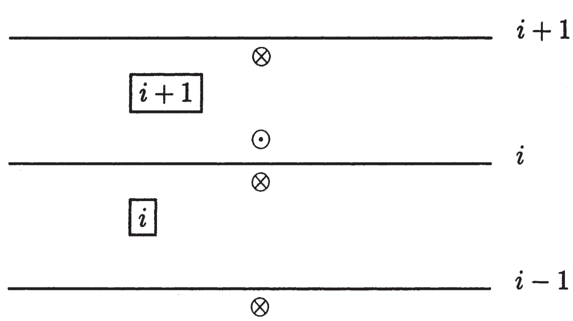

Fig. 2. Flexibility matrix sample points.

$$\Delta u^i = R_{bt}^{i+1} p_t^{i+1} + (R_{bb}^{i+1} - R_{tt}^i) p_t^i - R_{tb}^i p_t^{i-1} + R_{bb}^{i+1} \Delta p^i - R_{tb}^i \Delta p^{i-1}. \tag{3.22}$$

Since the quantities Δu^i, Δp^i, and Δp^{i-1} are specified in a typical problem, separating the known from the unknown quantities, we can rewrite (3.22) in the form of a set of vector recurrence relations or difference equations for the tractions p_t at the top of three successive layers (these points are marked by the $\otimes$ symbol in Fig. 2)

$$B^i p_t^{i+1} + C^i p_t^i + A^i p_t^{i-1} = \Delta^i, \tag{3.23}$$

where $B^i = R_{bt}^{i+1},\ C^i = (R_{bb}^{i+1} - R_{tt}^i),\ A^i = -R_{tb}^i$, and $\Delta^i = \Delta u^i - R_{bb}^{i+1} \Delta p^i + R_{tb}^i \Delta p^{i-1}$.

Once the traction vectors p_t^i at the top of each of the layers have been determined, the traction vectors below each of the layers can be readily determined from (3.19). An efficient recursive procedure to solve the block tri-diagonal system (3.23) can be obtained by using a block LU decomposition (see [9]). The flexibility relation (3.14) can be used to determine the displacements on either side of each of the interfaces.

4. The rescaled flexibility equations for thin layers

For very thin layers or small wavenumbers, the parameter $z = kd \ll 1$, and for the s-subsystem the coefficients in the flexibility matrices R_{tt}^l, R_{tb}^l, R_{bt}^l, and R_{bb}^l can be as large as $\mathrm{O}(z^{-3})$ while for the t-subsystem the coefficients can be as large as $\mathrm{O}(z^{-1})$. In this case, the expressions for the flexibility matrices given in (3.15)–(3.18) are not suitable for determining the appropriate flexibility matrices because the denominator in these equations involves a term $\mathrm{O}(z^4)$ while the largest numerator coefficient involves terms of the size $\mathrm{O}(z)$. Taking products of this form can cause poor cancellation effects. The strategy we adopt in this section is to derive a new set of scaled difference equations for very thin layers that are equivalent to the ones they replace but which have substantially improved conditioning. The key idea in this process is to expand the terms in the flexibility matrices in power series that are valid for small z. We then identify the dominant terms in these matrices, whose coefficients are now expressed as power series, and determine the inverse of the lowest order matrix, whose largest coefficient is $\mathrm{O}(z)$. We then rescale all the equations for that particular interface by multiplying them by this lowest order inverse matrix to obtain an equation that is much better conditioned. Only the interfaces involving very thin layer products z need to be rescaled in this way, while the remaining equations which may deal with very thick layers for example, can be evaluated using the expressions given in (3.15)–(3.18).

In establishing this procedure, we identify four possible types of thin layer interactions. The first class involves the situation in which the layer below the interface is thinner than the layer above the interface. There are two possible subcases. The first in which the lower layer is much thinner than the upper layer – to the extent that a power series expansion in the lower layer is appropriate, whereas a power series expansion in the upper layer is not valid. The second subclass involves the case in which the bottom layer is thinner, but the top layer is also thin enough for a power series expansion of its flexibility matrices to be necessary. The second class involves the situation in which the layer above the interface is thinner than the layer below the interface. This also divides into two subcases as did the first class.

The starting point for all the cases that follow is the difference equation for the ith layer which we express in the following form:

$$R_{bt}^{i+1} p_t^{i+1} + (R_{bb}^{i+1} - R_{tt}^i) p_t^i - R_{tb}^i p_t^{i-1} = \Delta u^i - R_{bb}^{i+1} \Delta p^i + R_{tb}^i \Delta p^{i-1} = \Delta^i. \tag{4.1}$$

Class A: Lower layer is thinner. Let the lower layer be assigned the index i and let d_i be its thickness. The upper layer then has the index $i+1$ and its thickness is represented by d_{i+1}, while the interface between the two layers has the index i (see Fig. 2). Let $z_i = kd_i$, $z_{i+1} = kd_{i+1}$, and let z_T be the threshold below which the power series expansion defined below in (4.2) is valid, and below which the expressions given in (3.15)–(3.18) may give poor results. The magnitude of z_T will depend on the number of terms that have been maintained in the power series expansion as well as the precision required.

Case 1. $d_i \ll d_{i+1}$ and $z_i < z_T < z_{i+1}$.

In this case the terms R^i_{tt} and R^i_{tb} for the s-subsystem are going to involve some coefficients that blow up like z_i^{-3}. We therefore expand each of the coefficients in these two flexibility matrices in powers of z_i:

$$R^i_{tt} = \frac{l_5 z^{-3}}{f^2}\begin{bmatrix} 6+\frac{6}{5}z^2+\frac{32}{175}z^4+\cdots & 3z+(\frac{3}{5}-\frac{f}{l_5})z^3+\cdots \\ 3z+(\frac{3}{5}-\frac{f}{l_5})z^3+\cdots & 2z^2+\frac{2}{15}z^4+\cdots \end{bmatrix}. \tag{4.2}$$

Here we have, for the sake of brevity, omitted the indices on z, f, and l_5 that indicate that they apply to the ith layer.

We now identify the leading order term in (4.2):

$$R^{i0}_{tt} = \frac{l_5 z^{-3}}{f^2}\begin{bmatrix} 6 & 3z \\ 3z & 2z^2 \end{bmatrix}$$

and determine its inverse

$$(R^{i0}_{tt})^{-1} = \frac{f^2}{l_5}\begin{bmatrix} \frac{2}{3}z^3 & -z^2 \\ -z^2 & 2z \end{bmatrix}. \tag{4.3}$$

Now if we represent by $\widetilde{R}^i_{tt}$ the higher-order terms that remain after the leading order term R^{i0}_{tt} has been subtracted off then we can rewrite (4.2) in the form (only retaining terms that are larger than $\mathrm{O}(z^6)$)

$$\begin{aligned} R^i_{tt} &= R^{i0}_{tt} + \widetilde{R}^i_{tt} \\ &= R^{i0}_{tt}(I + (R^{i0}_{tt})^{-1}\widetilde{R}^i_{tt}) \\ &= R^{i0}_{tt}\left\{\begin{bmatrix} 1+(\frac{1}{5}+\frac{f}{l_5})z^2+\frac{17}{175}z^4+\cdots & 0+(\frac{4}{15}-\frac{2f}{3l_5})z^3+\cdots \\ 0-\frac{2f}{l_5}z-\frac{2}{15}z^3-\cdots & 1-(\frac{1}{3}-\frac{f}{l_5})z^2-\frac{1}{45}z^4+\cdots \end{bmatrix}\right\}. \end{aligned} \tag{4.4}$$

For the other flexibility matrix R^i_{tb} we have the following power series expansion:

$$R^i_{tb} = \frac{l_5 z^{-3}}{f^2}\begin{bmatrix} -6-\frac{6}{5}z^2+\frac{47}{700}z^4+\cdots & 3z+\frac{1}{10}z^3-\cdots \\ -3z-\frac{1}{10}z^3+\cdots & z^2-\frac{1}{30}z^4-\cdots \end{bmatrix}$$

which when multiplied by $(R^{i0}_{tt})^{-1}$ yields

$$(R^{i0}_{tt})^{-1}R^i_{tb} = \begin{bmatrix} -1-\frac{7}{10}z^2+\frac{39}{1400}z^4+\cdots & z+\frac{1}{10}z^3-\cdots \\ z-\frac{1}{30}z^3-\cdots & -1-\frac{1}{6}z^2+\frac{1}{72}z^4+\cdots \end{bmatrix}. \tag{4.5}$$

The appropriate rescaled equation in this case is

$$\overbrace{(R^{i0}_{tt})^{-1}R^{i+1}_{bt}}^{B^i}p^{i+1}_t + \overbrace{\left\{(R^{i0}_{tt})^{-1}R^{i+1}_{bb} - (I+(R^{i0}_{tt})^{-1}\widetilde{R}^i_{tt})\right\}}^{C^i}p^i_t + \overbrace{-(R^{i0}_{tt})^{-1}R^i_{tb}}^{A^i}p^{i-1}_t = (R^{i0}_{tt})^{-1}\Delta^i. \tag{4.6}$$

We observe that the problematic terms R^i_{tt} and R^i_{tb} of the unscaled equations (4.1) are now reduced to coefficient matrices that are O(1) or smaller. Multiplication by $(R^{i0}_{tt})^{-1}$ has the effect of making the remaining product terms smaller so that for very small z the rescaled equations (4.6) are much better conditioned. The rescaled forcing term on the right-hand side of (4.6), which we have represented symbolically as $(R^{i0}_{tt})^{-1}\Delta^i$, also has a singular term of the form $(R^{i0}_{tt})^{-1}R^i_{tb}\Delta p^{i-1}$ for which the power series expansion (4.5) of the matrix denoted by A^i (4.6) should be used.

For the t-subsystem the flexibility coefficients R_{tt} and R_{tb} only blow up as $\mathrm{O}(z^{-1})$ so rescaling is not as critical as it was in the case of the s-subsystem. In this case corresponding expansions of the flexibility coefficients are:

$$(R_{tt}^{i0})^{-1}R_{tt}^{i} = (1 + (R_{tt}^{i0})^{-1}\widetilde{R}_{tt}^{i}) = \left\{1 + \frac{1}{3}z^2 - \frac{1}{45}z^4 + \cdots\right\}, \tag{4.7}$$

$$(R_{tt}^{i0})^{-1}R_{tb}^{i} = -\left\{1 - \frac{1}{6}z^2 + \frac{7}{360}z^4 - \cdots\right\}, \tag{4.8}$$

where $(R_{tt}^{i0}) = \frac{2}{f^i}z^{-1}$ and no expansion of the terms $(R_{tt}^{i0})^{-1}R_{bb}^{i+1}$ and $(R_{tt}^{i0})^{-1}R_{bt}^{i+1}$ in (4.6) is necessary.

Case 2. $d_i < d_{i+1} \ll 1$ and $z_i < z_{i+1} < z_T$.

In this case the coefficient matrices R_{tt}^{i} and R_{tb}^{i} for the *s*-subsystem involve some coefficients that blow up like z_i^{-3}, while R_{bb}^{i+1} and R_{bt}^{i+1} involve some coefficients that blow up like z_{i+1}^{-3}. In this case let $d_{i+1} = d_i(1+\epsilon)$ so that

$$z_{i+1} = z_i(1+\epsilon)$$

We expand the terms R_{tt}^{i} and R_{bb}^{i+1} simultaneously and extract the lowest order term R_{tt}^{i0} of the matrix R_{tt}^{i} to yield

$$R_{bb}^{i+1} - R_{tt}^{i} = R_{tt}^{i0}\left\{(R_{tt}^{i0})^{-1}R_{bb}^{i+1} - (I + (R_{tt}^{i0})^{-1}\widetilde{R}_{tt}^{i})\right\} \tag{4.9}$$

The power series expansion of $(I + (R_{tt}^{i0})^{-1}\widetilde{R}_{tt}^{i})$ is given by (4.4) in which the symbol z has been replaced by z_i. The other remaining term $(R_{tt}^{i0})^{-1}R_{bb}^{i+1}$ in (4.9) has the following power series expansion:

$$\begin{aligned}
\left[(R_{tt}^{i0})^{-1}R_{bb}^{i+1}\right]_{11} &= \gamma\left[-7 - 3\epsilon + \frac{1}{5}z_{i+1}^2(-7 - 3\epsilon + 5(1+\epsilon)\alpha) - \frac{1}{525}z_{i+1}^4(77 + 13\epsilon) + \cdots\right],\\
\left[(R_{tt}^{i0})^{-1}R_{bb}^{i+1}\right]_{12} &= \gamma\left[2z_{i+1}(2+\epsilon) + \frac{2}{15}z_{i+1}^3(4 + \epsilon - 5\alpha) + \cdots\right],\\
\left[(R_{tt}^{i0})^{-1}R_{bb}^{i+1}\right]_{21} &= \gamma(1+\epsilon)\left[\frac{12+6\epsilon}{z_{i+1}} + \frac{2}{5}z_{i+1}(6 + 3\epsilon - 5(1+\epsilon)\alpha) + \frac{2}{525}z_{i+1}^3(61 + 13\epsilon) + \cdots\right],\\
\left[(R_{tt}^{i0})^{-1}R_{bb}^{i+1}\right]_{22} &= \gamma(1+\epsilon)\left[-7 - 4\epsilon - \frac{1}{15}z_{i+1}^2(13 + 4\epsilon - 15\alpha) - \frac{1}{1575}z_{i+1}^4(43 + 4\epsilon) + \cdots\right],
\end{aligned} \tag{4.10}$$

where

$$\gamma = \frac{l_5^{i+1}(f^i)^2}{l_5^i(f^{i+1})^2}(1+\epsilon)^{-3}, \quad \alpha = \frac{f^{i+1}}{l_5^{i+1}}.$$

The series expansion for $(R_{tt}^{i0})^{-1}R_{tb}^{i}$ is given in (4.5), while the power series expansion for $(R_{tt}^{i0})^{-1}R_{bt}^{i+1}$ is as follows:

$$\begin{aligned}
\left[(R_{tt}^{i0})^{-1}R_{bt}^{i+1}\right]_{11} &= \gamma\left[7 + 3\epsilon + \frac{1}{10}z_{i+1}^2(9 + 3\epsilon) - \frac{1}{4200}z_{i+1}^4(259 + 71\epsilon) + \cdots\right],\\
\left[(R_{tt}^{i0})^{-1}R_{bt}^{i+1}\right]_{12} &= \gamma\left[z_{i+1}(3+\epsilon) + \frac{1}{30}z_{i+1}^3(1-\epsilon) - \cdots\right],\\
\left[(R_{tt}^{i0})^{-1}R_{bt}^{i+1}\right]_{21} &= \gamma(1+\epsilon)\left[\frac{-12-6\epsilon}{z_{i+1}} - \frac{1}{5}z_{i+1}(7+\epsilon) + \frac{1}{2100}z_{i+1}^3(212 + 71\epsilon) + \cdots\right],\\
\left[(R_{tt}^{i0})^{-1}R_{bt}^{i+1}\right]_{22} &= \gamma(1+\epsilon)\left[-5 - 2\epsilon - \frac{1}{30}z_{i+1}^2(1 - 2\epsilon) + \frac{1}{12600}z_{i+1}^4(251 + 38\epsilon) + \cdots\right].
\end{aligned} \tag{4.11}$$

The scaled equations in this case are the same as those given in (4.6), but in this case the terms $(R_{tt}^{i0})^{-1}R_{bb}^{i+1}$ and $(R_{tt}^{i0})^{-1}R_{bt}^{i+1}$ are replaced by their power series expansions given in (4.10) and (4.11), respectively. By contrast, in case AI it was sufficient to evaluate the matrices R_{bb}^{i+1} and R_{bt}^{i+1} using (3.15)–(3.18), which are then multiplied by the low-order inverse $(R_{tt}^{i0})^{-1}$ which is given in (4.3).

For the t-subsystem the flexibility coefficients R^i_{tt} and R^i_{tb} blow up as $O(z_i^{-1})$ while the flexibility coefficients R^{i+1}_{bb} and R^{i+1}_{bt} blow up as $O(z_{i+1}^{-1})$. We perform a similar combined expansion of the flexibility coefficients and extract the leading order term of the expansion for R^i_{tt} to obtain the scaling factor $R^{i0}_{tt} = \frac{2}{f^i} z_i^{-1}$. The expansions for $(R^{i0}_{tt})^{-1} R^i_{tt}$ and $(R^{i0}_{tt})^{-1} R^i_{tb}$ are the same as those given in (4.7) and (4.8). The expansions for the remaining coefficients in this case are:

$$(R^{i0}_{tt})^{-1} R^{i+1}_{bb} = -\frac{f^i}{f^{i+1}(1+\epsilon)}\left\{1 + \frac{1}{3} z^2_{i+1} - \frac{1}{45} z^4_{i+1} + \cdots\right\},$$

$$(R^{i0}_{tt})^{-1} R^{i+1}_{bt} = \frac{f^i}{f^{i+1}(1+\epsilon)}\left\{1 - \frac{1}{6} z^2_{i+1} + \frac{7}{360} z^4_{i+1} - \cdots\right\},$$

where $(R^{i0}_{tt}) = \frac{2}{f^i} z_i^{-1}$.

Class B: *Upper layer is thinner*. For this class we repeat the process described above but with the roles of i and $i+1$ reversed.

5. Numerical results

5.1. Performance of the rescaled flexibility technique

In this subsection we illustrate the performance of the rescaled equation approach for very small kd values. In order to be able to check the solution, we choose an example that has only moderately thick layers (d values) and achieve small kd products by evaluating the spectral coefficients for small wavenumbers k. In this problem, the spectral coefficients calculated using the unscaled equations and the scaled equations can be compared to those that have been calculated using the stiffness matrix method. We consider the four layer 2D problem shown in Fig. 3.

We evaluate the spectral coefficients for a 1 m vertical displacement discontinuity (DD) element (see [4]) using the unscaled flexibility matrix method, the scaled flexibility matrix method, and the stiffness matrix method. In Fig. 4 we plot the maximum errors in the spectral coefficients over all of the layers due to the rescaled and unscaled flexibility equations. The errors are determined by subtracting the spectral coefficients using the unscaled and rescaled flexibility equations from those obtained using the stiffness matrix method, which will be more accurate for this particular class of problem due to the small wavenumbers k . The rescaled spectral coefficients are more accurate for $0 < k < 0.08$ after which the unscaled equations are

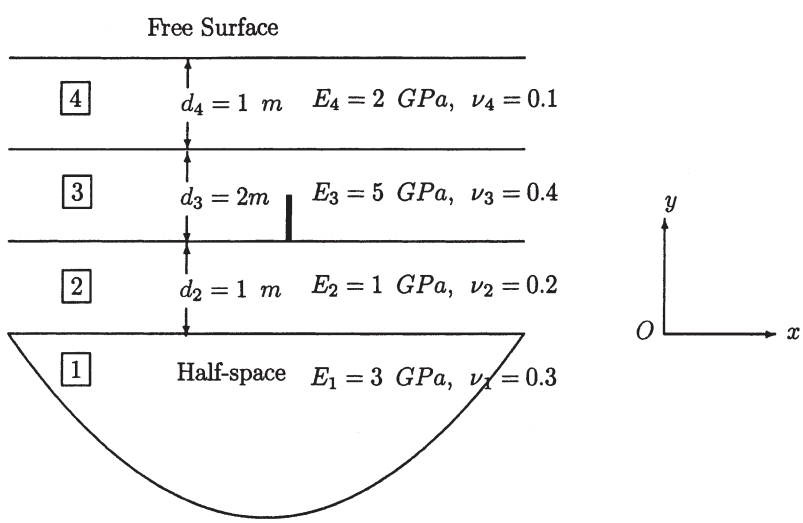

Fig. 3. Geometry for rescaled test problem.

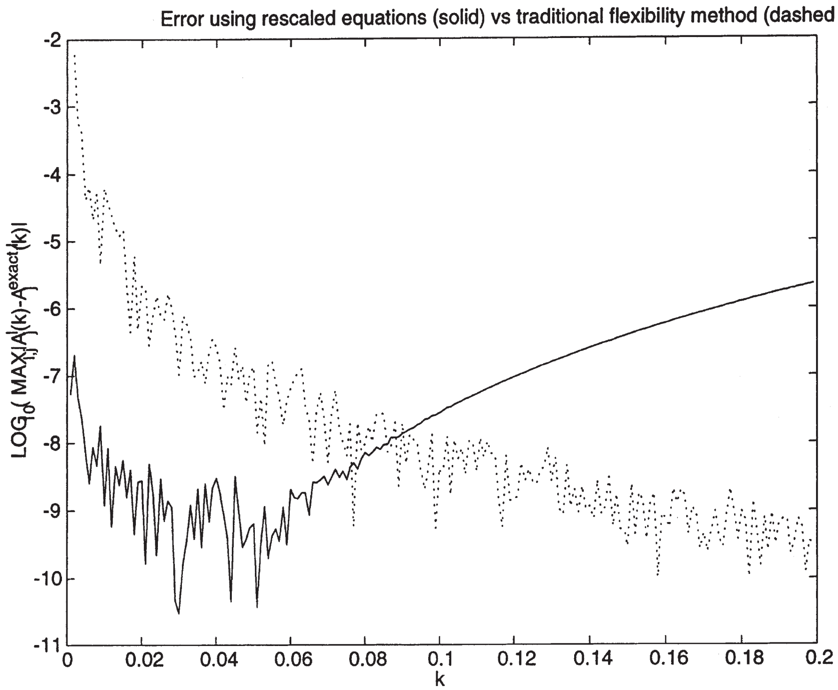

Fig. 4. Error in spectral coefficients for rescaled and unscaled flexibility matrices.

more accurate, i.e., for $0.08 < k$. The spectral coefficients calculated using the rescaled equations are less accurate for wavenumbers in the range $0.08 < k$ because the rescaled equations rely on power series expansions in the parameter kd, which are truncated assuming that kd is small. The truncation error in this case starts to become larger than the round-off error and increases for larger values of kd. We observe also that the error for the spectral coefficient for the rescaled equations in the region $0 < k < 0.08$ oscillates randomly indicating that the error is due to round-off. It is interesting to note that the errors in the spectral coefficients calculated using the unscaled equations show a randomness over the whole range of the k values shown in Fig. 4 because the errors are due to round-off. This is even true for very small wavenumbers in which the errors become unacceptably large due to the round-off errors caused by cancellation when the formulae (3.15)–(3.18) are used.

5.2. A penny-shaped crack intersecting a very thin layer and a thick layer

In this example we consider a penny-shaped crack with a radius of 1 m to be centered at the coordinates $(0, 2.1)$ in the 3D layered material shown in Fig. 5. We note that for clarity of presentation, the layer thicknesses have not been drawn to-scale in the figure but have been indicated in the figure. A to-scale representation of the crack is shown in Fig. 6. A constant pressure of 1 MPa is applied to the crack surfaces. This geometry has been chosen so that the thin 0.2 m layer is sandwiched between a 2 m layer and a much thicker 200 m layer. To resolve this mixture of length scales would require that kd products as small as 0.5×10^{-3} need to be evaluated for the thinnest of the layers, while the kd products for the thicker layers can be relatively large. This precludes the use of the stiffness matrix method and the propagator matrix method. The rescaled flexibility matrix method easily handles this large range of length scales within the same framework.

We solved the crack problem using two different meshes. The coarser mesh uses square piecewise constant DDs of size 0.2 m as shown in Fig. 6. The boundaries of the crack are modeled accurately using a fixed Eulerian mesh and a specialized truncation error correction (see [15]). In the case of the second finer mesh, we subdivide each of the coarser cells into four cells.

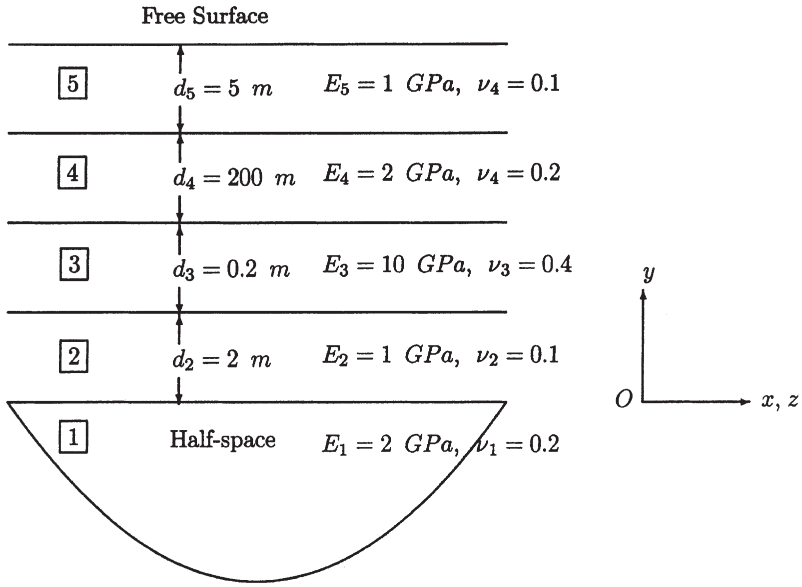

Fig. 5. Geometry for thin layer and thick layer test problem.

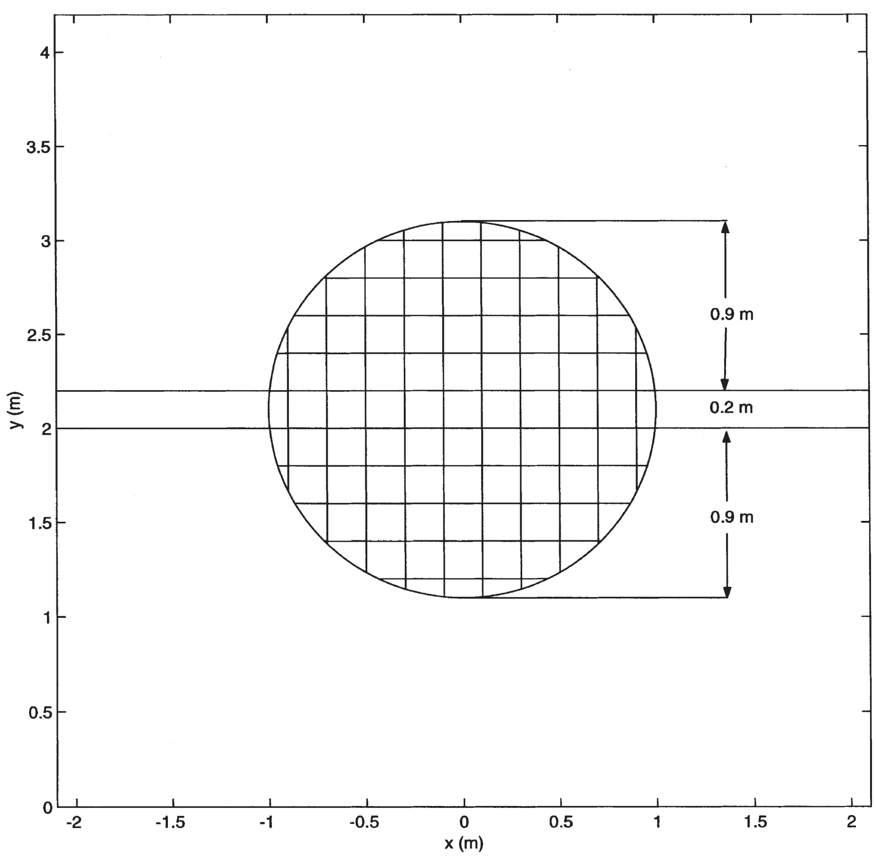

Fig. 6. The coarse discretization used to model the penny-shaped crack straddling a thin layer.

In Fig. 7 we plot the width profiles obtained using the two different discretizations. We observe that there is considerable pinching due to the effect of the very stiff and very thin layer. We also observe that there is close agreement between the two sets of results obtained using the two different discretizations.

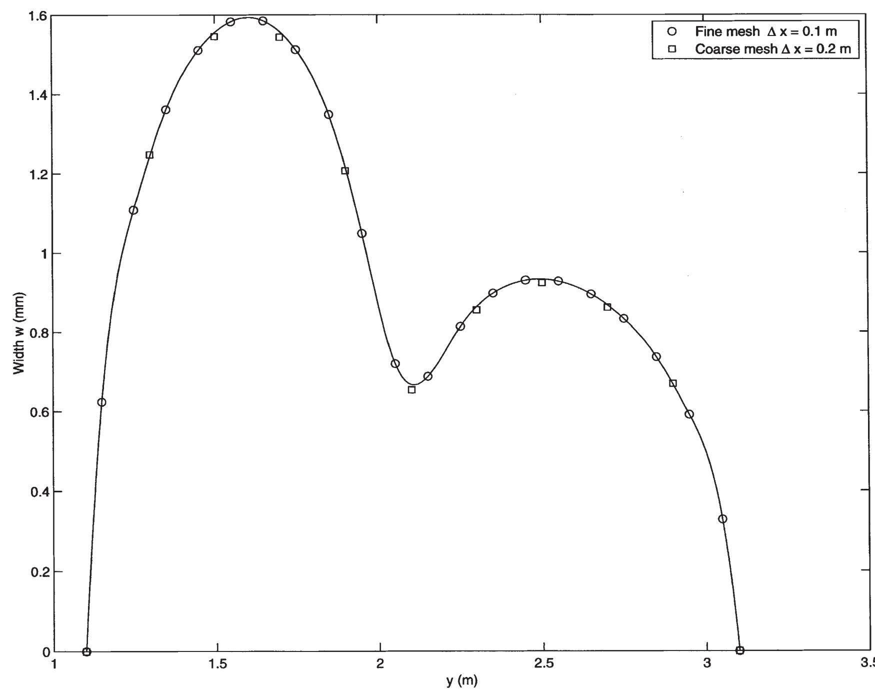

Fig. 7. Width profile along the line $x = 0$ for the coarse and the fine discretizations of the crack.

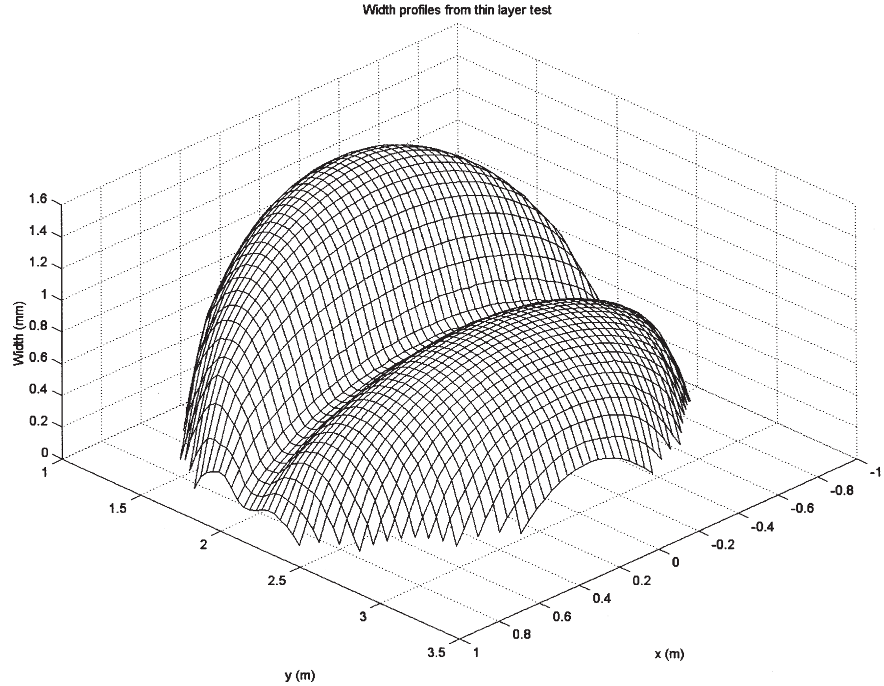

Fig. 8. Width profile for the penny-shaped crack problem obtained using the fine mesh.

In Fig. 8 we provide a 3D surface plot of the width profile that was obtained using the finer mesh. The pinching of the width profile due to the thin stiff layer can be clearly seen. This plot emphasizes the importance of having the ability to model very thin layers particularly if the elastic moduli of these thin layers are much larger than the neighboring layers. For the coarse mesh, the total solution process involving 95 piecewise constant DD elements took 8.8 s on a 200 MHz Pentium processor. For the fine mesh, the total solution process involving 332 piecewise constant DD elements took 27.4 s on a 200 MHz Pentium processor.

6. Conclusions

In this paper we have introduced the rescaled flexibility matrix method that can be used to solve the algebraic equations for a layered medium when there are very different length scales active in the problem. None of the traditional formulations of the systems of algebraic equations for layered elastic media, i.e., the stiffness matrix method, the propagator matrix method, or flexibility matrix method, is suited to solving layer problems in which there are very big differences in the length scales of the problem. The stiffness matrix method and the propagator matrix method are both suitable for solving problems in which there are not too many relatively thin layers so that the total extent of the layered elastic medium is limited. This is due to the large exponential terms of the kd products that need to be evaluated if the vertical dimension of the pack of layers becomes large. In contrast the flexibility matrix method is able to deal with large kd products but the equations become ill-conditioned and susceptible to round-off errors if the problem involves kd products that are very small. This situation can arise when there is a big range of length scales in the problem, such as when a very thin layer is next to a very thick layer.

In the rescaled flexibility matrix method described in this paper, power series expansions are used to identify the most singular terms in the standard flexibility matrices. Using these singular terms all the equations that are susceptible to round-off errors are rescaled by pre-inverting the singular submatrices analytically and using power series expansions to evaluate the regular terms that remain. Only those equations that are susceptible to round-off errors need to be dealt with using the rescaled approach, while the remaining equations, that may be associated with very thick layers, need not be rescaled and can be treated by using the standard formulae for evaluating the flexibility submatrices. Thus by separating the different length scales and constructing the flexibility matrices using expressions appropriate for the length scales involved, we are able to solve boundary value problems with a large range of length scales. The technique we describe can be used to solve problems involving layers that are 10 times thinner than can be treated by means of the standard flexibility matrix method.

We have provided two numerical examples demonstrating the performance of the rescaled flexibility method. The first example demonstrates the accuracy of the rescaled flexibility matrix approach when applied to a problem in which the layers have thicknesses $d = O(1)$ but for which a small range of wavenumbers $0 < k < 0.2$ close to 0 are considered in order to achieve small kd products. Because only a small number of moderately thick layers are used in this test problem and only small wavenumbers are considered, it is possible to obtain an accurate reference solution using the stiffness matrix method, which is expected to give an accurate result in this regime. This example makes it possible to compare the spectral coefficients calculated using the traditional flexibility matrix method with those calculated using the rescaled flexibility matrix method. The rescaled flexibility matrix method was shown to produce results that were typically 3 and up to 5 orders of magnitude more accurate than the standard flexibility matrix method for kd products that were less than 0.08. The level of the pointwise error was typically 10^{-8} and never larger than 10^{-7} for this range of kd values. For kd products that are larger than 0.08, the rescaled flexibility method gave poor results because the truncation error in the power series expansions for the rescaled flexibility matrix method became too large. However, for $kd > 0.08$ the standard flexibility matrix method is able to provide a solution with an error that was smaller than 10^{-8}. Thus, making use of rescaled flexibility matrices in conjunction with the standard flexibility matrix method, it is possible to obtain a unified algorithm that can solve boundary value problems with many different length scales.

In the second example the rescaled flexibility matrix method is used to solve a problem in which there is a very thin layer 0.2 m thick, sandwiched between a moderate layer 2 m thick and a very thick 200 m layer. A penny-shaped crack of 1 m radius was centered on the very thin layer in such a way that it intersected the two neighboring layers. To capture the far-field behavior of the problem, the rescaled flexibility matrix method was used to determine the spectral coefficients for the problem. Two distinct discretizations for this problem were used and both solutions showed good agreement. This example also clearly demonstrates the dramatic effect the inclusion of very thin layers can have.

We have thus provided a detailed description of the rescaled flexibility matrix method and have demonstrated that it provides a very useful procedure to solve boundary value problems for layered elastic materials in which there are many different length scales active in the problem.

Acknowledgements

The authors wish to thank Schlumberger for permission to publish this work.

Appendix A. Definition of the Fourier transform

Let $g(x,z) \in L^1(R^2)$ then we define the double Fourier transform (FT) of $g(x,z)$ to be

$$\widehat{g}(m,n) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(mx+nz)} g(x,z)\, \mathrm{d}x\, \mathrm{d}z \tag{A.1}$$

and the corresponding inversion formula is

$$g(x,z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(mx+nz)} \widehat{g}(m,n)\, \mathrm{d}m\, \mathrm{d}n. \tag{A.2}$$

The FT of the derivative of a function may be readily established by using the definition and integrating by parts to obtain:

$$\partial_x \widehat{g(x,z)} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(mx+nz)} \partial_x g(x,z)\, \mathrm{d}x\, \mathrm{d}z = -im\widehat{g}(m,n). \tag{A.3}$$

Thus taking a derivative with respect to x (respectively, z) in the spatial domain is reduced in the wavenumber domain to multiplication by the wavenumber m (respectively, n). This is used to reduce the system of PDEs governing the stress and displacement fields to a system of ODEs that are parameterized by the wavenumbers m and n.

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