Numerics for PDE using EXCEL

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First derivative Approximations

$$f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2!} f''(x) + \ldots$$

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x) + \frac{\Delta x}{2} f''(x) + \ldots .$$

$$f(x-\Delta x) = f(x) - \Delta x f'(x) + \frac{\Delta x^2}{2!} f''(x) - \ldots$$

$\ominus$

$$\frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} = f'(x) + \frac{\Delta x^2}{3!} f^{(3)}(\xi)$$

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x) + \frac{\Delta x}{2} f''(x) + \ldots .$$

Second derivative Approximation

$$f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2!} f''(x) + \ldots$$

$$f(x-\Delta x) = f(x) - \Delta x f'(x) + \frac{\Delta x^2}{2!} f''(x) - \ldots$$

$\oplus$

$$f(x+\Delta x) + f(x-\Delta x) = 2f(x) + \Delta x^2 f''(x) +$$

$$\frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2} = f''(x) + \frac{1}{12}\Delta x^2 f^{(4)}(\xi)$$

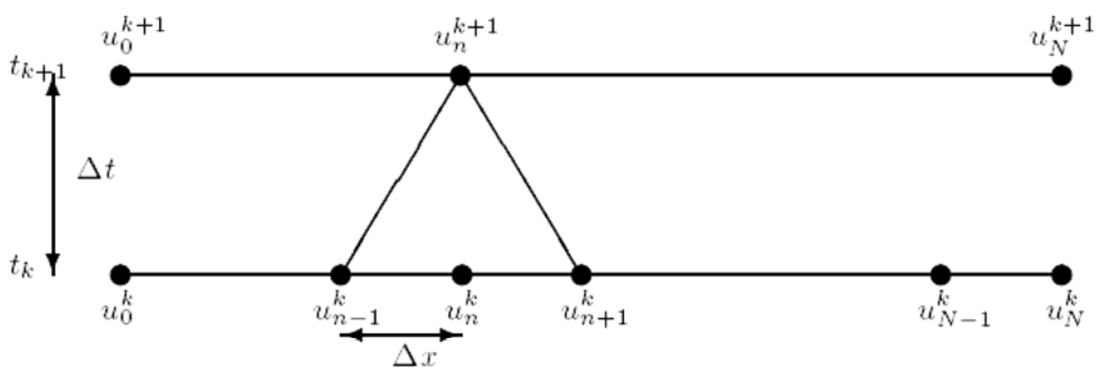
1D heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < 1, t > 0$$

BC: $\quad u(0,t) = 0 \quad u(1,t) = 0$

IC: $\quad u(x,0) = f(x)$

$$u(x_n, t_k) \simeq u_n^k$$



Discrete from of 1D heat equation

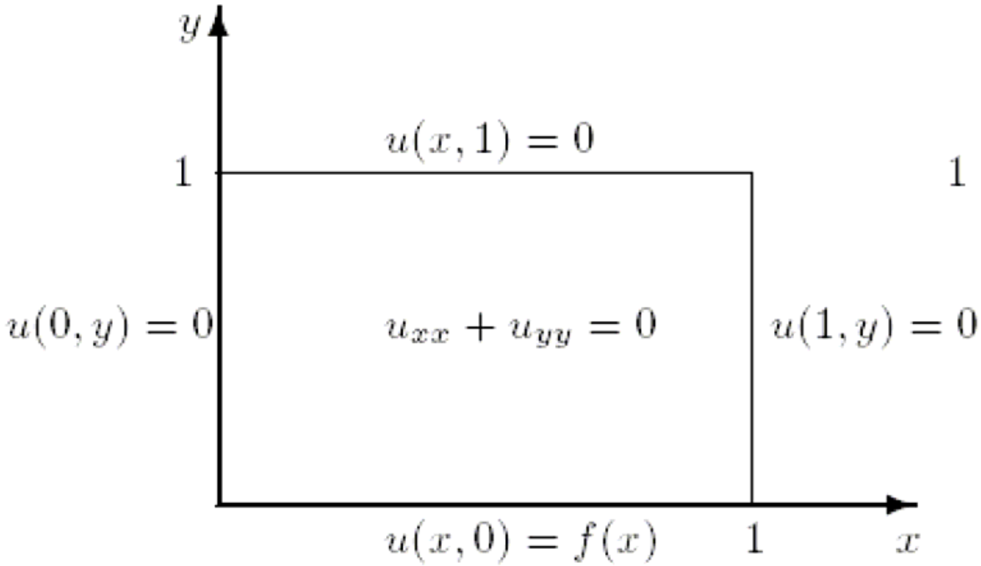
$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

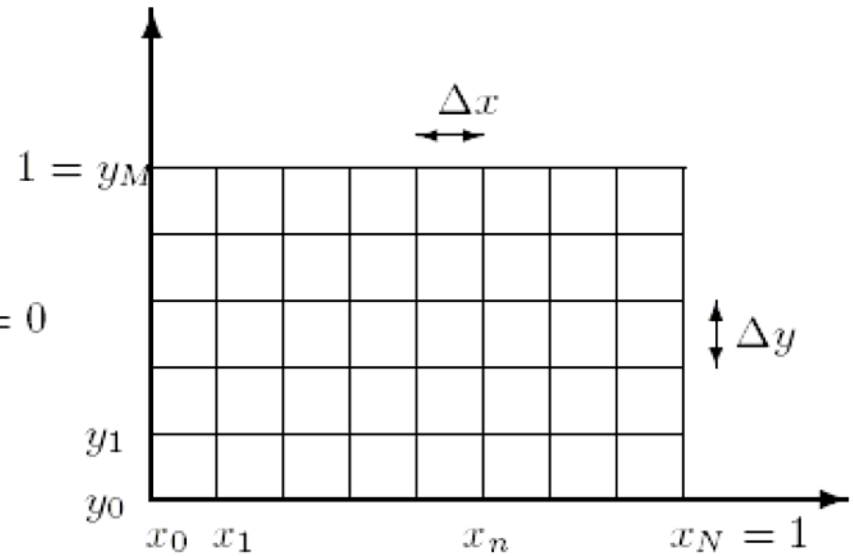
$$\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} \simeq \alpha^2 \left(\frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2} \right)$$

$$u_n^{k+1} = u_n^k + \alpha^2 \left(\frac{\Delta t}{\Delta x^2} \right) \left(u_{n+1}^k - 2u_n^k + u_{n-1}^k \right)$$

Laplace's Equation

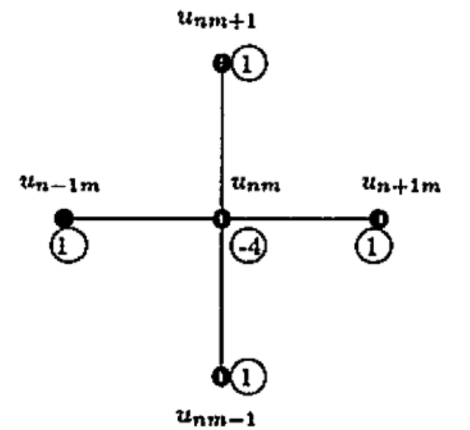
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$





Discretization of Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



$$u(x_n, y_m) \simeq u_{nm}$$

$$\frac{u_{n+1m} - 2u_{nm} + u_{n-1m}}{\Delta x^2} + \frac{u_{nm+1} - 2u_{nm} + u_{nm-1}}{\Delta y^2} = 0$$

$$\Delta x = \Delta y$$

$$uu_{nm} = \frac{u^k_{n+1m} + u^k_{n-1m} + u^k_{nm+1} + u^k_{nm-1}}{4}$$

Features of point iterative methods

- Consider solving: $w'' = 0,\ w(0) = 1,\ w(1) = 0$
- By iterating the difference equations:

$$\frac{w_{n+1} - 2w_n + w_{n-1}}{\Delta x^2} = 0,\ w_0 = 1,\ w_N = 0$$

- Using Jacobi iteration:

$$\frac{w_n^{k+1} - w_n^k}{\Delta t} = \frac{\Delta x^2}{2\Delta t}\,\frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{\Delta x^2}$$

- Iterate to equilibrium using diffusion equation:

$$\frac{\partial w}{\partial t} = D\frac{\partial^2 w}{\partial x^2}$$

The wave equation

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < L$$

BC: $\quad u(0,t) = 0, \qquad u(L,t) = 0$

IC: $\quad u(x,0) = f(x), \qquad u_t(x,0) = g(x)$

$$\frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\Delta t^2} = c^2 \left(\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{\Delta x^2} \right)$$