One dimensional models of hydraulic fracture

Anthony Peirce (UBC)

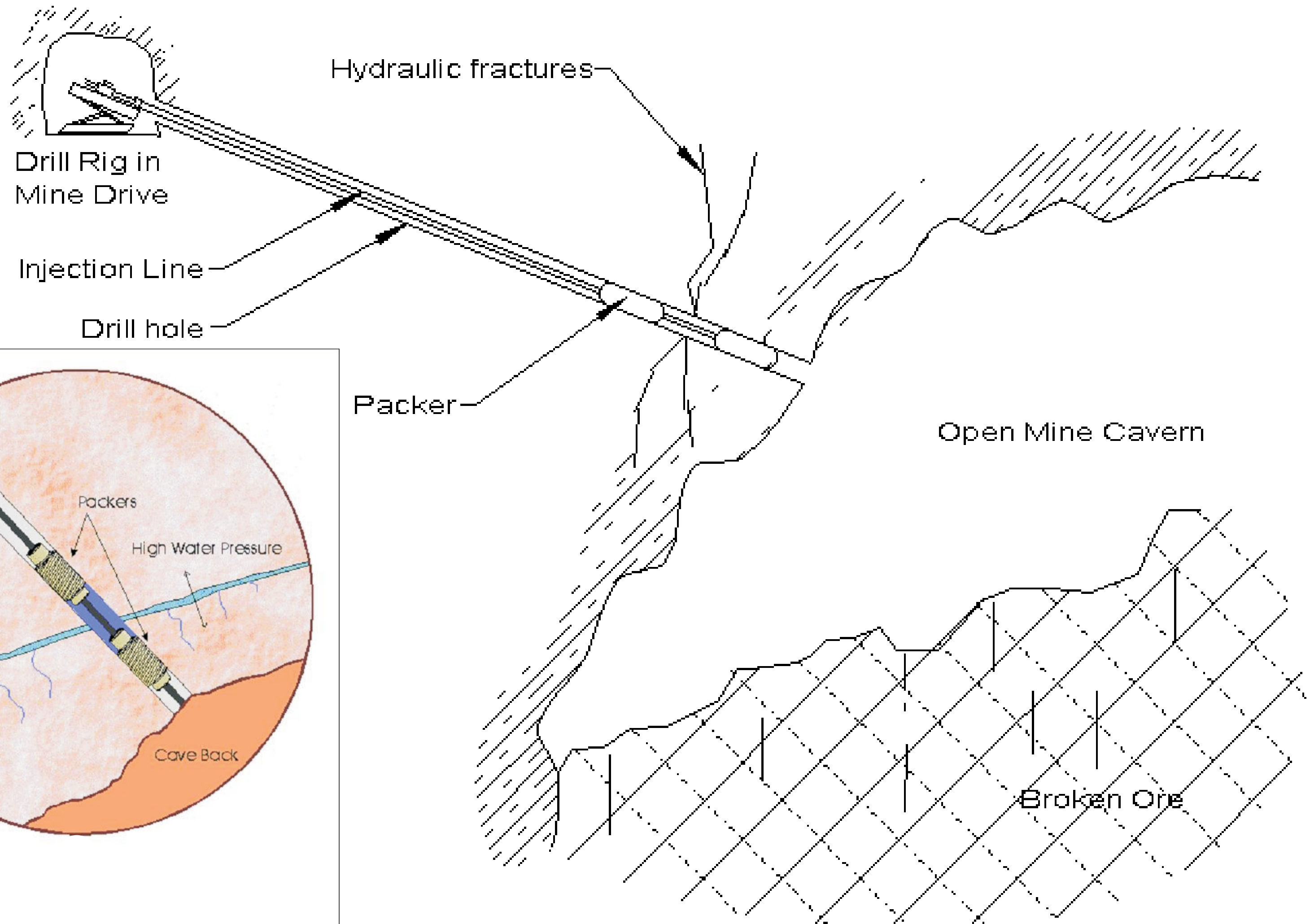
Collaborators:
Jose` Adachi (SLB)
Shira Daltrop (UBC)
Emmanuel Detournay (UMN)

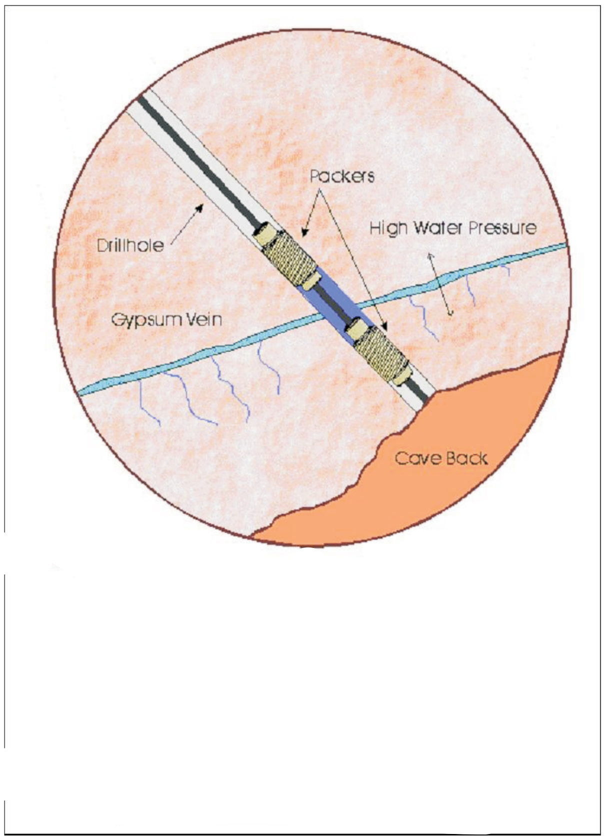
WITS University
1 April 2009

Outline

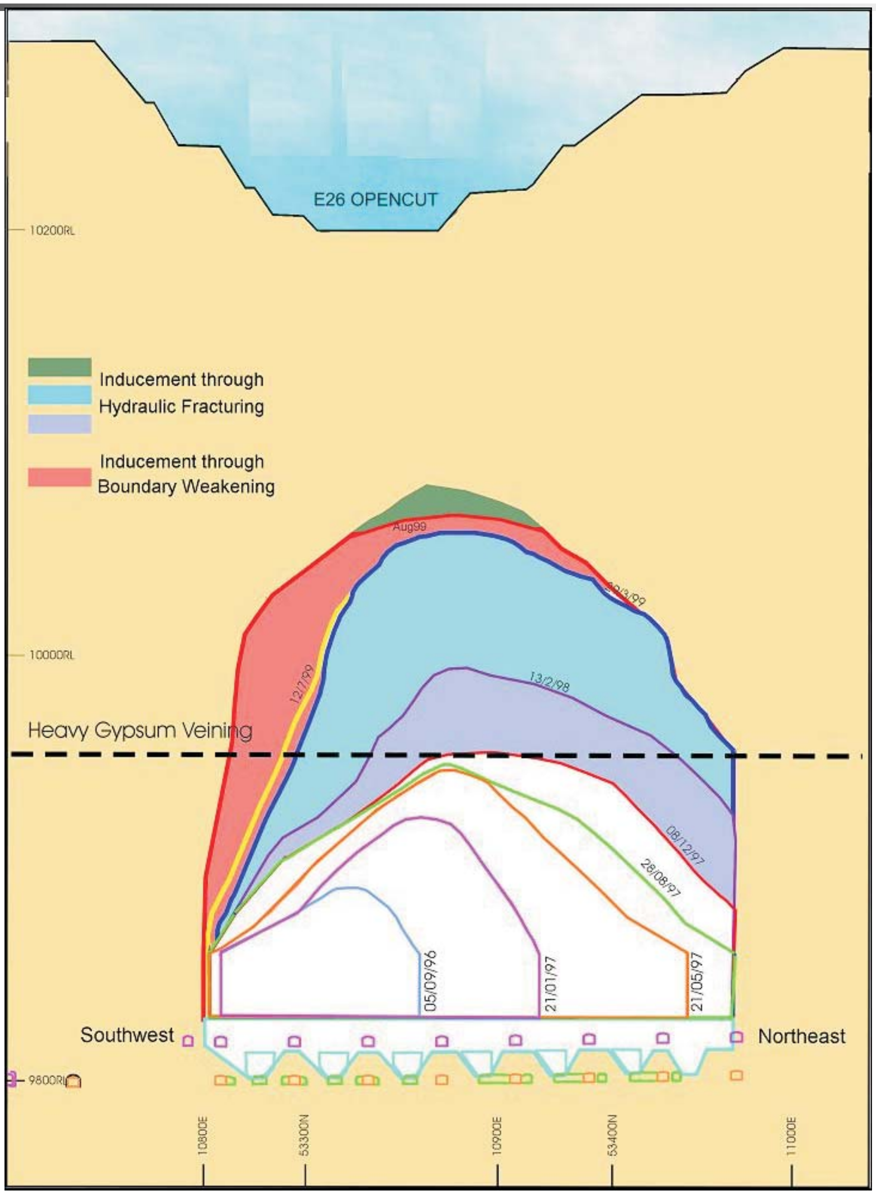
- The HF problem and 2D models
- Slender geometries -> the possibilities for 1D models
- The classic PKN model – porous medium eq – limitations
- Extension of PKN to include toughness - 1D integro-PDE
- P3D model – PKN methodology ->pseudo 3D
- Conclusions

HF Examples - block caving

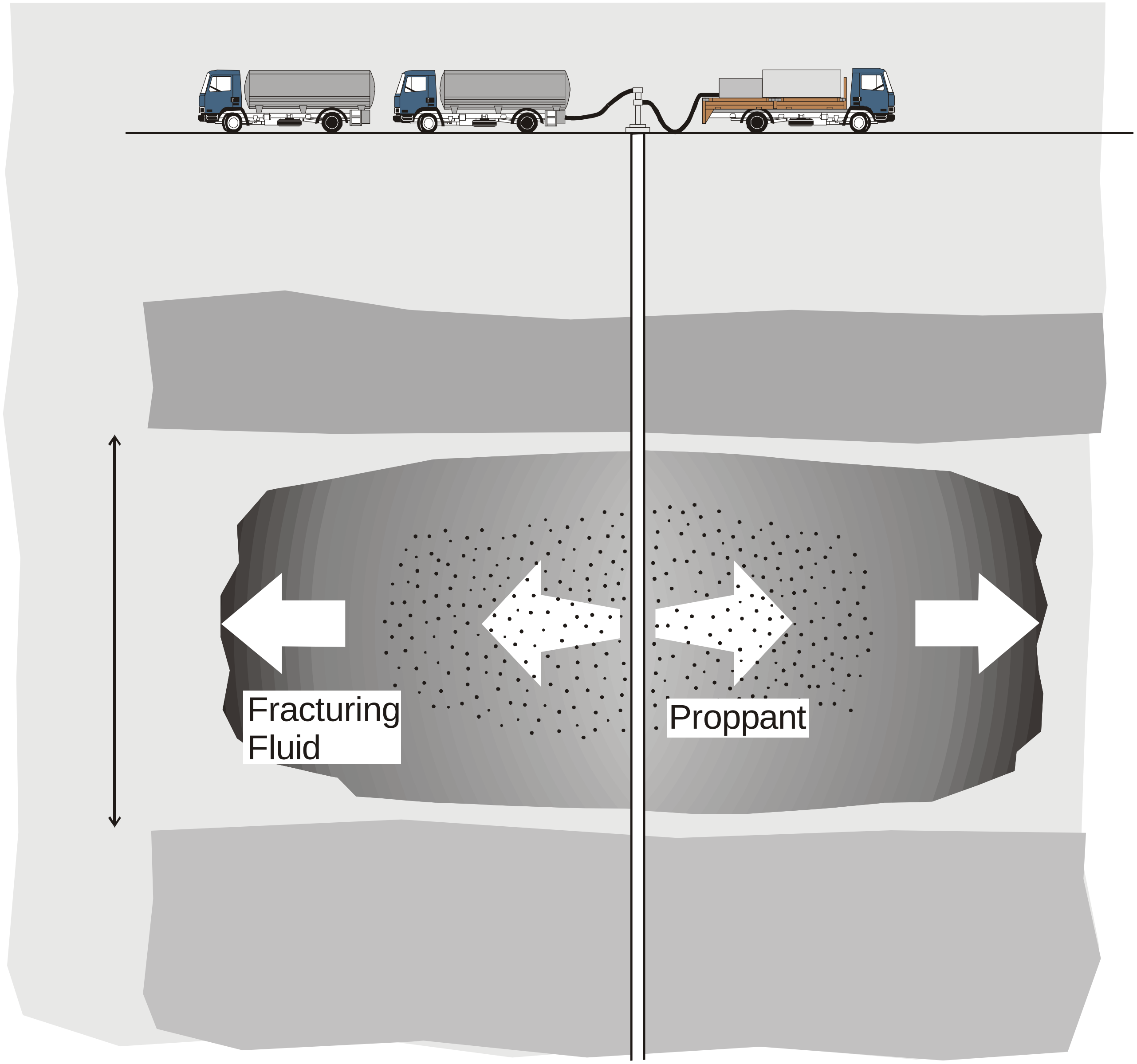




HF Example – caving (Jeffrey, CSIRO)

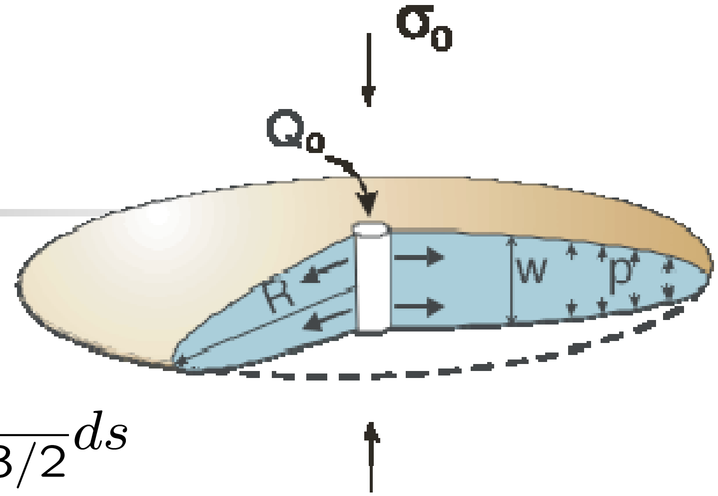


Oil well stimulation



Lab test with stress contrast (Bunger)

2-3D HF Equations



- Elasticity (non-locality)

$$p_f - \sigma_o = -\frac{E'}{8\pi}\int_{S(t)} \frac{w(x',y',t)}{\left[(x'-x)^2+(y'-y)^2\right]^{3/2}}ds$$

$$\Rightarrow \quad p_f - \sigma_o = Cw$$

- Lubrication (non-linearity)

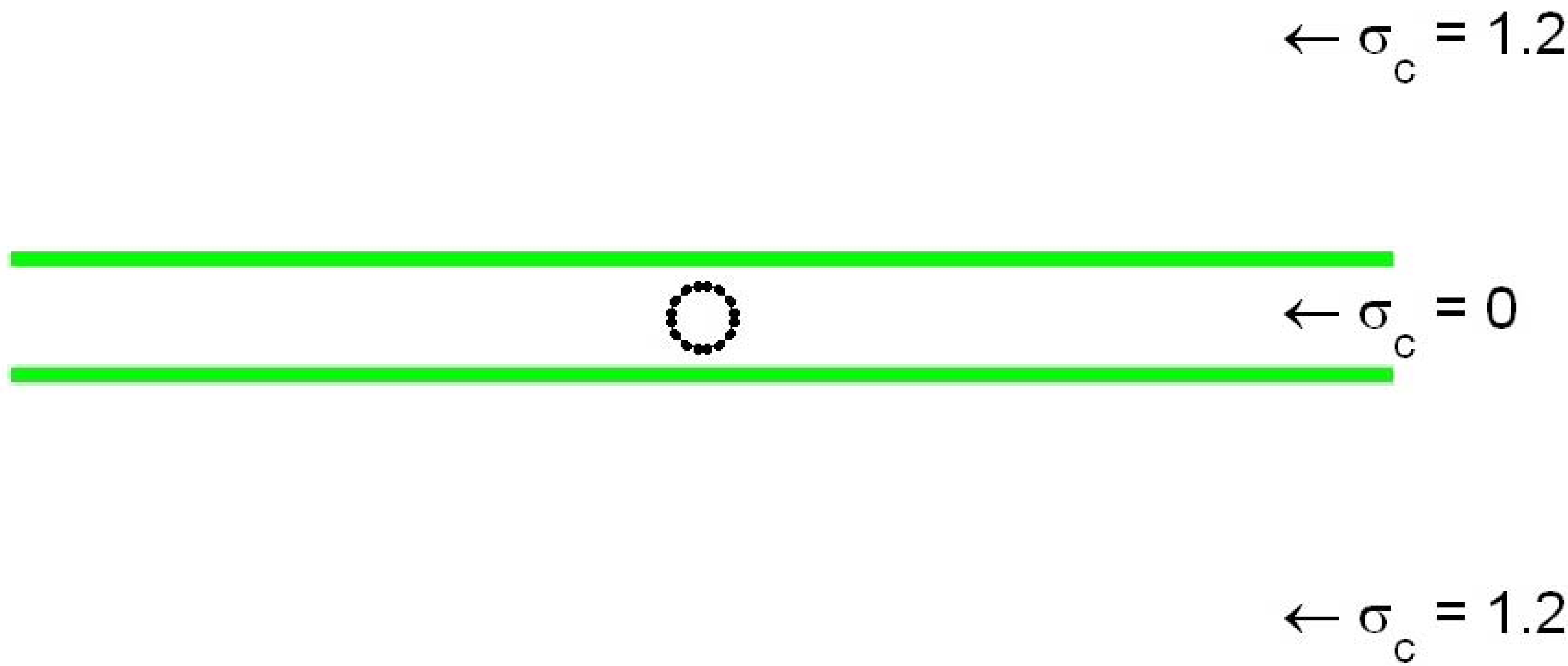
$$\frac{\partial w}{\partial t} = \nabla\cdot\left(\frac{w^3}{\mu'}\nabla p_f\right) - \frac{C'}{\sqrt{t-t_0(x,y)}} + Q(t)\delta(x,y)$$

$$\frac{\partial w}{\partial t} = A(w)Cw + S$$

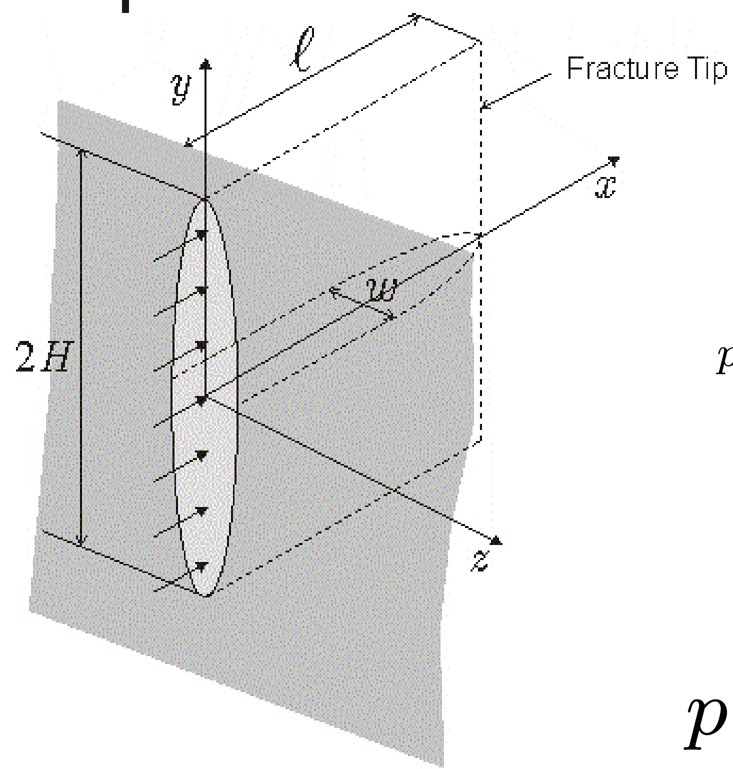
$$\Rightarrow \quad \frac{\Delta w}{\Delta t} = A(w)p_f + S$$

- Boundary conditions at moving front (free boundary)

$$\lim_{s\to 0}\frac{w}{s^{1/2}} = \frac{K'}{E'}, \qquad \lim_{s\to 0} w^3\frac{\partial p_f}{\partial s} = 0 \qquad v = \lim_{s\to 0} -\frac{1}{\mu'}w^2\nabla p$$


← σ_c = 1.2
← σ_c = 0
← σ_c = 1.2

The PKN Model



$$p(x,y) = -\frac{E'}{8\pi}\int_{-H}^{H}\int_{-\infty}^{\infty}\frac{w(x',y')}{\left[(x'-x)^2+(y'-y)^2\right]^{3/2}}dx'dy'$$

$$w(x',y') = w(y')$$

$$p(y) = -\frac{E'}{8\pi}\int_{-H}^{H} w(y')\int_{-\infty}^{\infty}\frac{dx'}{\left[(x'-x)^2+(y'-y)^2\right]^{3/2}}dy'$$

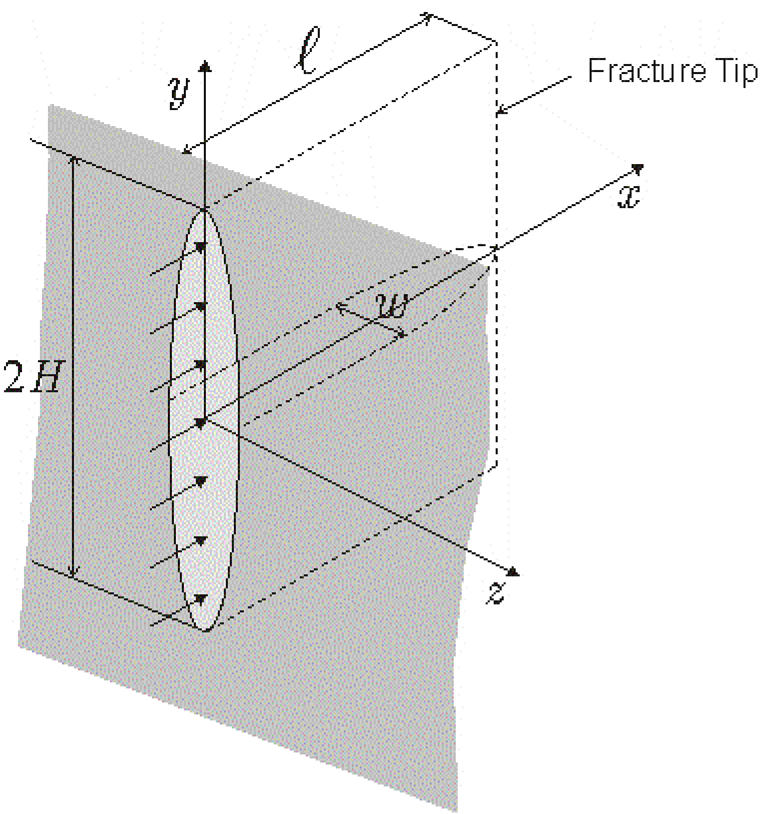
$$p(y) = -\frac{E'}{4\pi}\int_{-H}^{H}\frac{w(y')}{(y'-y)^2}dy'$$

$$p(y) = p_0$$

$$w(y) = \frac{4Hp_0}{E'}\sqrt{1-\frac{y^2}{H^2}} = w_0\sqrt{1-\frac{y^2}{H^2}}$$

$$\bar{w}(x) \approx \frac{4Hp(x)}{E'}\ \frac{1}{2H}\int_{-H}^{H} dy \cdot \sqrt{1-\frac{y^2}{H^2}} = w_0(x)\left(\frac{\pi}{4}\right)$$

Assumptions behind the PKN Model



- Pressure independent of y :
$$p(x,y) = p(x)$$
- Vertical sections approximately in a state of plane strain:
$$w(x,y) \approx \frac{4Hp(x)}{E'}\sqrt{1-\frac{y^2}{H^2}}$$
$$= w_0(x)\sqrt{1-\frac{y^2}{H^2}}$$
- PKN model:
$$p(x) = \frac{E'}{4H}w_0(x) = \frac{E'}{\pi H}\bar{w}(x)$$
  - Local elasticity equation
  - $w(\pm l,t) = 0 \rightarrow p(\pm l,t) = 0$

Can't model pressure singularities for example when $K_{Ic} \neq 0$

PKN – Averaging the Lubrication Eq

$$\frac{\partial w}{\partial t} = \nabla \cdot \left(\frac{w^3}{\mu'} \nabla p \right) + Q(t)\delta(x)\delta(y)$$

$\frac{1}{2H} \int_{-H}^{H} dy \cdot$

$$\frac{\partial}{\partial t} \left(\frac{1}{2H} \int_{-H}^{H} w dy \right) = \frac{1}{\mu'} \frac{\partial}{\partial x} \left(\frac{1}{2H} \int_{-H}^{H} w^3 dy \frac{\partial p}{\partial x} \right) + \frac{Q(t)\delta(x)}{2H}$$

$$w(x, y) = w_0(x) \sqrt{1 - \frac{y^2}{H^2}}$$

$$\frac{\partial}{\partial t} \left(w_0(x) \frac{\pi}{4} \right) = \frac{1}{\mu'} \frac{\partial}{\partial x} \left(\frac{3\pi}{16} w_0^3(x) \frac{\partial p}{\partial x} \right) + \frac{Q(t)\delta(x)}{2H}$$

$$\frac{\partial w_0}{\partial t} = \frac{1}{16\mu} \frac{\partial}{\partial x} \left(w_0^3 \frac{\partial p}{\partial x} \right) + \frac{2Q(t)\delta(x)}{\pi H}$$

$$\bar{w}(x) = w_0(x) \left(\frac{\pi}{4} \right)$$

$$\frac{\partial \bar{w}}{\partial t} = \frac{1}{\pi^2 \mu} \frac{\partial}{\partial x} \left(\bar{w}^3 \frac{\partial p}{\partial x} \right) + \frac{Q(t)\delta(x)}{2H}$$

Scaled lubrication & similarity solution

$$\tau = \frac{t}{t_*} \quad \xi = \frac{x}{\ell_* \gamma(\tau)} \quad \Omega = \frac{w_0}{w_*} \quad \Pi = \frac{p}{p*} = \frac{4H}{E' w_*} p$$

$$\frac{\partial w_0}{\partial t} = \frac{1}{16\mu} \frac{\partial}{\partial x}\left(w_0^3 \frac{\partial p}{\partial x} \right) + \frac{2Q_0}{\pi H} \delta(x) \qquad \mathcal{G}_m = \frac{64\mu H \ell_*^2}{t_* E' w_*^3}, \mathcal{G}_v = \frac{t_* Q_0}{\pi H w_* \ell_*}$$

$$\frac{\partial \Omega}{\partial \tau} - \xi \frac{\dot{\gamma}}{\gamma} \frac{\partial \Omega}{\partial \xi} = \frac{1}{4\gamma^2 \mathcal{G}_m} \frac{\partial^2}{\partial \xi^2} \Omega^4 + \mathcal{G}_v \frac{\delta(\xi)}{\gamma}$$

$$\Omega^4 = T(\tau)\phi(\xi)$$

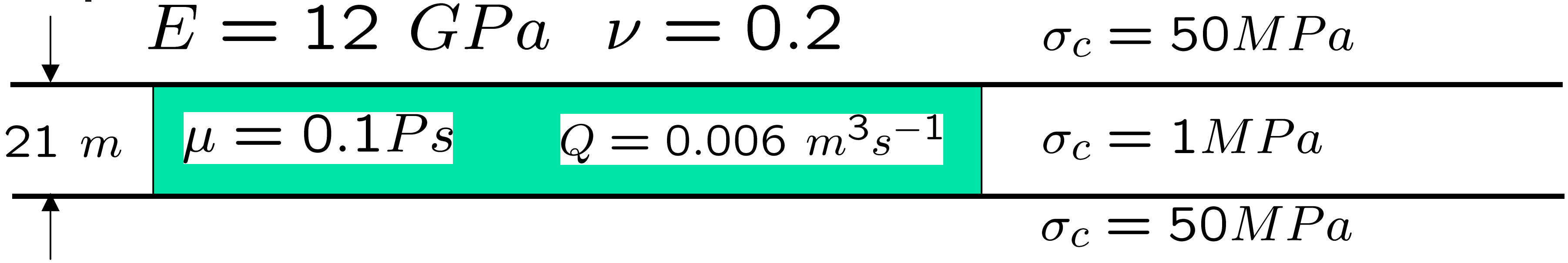
$$T(\tau)\phi'(0) = -4\mathcal{G}_m \mathcal{G}_v \gamma(\tau) \qquad \phi(1) = 0 = \phi'(1)$$

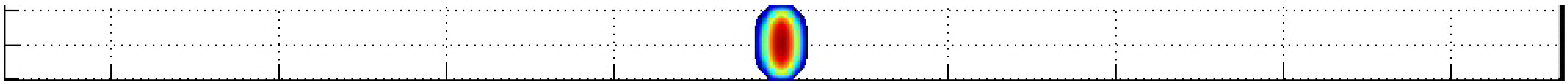
$$\gamma^{1/4} \dot{\gamma} \left(\phi - \xi \phi' \right) = \frac{1}{\mathcal{G}_m} \phi'' \phi^{3/4}$$

$$\phi = B(1-\xi)^\alpha \Rightarrow \alpha - 1 = \alpha - 2 + \frac{3}{4}\alpha \Rightarrow \alpha = \frac{4}{3}$$

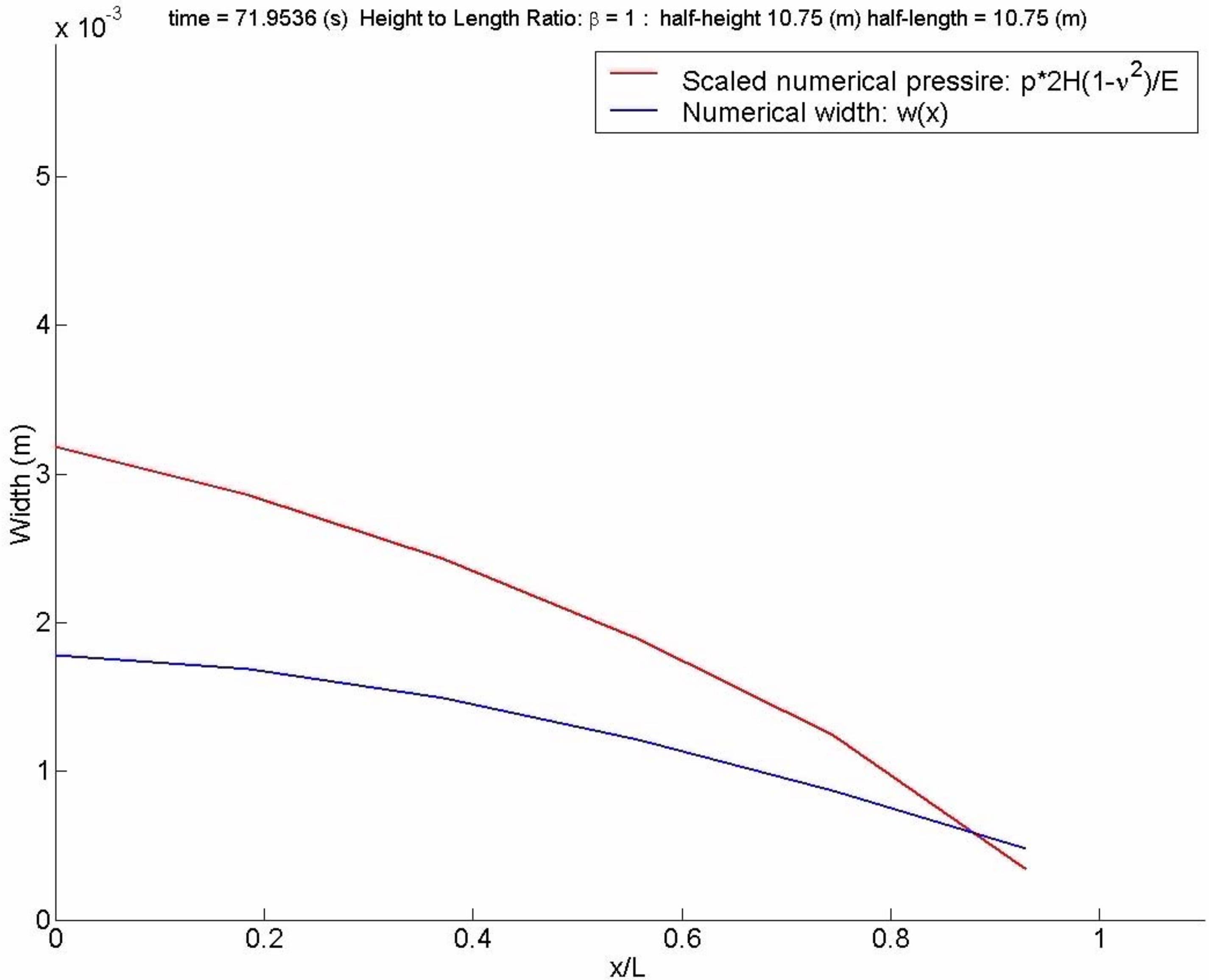
$$\Omega = \left(3\lambda \mathcal{G}_m \mathcal{G}_v^2 \tau \right)^{1/5} (1-\xi)^{1/3}$$

Numerical soln of a finger-like frac

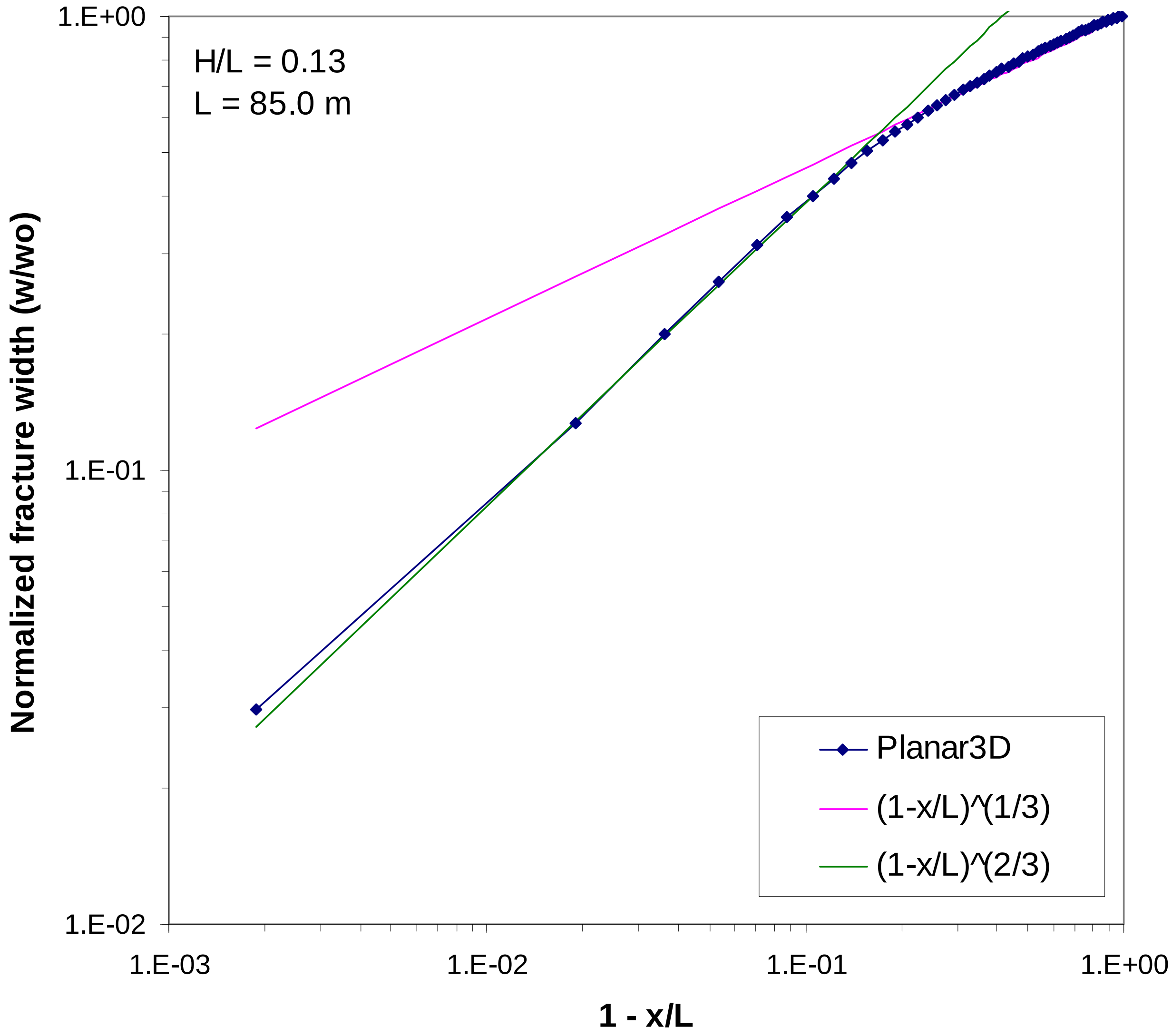




Width and scaled pressures



Asymptotics of numerical soln



Extended PKN: 2D integral eq $\longrightarrow$ 1D integral eq

- Integral eq for a pressurized rectangular crack

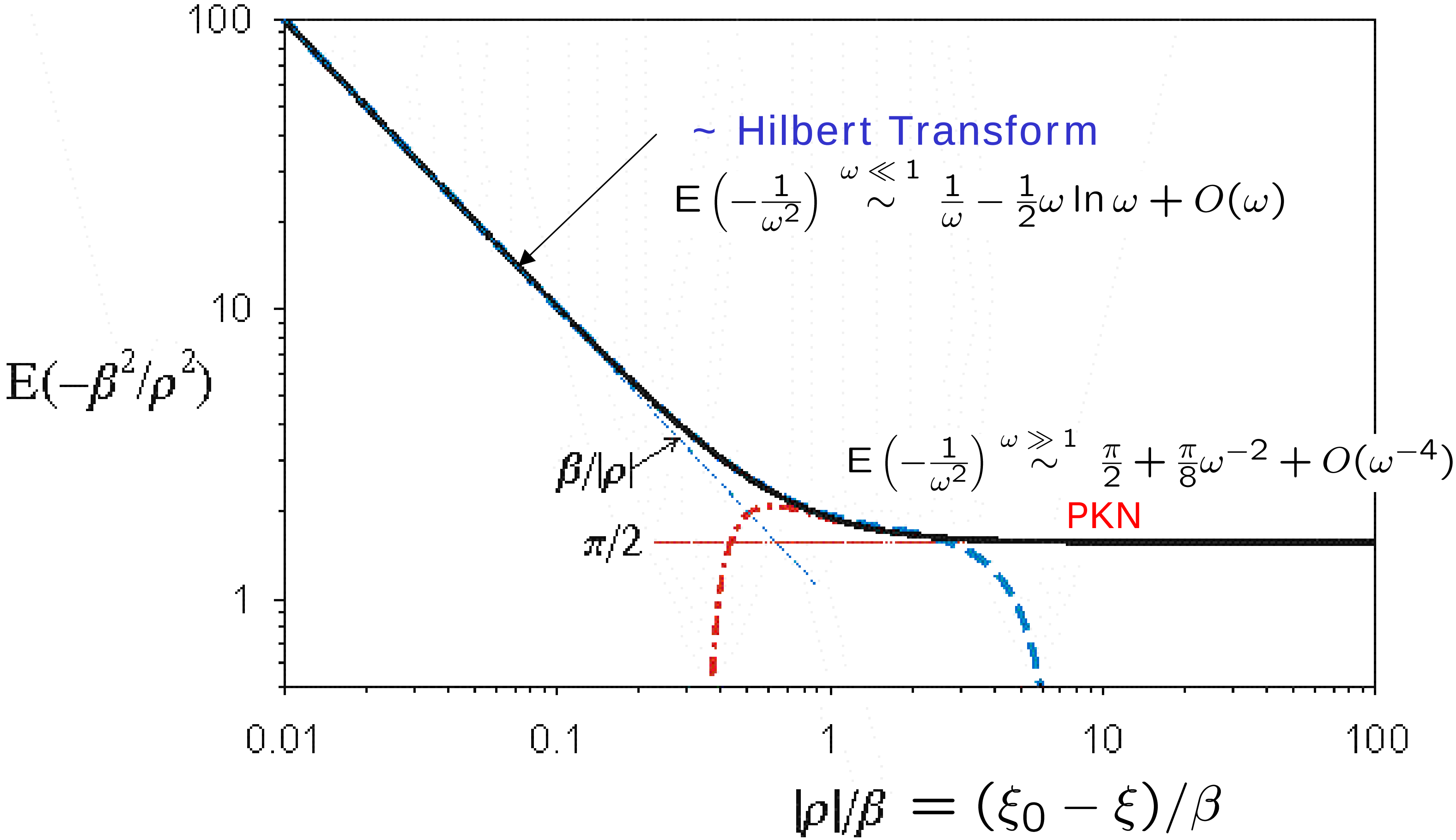
$$p(x,y) = -\frac{E'}{8\pi}\int_{-l}^{l}\int_{-H}^{H}\frac{w_0(\bar{x})\sqrt{1-\frac{\bar{y}^2}{H^2}}}{[(x-\bar{x})^2+(y-\bar{y})^2]^{\frac{3}{2}}}d\bar{y}d\bar{x}$$

- Re-scale variables: $\beta = \frac{H}{l},\ \xi = \frac{x}{l},\ \eta = \frac{y}{l},\ \Omega = \frac{w_0}{w_*},\ \Pi = p\frac{H}{E' w_*}$

$$\Pi(\xi) = -\frac{\beta}{8\pi}\int_{-1}^{1}\Omega(\xi)\int_{-\beta}^{\beta}\frac{\sqrt{1-\frac{\eta^2}{H^2}}}{[(\xi-\xi_0)^2+\eta^2]^{\frac{3}{2}}}d\eta d\xi_0$$

$$= -\frac{1}{\pi}\int_{-1}^{1}\Omega'(\xi_0)sgn(\xi-\xi_0)E\left(-\frac{\beta^2}{(\xi-\xi_0)^2}\right)d\xi_0$$

Asymptotic behaviour of the kernel



Assumed behaviour of $\Omega(\xi)$

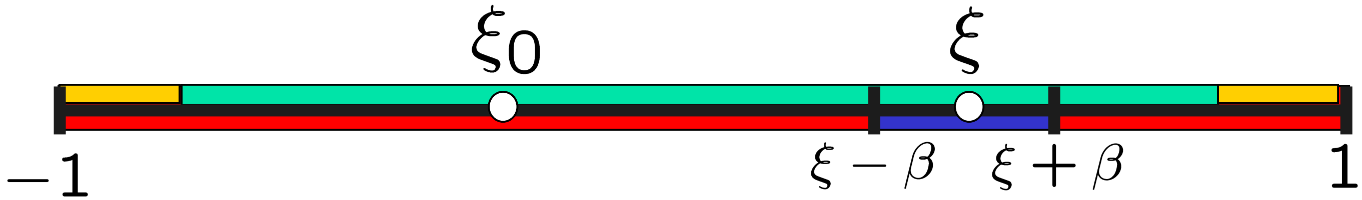
- Power law in the tip regions:

$$\Omega'(\xi_o) = A\alpha(1 \pm \xi)^{\alpha-1}$$

- Analytic away from the tips:

$$\Omega'(\xi_o) = \Omega'(\xi) + (\xi_o - \xi)\Omega''(\xi) + \ldots$$

Outer expansion $1 - \xi > \beta$



Hilbert Transform Region $\frac{\xi_o - \xi}{\beta} \ll 1$

$$I_1(\xi;\beta) = -\frac{1}{\pi}\int_{\xi-\beta}^{\xi+\beta}\left[\frac{\beta}{\xi_o - \xi} - \frac{1}{2\beta}(\xi_o - \xi)\ln\frac{|\xi_o - \xi|}{\beta} + \ldots\right]\Omega'(\xi_o)d\xi_0$$

PKN Region $\frac{\xi_o - \xi}{\beta} \gg 1$

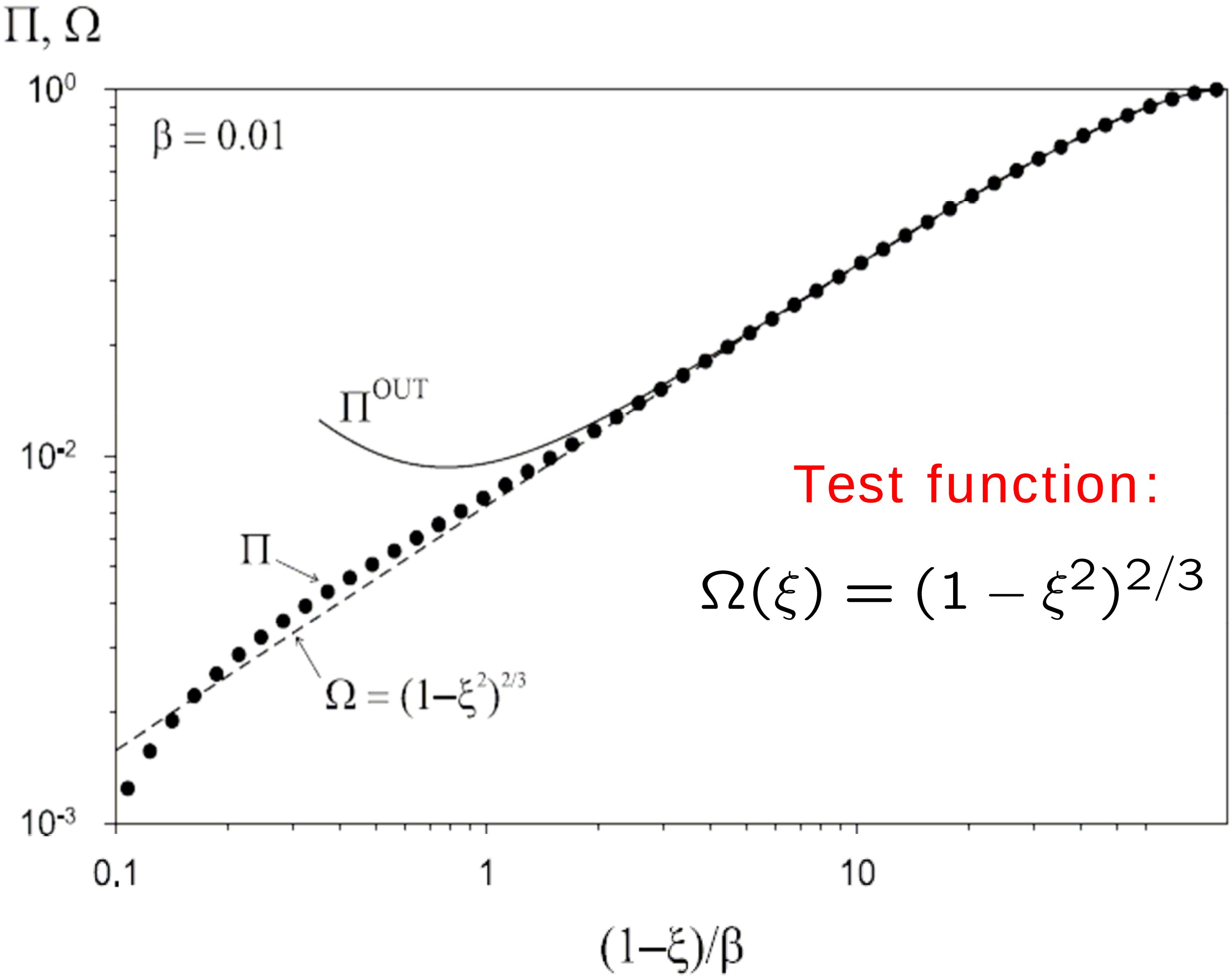
$$I_2(\xi;\beta) = \frac{1}{2}\left\{\int_{-1}^{\xi-\beta} - \int_{\xi+\beta}^{1}\left[1 + \frac{\beta^2}{4(\xi_o - \xi)^2} + \ldots\right]\Omega'(\xi_o)d\xi_0\right\}$$

$$\Pi(\xi;\beta) = I_1(\xi;\beta) + I_2(\xi;\beta)$$

$$= \Omega(\xi) + \frac{1}{4}\beta^2 \ln\beta\, \Omega''(\xi) + C_1\beta^2\Omega''(\xi)$$

PKN

Outer expansion

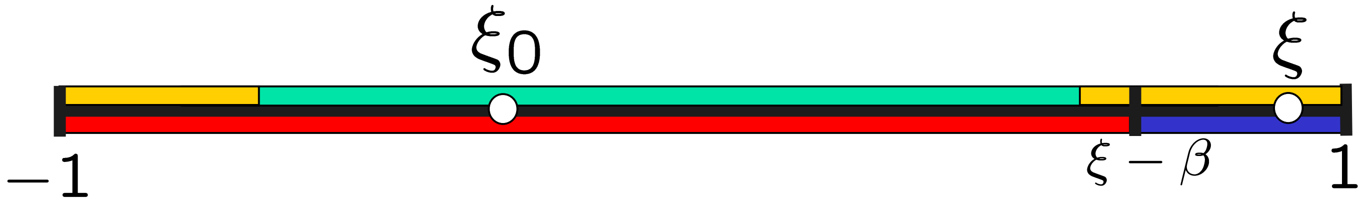


Test function:

$$\Omega(\xi) = (1 - \xi^2)^{2/3}$$

0

Inner expansion: $1-\xi<\beta$



Hilbert Transform Region $\frac{\xi_o-\xi}{\beta}\ll 1$

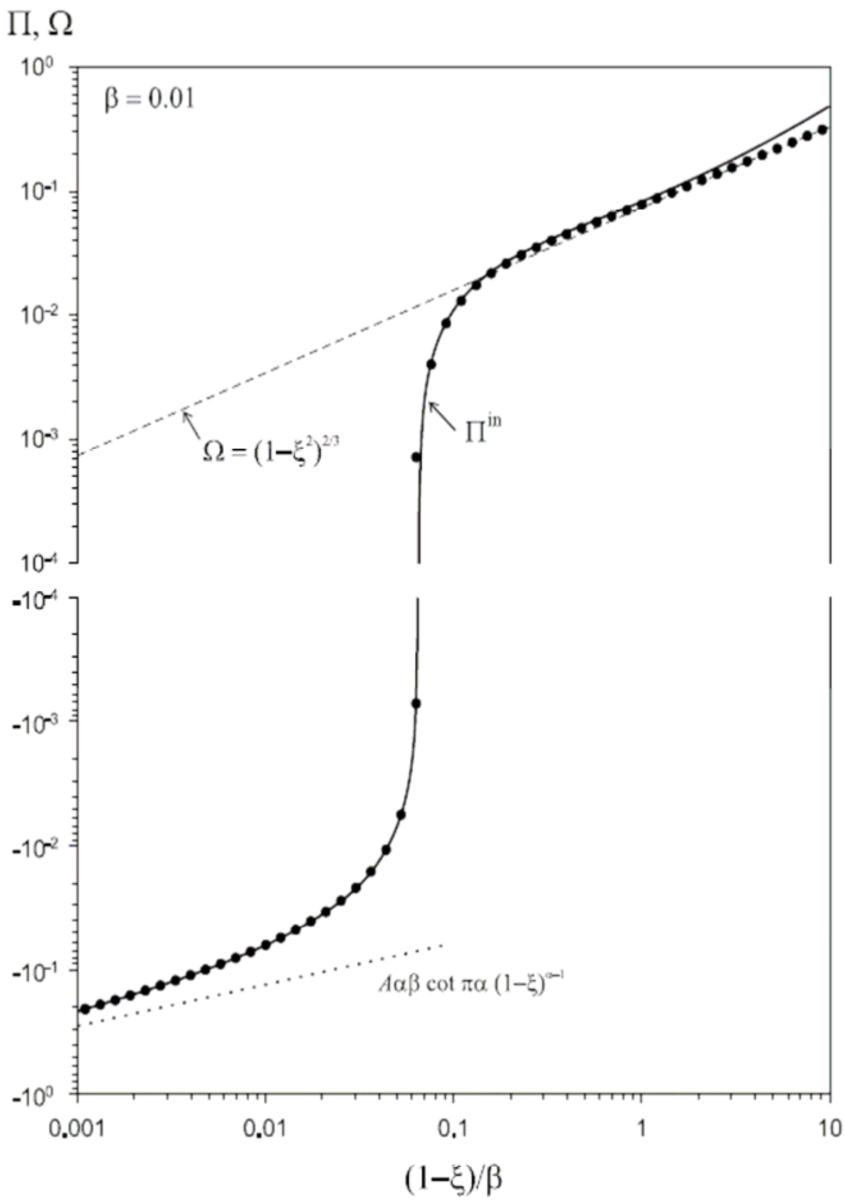
$$I_1(\xi;\beta)=-\frac{1}{\pi}\int_{\xi-\beta}^{1}\left[\frac{\beta}{\xi_o-\xi}-\frac{1}{2\beta}(\xi_o-\xi)\ln\frac{|\xi_o-\xi|}{\beta}+\ldots\right]\Omega'(\xi_o)d\xi_0$$

PKN Region $\frac{\xi_o-\xi}{\beta}\gg 1$

$$I_2(\xi;\beta)=\frac{1}{2}\int_{-1}^{\xi-\beta}\left[1+\frac{\beta^2}{4(\xi_o-\xi)^2}+\ldots\right]\Omega'(\xi_o)d\xi_0$$

$$\Pi(\xi;\beta)\sim\beta A\,\alpha\,(1-\xi)^{\alpha-1}\cot\pi\alpha+\beta^{\alpha}\lambda_0(\alpha)+O(\beta^{\alpha-1}(1-\xi))$$

Inner Expansion

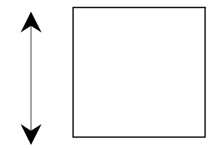


Discretizing the Elasticity Equation

$$p = -\frac{E'}{H\pi}\int_{-\ell}^{\ell} w_0'(x_0)\,sgn(x_0-x)\sqrt{1+\frac{H^2}{(x_0-x)^2}}\,E\left(\frac{H^2}{(x_0-x)^2+H^2}\right)dx_0$$

$$w_0(x) \approx \sum_n w_m H_m(x),$$

$$H_m(x) = 1$$



$$\Delta x = 2a$$

$$p(x_n) = -\frac{E'}{H\pi}\sum_m w_m G(x_n - x_m)$$

$$G(x_n-x_m) = \left[sgn(\chi - x)\sqrt{1+\frac{H^2}{(\chi-x_n)^2}}\,E\left(\frac{H^2}{(\chi-x_n)^2+H^2}\right)\right]_{\chi=x_m-a}^{\chi=x_m+a}$$

$$p = Cw$$

Discretizing the Fluid Flow Equation

$$\frac{\partial w_0}{\partial t} = \frac{1}{16\mu}\frac{\partial}{\partial x}\left(w_0^3\frac{\partial p}{\partial x}\right) + \frac{2Q_0}{\pi H}\delta(x)$$

$$\frac{\partial}{\partial t}\int_{x_n-a}^{x_n+a} w_0 dx = \frac{1}{16\mu}\left[w_0^3\frac{\partial p}{\partial x}\right]_{x_n-a}^{x_n+a} + \frac{2Q_0}{\pi H}\delta_{n0}$$

$$\dot{w}_n = \frac{1}{16\mu\Delta x}\left[w^3_{n+\frac{1}{2}}\left(\frac{p_{n+1}-p_n}{\Delta x}\right) - w^3_{n-\frac{1}{2}}\left(\frac{p_n-p_{n-1}}{\Delta x}\right)\right]$$

$$\frac{\Delta w}{\Delta t} = A(w)w + F\delta$$

EPKN Tip solutions – viscosity

$$\frac{\partial w_0}{\partial t} = \frac{1}{16\mu}\frac{\partial}{\partial x}\left({w_0}^3\frac{\partial p}{\partial x}\right) + \frac{2Q_0}{\pi H}\delta(x) \qquad p(x) = \frac{E'}{4H}w_0(x)$$

$$\hat{x} = \ell(t) - x \qquad \frac{\partial}{\partial x} = -\frac{\partial}{\partial \hat{x}}, \qquad \frac{\partial}{\partial t} = \dot{\ell}(t)\frac{\partial}{\partial \hat{x}} = V\frac{\partial}{\partial \hat{x}}$$

$$V\frac{\partial \hat{w}}{\partial \hat{x}} = \frac{1}{16\mu}\frac{\partial}{\partial \hat{x}}\left(\hat{w}^3\frac{\partial \hat{p}}{\partial \hat{x}}\right)$$

Close to the tip $\hat{w} \sim A\hat{x}^\alpha \qquad \hat{p} \sim \frac{E'}{4}A\alpha \cot \pi\alpha \hat{x}^{\alpha-1}$

$$V = \frac{1}{16\mu}\hat{w}^2\frac{\partial \hat{p}}{\partial \hat{x}} \simeq \frac{1}{16\mu}A^2\hat{x}^{2\alpha}\frac{E'}{4}A\alpha(\alpha-1)\cot \pi\alpha \hat{x}^{\alpha-2}$$

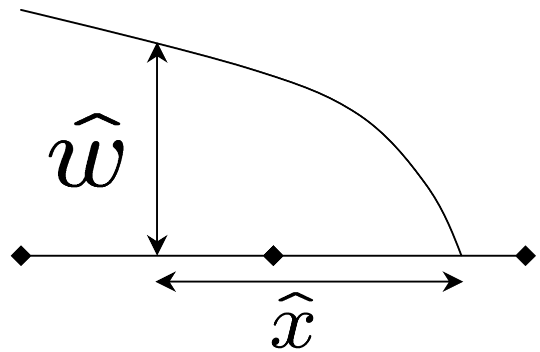
$$\alpha = 2/3 \qquad \theta_\alpha = \alpha(\alpha-1)\cot \pi\alpha$$

$$\hat{w} = \left(\frac{64\mu V}{E'\theta_\alpha}\right)^{\frac{1}{3}} \hat{x}^{\frac{2}{3}}$$

Locating the tip position

Viscosity Dominated: $\mathcal{G}_m = \mathcal{G}_v \Rightarrow \ell_* \sim t_*^{4/5}$

$$\hat{x} = \left[\hat{w} \left(\frac{E'\theta_\alpha}{64\mu V} \right)^{1/3} \right]^{3/2}$$



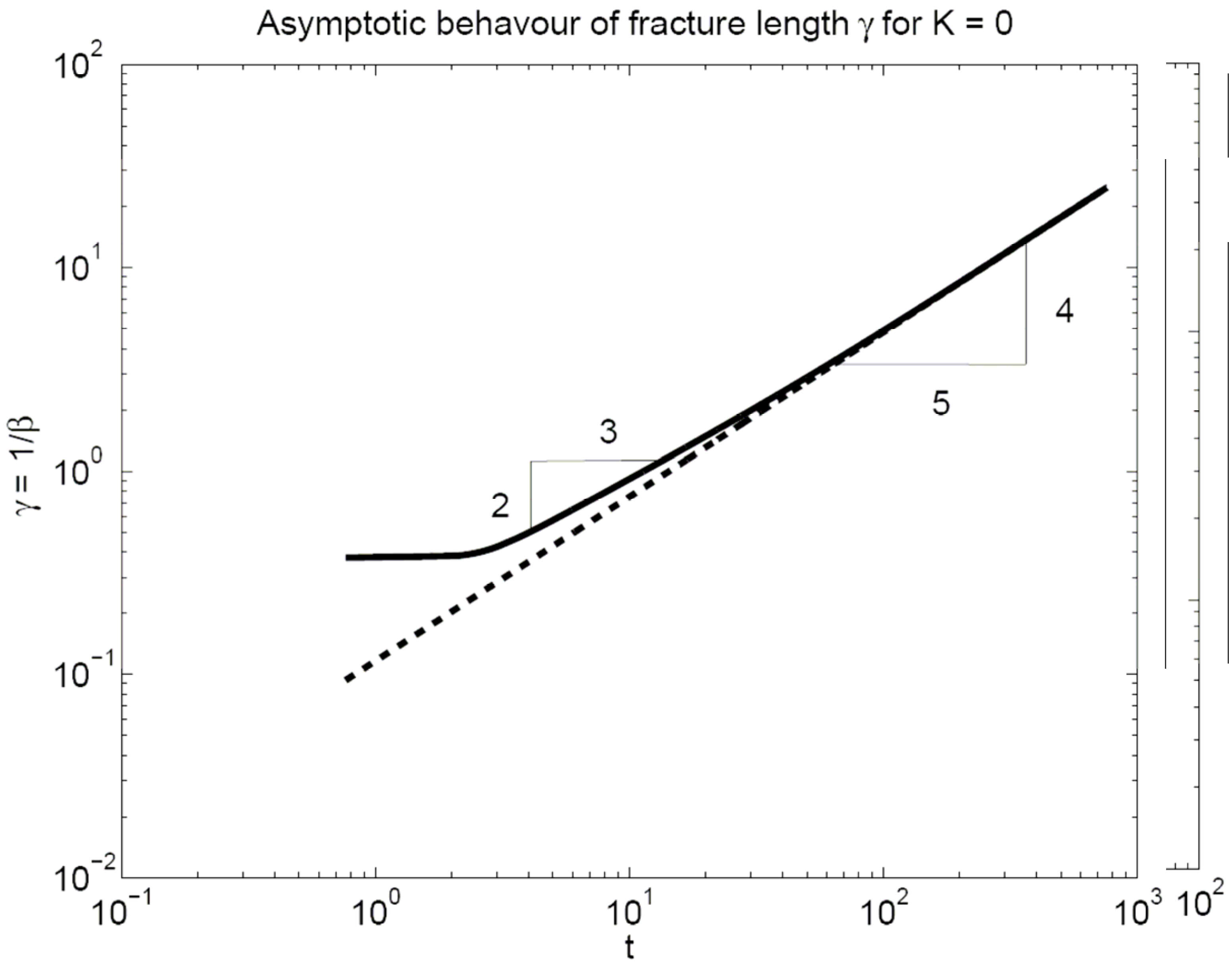
$$V = \frac{\hat{x} - \hat{x}_0}{\Delta t}$$

$$\hat{x}^3 - \hat{x}_0 \hat{x}^2 - b = 0 \qquad b = \frac{E'\theta_\alpha \hat{w}^3 \Delta t}{64\mu} > 0$$

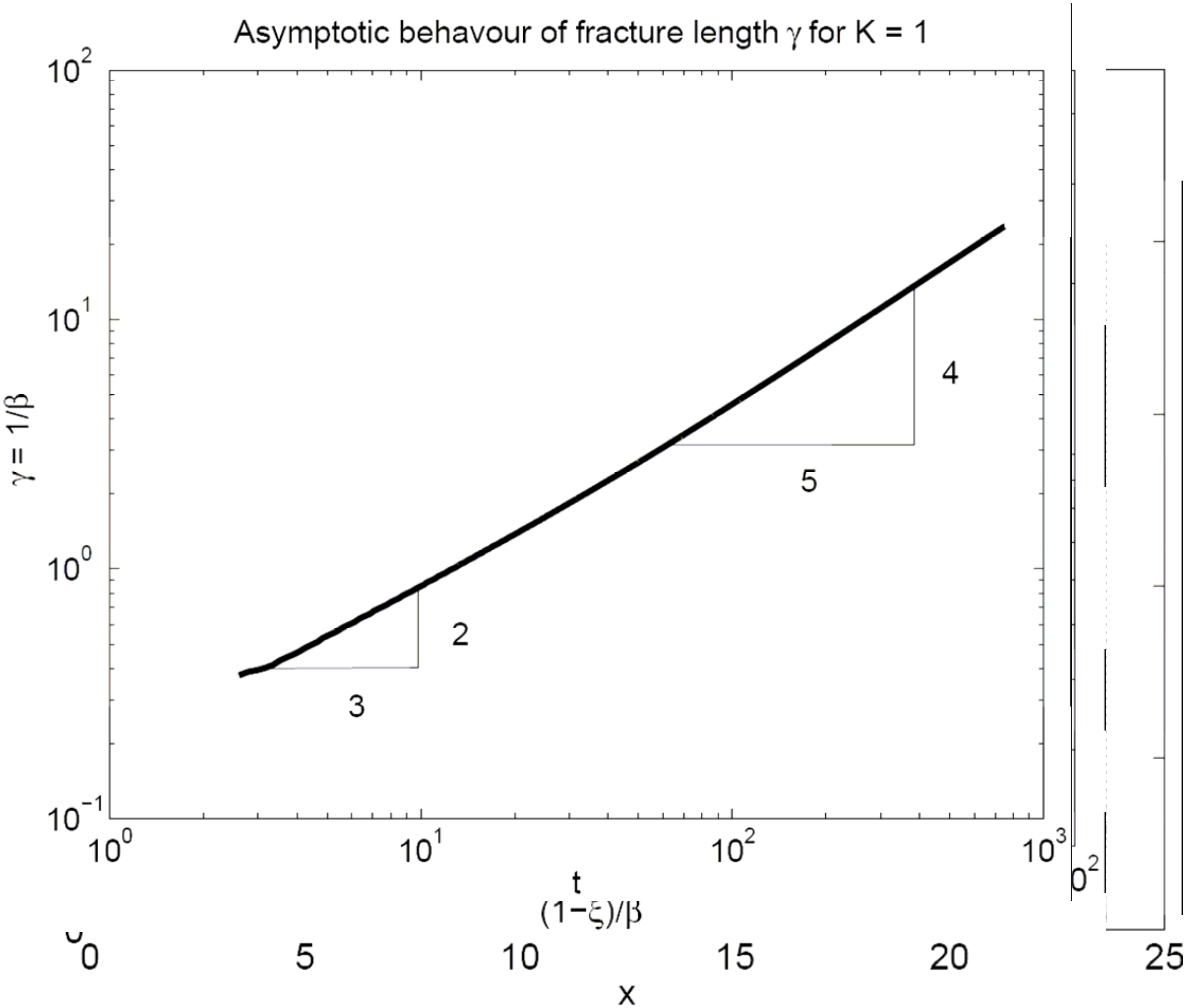
Toughness Dominated: $\mathcal{G}_k = \frac{K' \ell_*^{1/2}}{E' w_*} = \mathcal{G}_v \Rightarrow \ell_* \sim t_*^{2/3}$

$$\hat{w} \simeq \frac{K'}{E'} \hat{x}^{1/2} \qquad \hat{x} = \left(\frac{\hat{w} E'}{K'} \right)^2$$

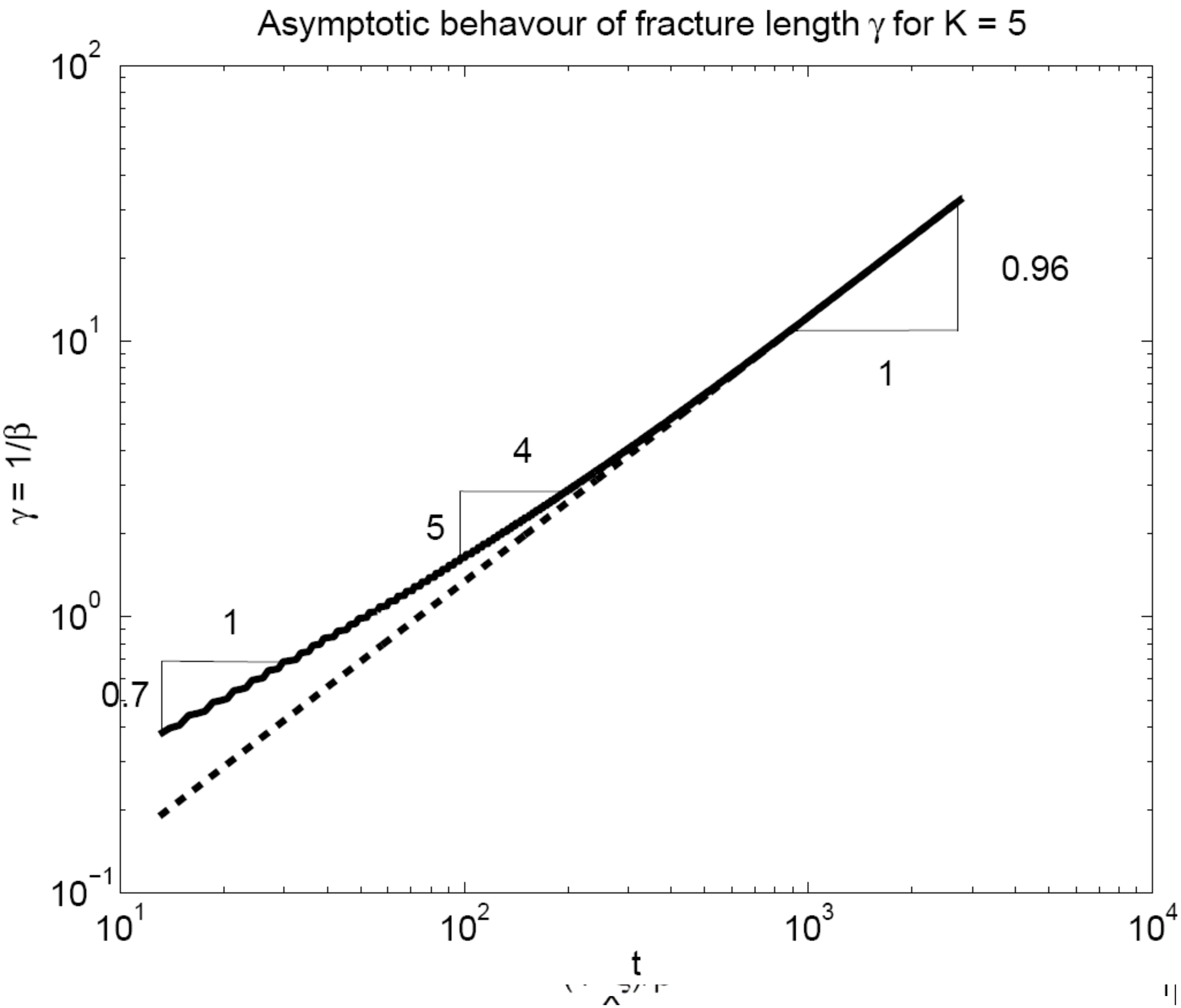
Numerical Results K=0



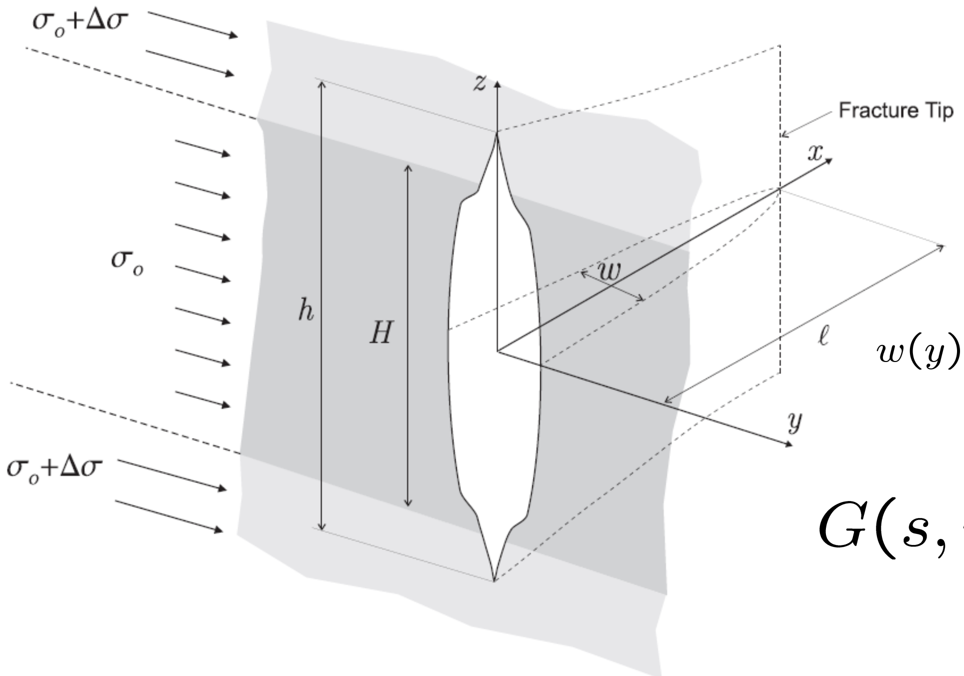
Numerical Results K=1



Numerical Results K=5



P3D models



$$p(y) - \sigma_c = -\frac{E'}{4\pi} \int_{-H/2}^{H/2} \frac{w(y')}{(y'-y)^2} dy'$$

$$w(y) = \frac{4}{\pi E'} \left[p \int_{0}^{H/2} G(s,y) ds + (p - \Delta\sigma) \int_{H/2}^{h/2} G(s,y) ds \right]$$

$$G(s,y) = \log \left| \frac{\sqrt{h^2 - 4y^2} + \sqrt{h^2 - 4s^2}}{\sqrt{h^2 - 4y^2} - \sqrt{h^2 - 4s^2}} \right|$$

$$w(y) = \frac{2}{E'} (p - \Delta\sigma) \sqrt{h^2 - 4y^2} + \frac{4\Delta\sigma}{\pi E'} \left\{ \sqrt{h^2 - 4y^2} \sin^{-1}\left(\frac{H}{h}\right) - y \log \left| \frac{H\sqrt{h^2-4y^2} + 2y\sqrt{h^2-H^2}}{H\sqrt{h^2-4y^2} - 2y\sqrt{h^2-H^2}} \right| + \frac{H}{2} \log \left| \frac{\sqrt{h^2-4y^2} + \sqrt{h^2-H^2}}{\sqrt{h^2-4y^2} - \sqrt{h^2-H^2}} \right| \right\}$$

$$K_{Ic} = K_I = \sqrt{\frac{8h}{\pi}} \left[p \int_{0}^{H/2} \frac{ds}{\sqrt{h^2 - 4s^2}} + (p - \Delta\sigma) \int_{H/2}^{h/2} \frac{ds}{\sqrt{h^2 - 4s^2}} \right]$$

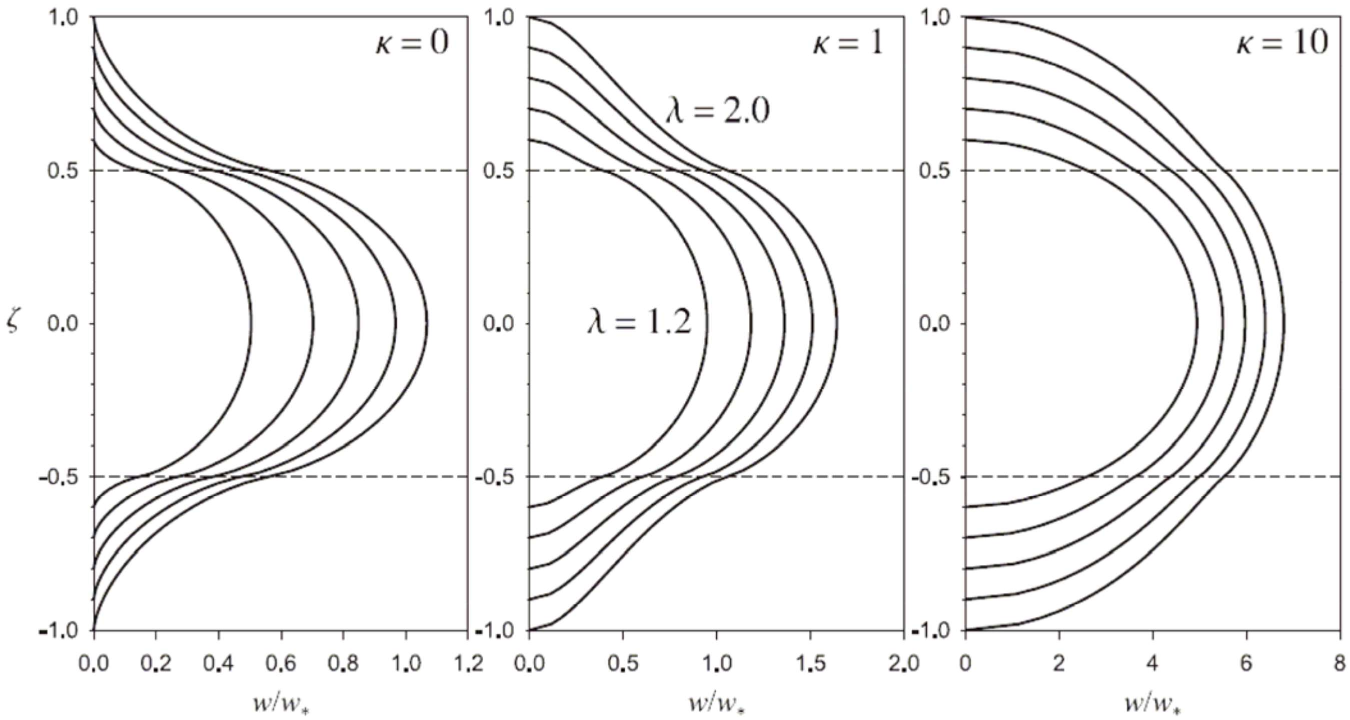
$$p - \Delta\sigma = \frac{\Delta\sigma}{\pi} \sqrt{\frac{H}{h}} \left(\sqrt{\frac{2\pi}{H}} \frac{K_{Ic}}{\Delta\sigma} \right) - \frac{2\Delta\sigma}{\pi} \sin^{-1}\left(\frac{H}{h}\right)$$

Scaled elasticity equations

$$p = \Delta\sigma\Pi \quad h = H\lambda \quad y = H\zeta \quad w = \left(\frac{\pi H}{2E'}\right)\Delta\sigma\Omega$$

$$\Omega(\zeta) = \frac{4}{\pi}\left[(\Pi-1)\sqrt{\lambda^2-4\zeta^2} + \frac{4}{\pi}\left\{\sqrt{\lambda^2-4\zeta^2}\sin^{-1}\left(\frac{1}{\lambda}\right) - \zeta\log\left|\frac{\sqrt{\lambda^2-4\zeta^2}+2\zeta\sqrt{\lambda^2-1}}{\sqrt{\lambda^2-4\zeta^2}-2\zeta\sqrt{\lambda^2-1}}\right| + \frac{1}{2}\log\left|\frac{\sqrt{\lambda^2-4\zeta^2}+\sqrt{\lambda^2-1}}{\sqrt{\lambda^2-4\zeta^2}-\sqrt{\lambda^2-1}}\right|\right\}\right]$$

$$(\Pi - 1) = \frac{\kappa}{\pi\sqrt{\lambda}} - \frac{2}{\pi}\sin^{-1}\left(\frac{1}{\lambda}\right) \qquad \kappa = \sqrt{\frac{2\pi}{H}}\frac{K_{Ic}}{\Delta\sigma}$$



Averaged equations

$$\bar{\Omega} = \int_{-\lambda/2}^{\lambda/2} \Omega d\zeta = \frac{1}{\pi}\left(\kappa\lambda^{3/2} + 2\sqrt{\lambda^2 - 1}\right) = \frac{\bar{w}}{w_*}$$

$$\frac{1}{H}\int_{-h/2}^{h/2} dy \cdot \quad \frac{\partial w}{\partial t} + \nabla \cdot q = Q(t)\delta(x)\delta(y) \qquad \frac{\partial \bar{w}}{\partial t} + \frac{\partial \bar{q}}{\partial x} = \frac{Q\delta(x)}{H}$$

$$q = -\frac{w^3}{\mu'}\nabla p \qquad \bar{q} = q_*\bar{\Psi} = \frac{1}{H}\int_{-h/2}^{h/2} w^3 dy \frac{\partial p}{\partial x}$$

$$x = \ell\xi = \ell_*\gamma\xi$$

$$\bar{\Psi} = -\left\{\frac{p_* w_*^3}{\pi^2 \mu \;\; \ell_* q_*}\right\}\left(\frac{\pi^2}{12\bar{\Omega}^3}\int_{-\lambda/2}^{\lambda/2} \Omega^3 d\zeta\right)\frac{\bar{\Omega}^3}{\gamma}\frac{\partial \Pi}{\partial \xi}$$

$$\frac{\partial \Pi}{\partial \xi} = \frac{\partial \Pi}{\partial \lambda}\frac{\partial \lambda}{\partial \bar{\Omega}}\frac{\partial \bar{\Omega}}{\partial \xi}$$

$$\bar{\Psi} = -\Phi(\lambda(\bar{\Omega}, \kappa))\frac{\bar{\Omega}^3}{\gamma L_*}\frac{\partial \bar{\Omega}}{\partial \xi}$$

Scaled lubrication equation

$$t = t_* \tau \quad x = \ell_* \gamma(\tau)\xi \quad \bar{w} = w_* \bar{\Omega}, \quad \bar{q} = q_* \bar{\Psi}$$

$$\frac{\partial \bar{w}}{\partial t} + \frac{\partial \bar{q}}{\partial x} = 0 \qquad \frac{\bar{q}}{q_*} = \bar{\Psi} = -\Phi(\lambda(\bar{\Omega}, \kappa))\frac{\bar{\Omega}^3}{\gamma L_*}\frac{\partial \bar{\Omega}}{\partial \xi}$$

$$\frac{\partial \bar{\Omega}}{\partial \tau} - \xi \frac{\dot{\gamma}}{\gamma}\frac{\partial \bar{\Omega}}{\partial \xi} = \frac{1}{\mathcal{G}_m}\frac{\partial}{\partial \xi}\left(\frac{\Phi(\lambda(\bar{\Omega},\kappa))}{\gamma^2}\bar{\Omega}^3\frac{\partial \bar{\Omega}}{\partial \xi}\right) + \mathcal{G}_v \frac{\delta(\xi)}{\gamma}$$

$$\mathcal{G}_v = \frac{t_* q_*}{w_* \ell_*} \quad q_* = \frac{Q}{H} \quad \mathcal{G}_m = \frac{\pi^2 \mu \ell_*^2}{t_* \Delta\sigma w_*^2}$$

$$\mathcal{G}_v = 1 = \mathcal{G}_m$$

$$\ell_* = \frac{\pi H^4 \Delta\sigma^4}{8\mu(E')^3 Q} \quad t_* = \frac{\pi^2 H^6 \Delta\sigma^5}{16\mu(E')^4 Q^2}$$

Collocation scheme to solve ODE

$$\frac{\partial\bar{\Omega}}{\partial\xi} = -\frac{\gamma\bar{\Psi}}{\Phi(\bar{\Omega},\kappa)\bar{\Omega}^3} = f(\bar{\Omega},\bar{\Psi})$$

$$\frac{\partial\bar{\Psi}}{\partial\xi} = -\frac{\xi\gamma\gamma_\tau}{\Phi(\bar{\Omega},\kappa)\bar{\Omega}^3}\bar{\Psi} - \gamma\bar{\Omega}_\tau = g(\bar{\Omega},\bar{\Psi})$$

$$\gamma\int_0^1 \bar{\Omega}(\xi,\tau)d\xi = \tau$$

$$\frac{dY}{d\xi} = F(\xi,Y)$$

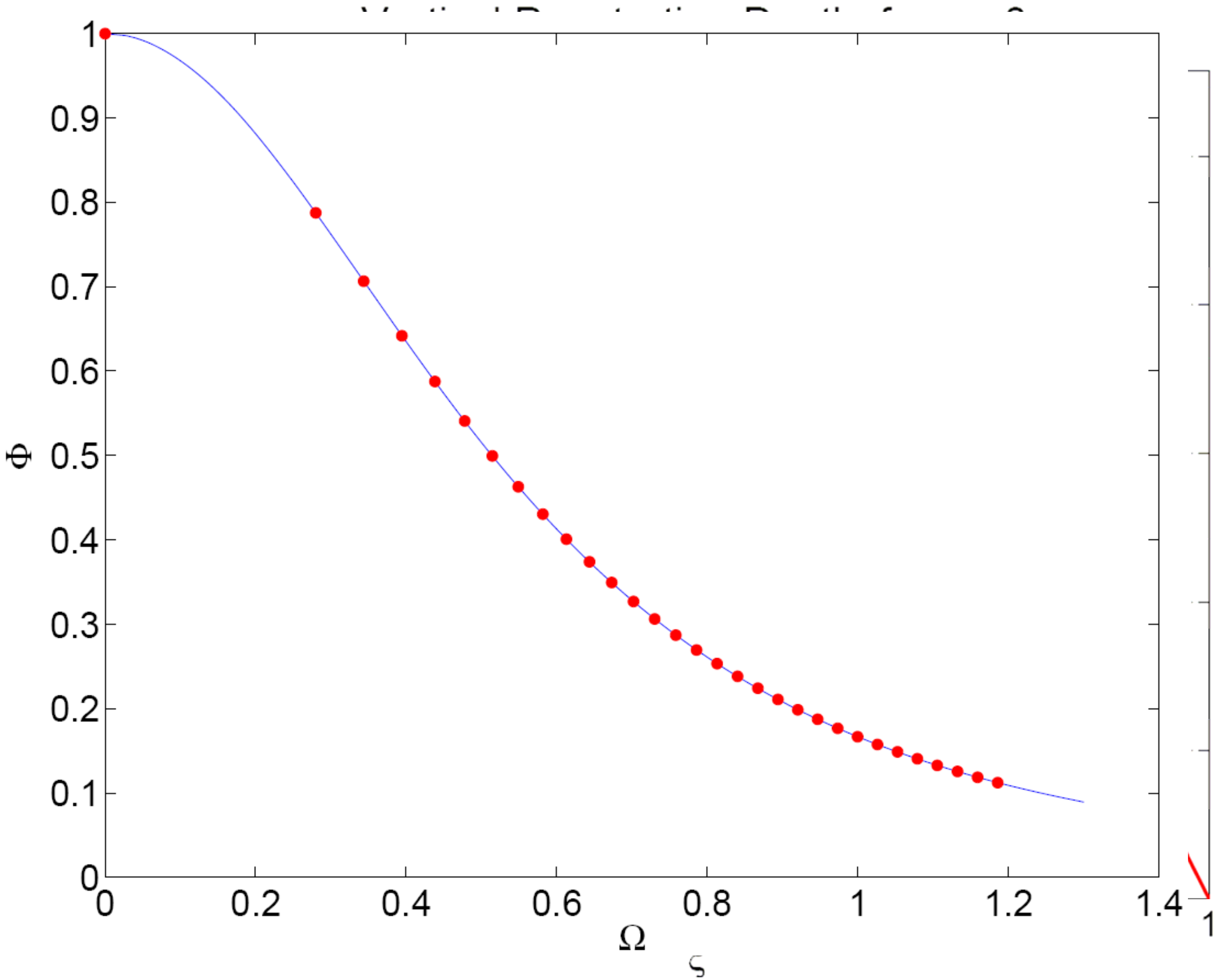
$$\bar{\Psi}_0 = 1$$

$$\bar{\Omega}(\xi_{N-1},\tau) = \bar{\Omega}_{PKN}(\xi_{N-1},\tau)$$

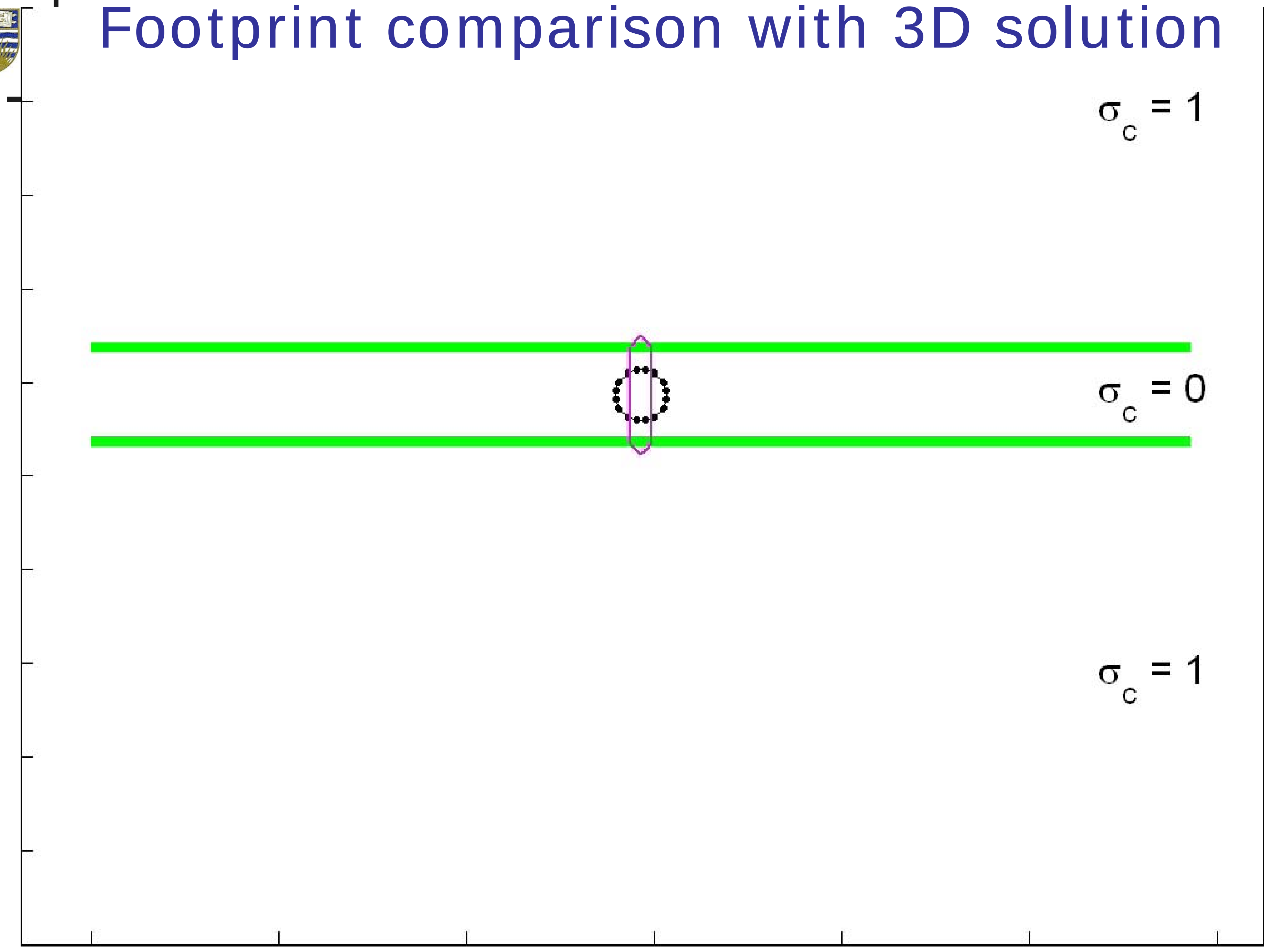
$$Y_{k+1} = Y_k + \int_{\xi_k}^{\xi_{k+1}} F(\xi,Y)d\xi$$

$$Y_{k+1} = Y_k + \frac{\Delta\xi}{6}\left[F_k + 4F_{k+\frac{1}{2}} + F_{k+1}\right] + O(\Delta\xi^5)$$

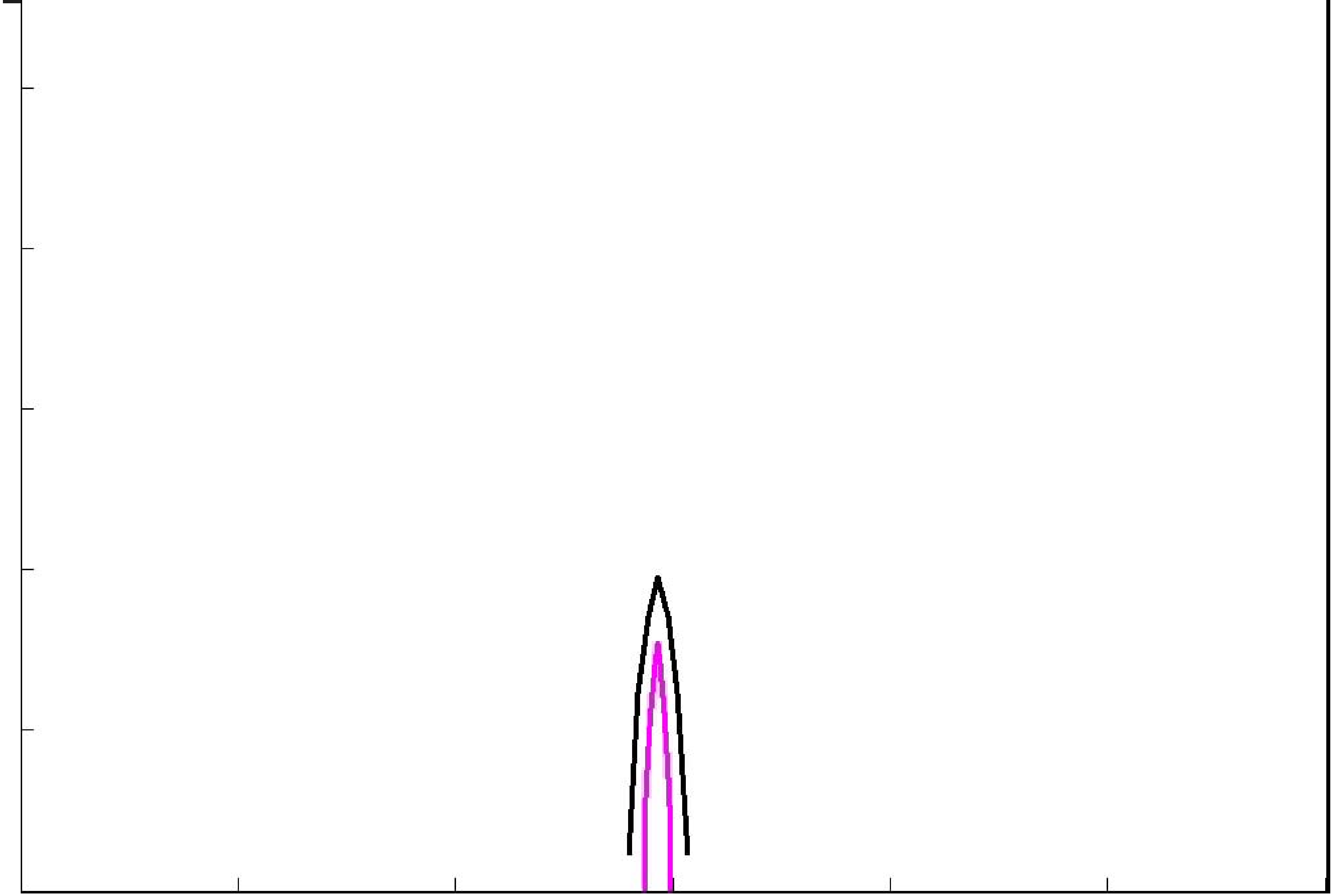
Numerical Results



Footprint comparison with 3D solution



Width comparison with 3D solution



Concluding remarks

- The HF problem and 2D models
- Slender geometries -> the possibilities for 1D models
- The classic PKN model – porous medium eq - limitations
- Extension of PKN to include toughness - 1D integro-PDE
  - Reduction of the 2D integral to a 1D nonlocal equation in the small aspect ratio limit
  - Asymptotics of the 1D kernel
  - Asymptotic analysis to determine the action of the integral operator in different regions of the domain:
    - Outer region: 1D integral equation - local PDE
    - Inner tip region: 1D integral equation - Hilbert Transform
  - Tip asymptotics and numerical solution
- P3D model – PKN methodology ->pseudo 3D
  - Plane strain width solution and averaging
  - Averaging and scaling the lubrication
  - Reduction to a convection diffusion equation
  - Numerical solution via collocation
  - Results and comparison with 3D solution

PKN Traveling Wave solution

$$\frac{\partial \bar{w}}{\partial t} = \frac{1}{\pi^2 \mu} \frac{\partial}{\partial x}\left(\bar{w}^3 \frac{\partial p}{\partial x}\right) + \frac{Q(t)\delta(x)}{2H} \qquad p(x) = \frac{E'}{\pi H}\bar{w}(x)$$

$$\frac{\partial \bar{w}}{\partial t} = \frac{\bar{E}}{H}\frac{1}{\bar{\mu}} \frac{\partial}{\partial x}\left(\bar{w}^3 \frac{\partial \bar{w}}{\partial x}\right) + \frac{Q(t)\delta(x)}{2H}$$

$$\hat{x} = \ell(t) - x$$

$$\frac{\partial}{\partial x} = -\frac{\partial}{\partial \hat{x}} \qquad \frac{\partial}{\partial t} = \dot{\ell}(t)\frac{\partial}{\partial \hat{x}} = V\frac{\partial}{\partial \hat{x}}$$

$$V\frac{\partial \hat{w}}{\partial \hat{x}} = \frac{\bar{E}}{H}\frac{1}{\bar{\mu}}\frac{\partial}{\partial \hat{x}}\left(\hat{w}^3 \frac{\partial \hat{w}}{\partial \hat{x}}\right) \qquad \hat{w}^2 \frac{\partial \hat{w}}{\partial \hat{x}} = \frac{H\bar{\mu}V}{\bar{E}}$$

$$\hat{w} = \left(3\frac{H\bar{\mu}V}{\bar{E}}\right)^{\frac{1}{3}} \hat{x}^{\frac{1}{3}}$$