Math 257-316 PDE Formula sheet - final exam

Trigonometric and Hyperbolic Function identities

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$ $\quad\sin^2 t + \cos^2 t = 1$
$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\beta\sin\alpha.$ $\quad\sin^2 t = \frac{1}{2}(1-\cos(2t))$
$\sinh(\alpha \pm \beta) = \sinh\alpha\cosh\beta \pm \sinh\beta\cosh\alpha$ $\quad\cosh^2 t - \sinh^2 t = 1$
$\cosh(\alpha \pm \beta) = \cosh\alpha\cosh\beta \pm \sinh\beta\sinh\alpha.$ $\quad\sinh^2 t = \frac{1}{2}(\cosh(2t)-1)$

Basic linear ODE's with real coefficients

	constant coefficients	**Euler eq**
ODE	$ay''+by'+cy=0$	$ax^2y''+bxy'+cy=0$
indicial eq.	$ar^2+br+c=0$	$ar(r-1)+br+c=0$
$r_1 \neq r_2$ real	$y = Ae^{r_1 x}+Be^{r_2 x}$	$y = Ax^{r_1}+Bx^{r_2}$
$r_1 = r_2 = r$	$y = Ae^{rx}+Bxe^{rx}$	$y = Ax^r + Bx^r\ln\lvert x\rvert$
$r = \lambda \pm i\mu$	$e^{\lambda x}[A\cos(\mu x)+B\sin(\mu x)]$	$x^\lambda[A\cos(\mu\ln\lvert x\rvert)+B\sin(\mu\ln\lvert x\rvert)]$

Series solutions for $y''+p(x)y'+q(x)y=0$ $(\star)$ around $x=x_0$.

Ordinary point x_0: Two linearly independent solutions of the form:

$$y(x) = \sum_{n=0}^\infty a_n(x-x_0)^n$$

Regular singular point x_0: Rearrange $(\star)$ as:
$(x-x_0)^2y''+[(x-x_0)p(x)](x-x_0)y'+[(x-x_0)^2q(x)]y=0$
If $r_1>r_2$ are roots of the indicial equation: $r(r-1)+br+c=0$ where $b=\lim_{x\to x_0}(x-x_0)p(x)$ and $c=\lim_{x\to x_0}(x-x_0)^2q(x)$ then a solution of $(\star)$ is

$$y_1(x) = \sum_{n=0}^\infty a_n(x-x_0)^{n+r_1} \quad \text{where } a_0=1.$$

The second linerly independent solution y_2 is of the form:
Case 1: If r_1-r_2 is neither 0 nor a positive integer:

$$y_2(x) = \sum_{n=0}^\infty b_n(x-x_0)^{n+r_2} \text{ where } \ b_0=1.$$

Case 2: If $r_1-r_2=0$:

$$y_2(x) = y_1(x)\ln(x-x_0)+\sum_{n=1}^\infty b_n(x-x_0)^{n+r_2} \text{ for some } b_1,b_2...$$

Case 3: If r_1-r_2 is a positive integer:

$$y_2(x) = ay_1(x)\ln(x-x_0)+\sum_{n=0}^\infty b_n(x-x_0)^{n+r_2} \text{ where } \ b_0=1.$$

Fourier, sine and cosine series

Let $f(x)$ be defined in $[-L,L]$then its Fourier series $Ff(x)$ is a $2L$-periodic function on $\mathbf{R}$: $Ff(x) = \frac{a_0}{2}+\sum_{n=1}^\infty\left\{a_n\cos(\frac{n\pi x}{L})+b_n\sin(\frac{n\pi x}{L})\right\}$
where $a_n = \frac{1}{L}\int_{-L}^L f(x)\cos(\frac{n\pi x}{L})\,dx$ and $b_n = \frac{1}{L}\int_{-L}^L f(x)\sin(\frac{n\pi x}{L})\,dx$

Theorem (Pointwise convergence) If $f(x)$ and $f'(x)$ are piecewise continuous, then $Ff(x)$ converges for every x to $\frac{1}{2}[f(x-)+f(x+)]$.

Parseval's indentity

$$\frac{1}{L}\int_{-L}^L |f(x)|^2dx = \frac{|a_0|^2}{2}+\sum_{n=1}^\infty\left(|a_n|^2+|b_n|^2\right).$$

For $f(x)$ defined in $[0,L]$, its cosine and sine series are

$$Cf(x) = \frac{a_0}{2}+\sum_{n=1}^\infty a_n\cos(\frac{n\pi x}{L}), \quad a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})\,dx,$$

$$Sf(x) = \sum_{n=1}^\infty b_n\sin(\frac{n\pi x}{L}), \quad b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})\,dx.$$

D'Alembert's solution to the wave equation

PDE: $u_{tt}=c^2u_{xx},\ -\infty<x<\infty,\ t>0$ **IC**: $u(x,0)=f(x),\ u_t(x,0)=g(x)$.
SOLUTION: $u(x,t)=\frac{1}{2}[f(x+ct)+f(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}g(s)ds$

Sturm-Liouville Eigenvalue Problems

ODE: $[p(x)y']'-q(x)y+\lambda r(x)y=0, \quad a<x<b.$
BC: $\alpha_1y(a)+\alpha_2y'(a)=0, \quad \beta_1y(b)+\beta_2y'(b)=0.$
Hypothesis: p, p', q, r continuous on $[a,b]$. $p(x)>0$ and $r(x)>0$ for $x\in[a,b]$. $\alpha_1^2+\alpha_2^2>0$. $\beta_1^2+\beta_2^2>0$.
Properties (1) The differential operator $Ly=[p(x)y']'-q(x)y$ is symmetric in the sense that $(f,Lg)=(Lf,g)$ for all f,g satisfying the BC, where $(f,g)=\int_a^b f(x)g(x)\,dx$. (2) All eigenvalues are real and can be ordered as $\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots$ with $\lambda_n\to\infty$ as $n\to\infty$, and each eigenvalue admits a unique (up to a scalar factor) eigenfunction ϕ_n.
(3) **Orthogonality**: $(\phi_m,r\phi_n)=\int_a^b\phi_m(x)\phi_n(x)r(x)\,dx=0$ if $\lambda_m\neq\lambda_n$.
(4) **Expansion**: If $f(x):[a,b]\to\mathbf{R}$ is square integrable, then

$$f(x)=\sum_{n=1}^\infty c_n\phi_n(x),\ a<x<b\ ,\ c_n=\frac{\int_a^b f(x)\phi_n(x)r(x)\,dx}{\int_a^b\phi_n^2(x)r(x)\,dx},\ n=1,2,\dots$$