

The University of British Columbia
 Final Examination - 8:30 December 7, 2017
Mathematics 257/316
 All Sections

Closed book examination

Time: 2.5 hours

Last Name _____ **First** _____ **Signature** _____

Student Number _____

Special Instructions:

No books, notes, or calculators are allowed. A formula sheet is attached.

Student Conduct during Examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (a) speaking or communicating with other candidates, unless otherwise authorized;
 - (b) purposely exposing written papers to the view of other candidates or imaging devices;
 - (c) purposely viewing the written papers of other candidates;
 - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1		20
2		20
3		20
4		20
5		20
Total		100

1. Consider the differential equation

$$3x^2(x+2)y'' + 7xy' - 2y = 0 \quad (1)$$

- (a) Classify the points $-\infty < x < \infty$ as ordinary points, regular singular points, or irregular singular points.
- (b) What form of expansion would you use around the point $x_0 = -2$? What is the minimal radius of convergence of this series?
- (c) Find two values of r such that there are solutions of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.
- (d) Use the series expansion in (c) to determine two independent solutions of (1). You only need to calculate the first three non-zero terms in each case.

[total 20 marks]

(Question 1 Continued)

(Question 1 Continued)

2. Consider the following diffusion initial-boundary value problem

$$\begin{aligned}u_t &= u_{xx}, \quad 0 < x < \pi, \quad t > 0 \\u(0, t) &= 0 = u(\pi, t) \\u(x, 0) &= x\end{aligned}\tag{2}$$

- (a) Determine the solution to (2) by separation of variables. [10 marks]
- (b) Briefly describe how you would use the method of finite differences to obtain an approximate solution to this boundary value problem that is accurate to $O(\Delta x^2, \Delta t)$ terms. Use the notation $u_n^k \simeq u(x_n, t_k)$ to represent the nodal values on the finite difference mesh. [6 marks]
- (c) Use the solution $u_n^k = G^k e^{in\theta}$ to derive a condition for the stability of this scheme. [4 marks]

[total 20 marks]

(Question 2 Continued)

(Question 2 Continued)

3. Solve the following initial boundary value problem for the wave equation subject to a periodic forcing with $\omega \notin \{1, 2, \dots\}$:

$$\begin{aligned}u_{tt} &= u_{xx} + \sin \omega t \sin (3x), & 0 < x < \pi, & t > 0 \\u(0, t) &= 0 \text{ and } u(\pi, t) = 0, & t > 0 \\u(x, 0) &= \sin x, \quad u_t(x, 0) = 0, & 0 < x < \pi\end{aligned}$$

[total 20 marks]

(Question 3 Continued)

(Question 3 Continued)

4. Consider the eigenvalue problem

$$\begin{aligned}x^2 y'' + xy' + \lambda y &= 0 \\ y'(1) &= 0 = y'(e^\pi)\end{aligned}$$

- (a) Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions. [8 marks]
- (b) Use the eigenfunctions in (a) to solve the following mixed boundary value problem for Laplace's equation on the semi-annular region:

$$\begin{aligned}u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0, & 1 < r < e^\pi, & \quad 0 < \theta < \pi \\ u(r, 0) &= 0 & \text{and} & \quad u(r, \pi) = f(r) \\ \frac{\partial u(1, \theta)}{\partial r} &= 0 & \text{and} & \quad \frac{\partial u(e^\pi, \theta)}{\partial r} = 0\end{aligned}$$

[12 marks]

[total 20 marks]

(Question 4 Continued)

(Question 4 Continued)

5. We wish to determine how long a steel beam will take to lose its structural integrity when one end is subjected to a fire of increasing intensity. Consider the following one dimensional model in which the left boundary condition represents the heat flux due to the fire and the right boundary condition represents the heat lost to the environment. Solve the inhomogeneous heat conduction problem subject to time dependent boundary conditions:

$$\begin{aligned}u_t &= u_{xx} - x, \quad 0 < x < 1, \quad t > 0 \\u_x(0, t) &= -t, \quad \text{and} \quad \frac{\partial u(1, t)}{\partial x} = -u(1, t) \\u(x, 0) &= x^2.\end{aligned}$$

- (a) Determine a simple function $w(x, t)$ that satisfies the inhomogeneous boundary conditions. [4 marks]
- (b) Now let $u(x, t) = w(x, t) + v(x, t)$ and determine the boundary value problem satisfied by $v(x, t)$. [4 marks]
- (c) Now determine a steady-state solution $\omega(x)$ for the equation for $v(x, t)$. Let $v(x, t) = \omega(x) + \phi(x, t)$, and determine the boundary value problem satisfied by $\phi(x, t)$. [4 marks]
- (d) Complete the solution to the problem by using separation of variables to solve the boundary value problem for $\phi(x, t)$. Determine the equation satisfied by the eigenvalues and illustrate the solutions graphically - you need not obtain an explicit expression for the eigenvalues. [8 marks]

[total 20 marks]

(Question 5 Continued)

(Question 5 Continued)

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