The University of British Columbia

Final Examination - 8:30 December 7, 2017

Mathematics 257/316

All Sections

Closed book examination | Time: 2.5 hours

Last Name ________________ **First** ____________ **Signature** ________________

Student Number ________________

Special Instructions:

No books, notes, or calculators are allowed. A formula sheet is attached.

1		20
2		20
3		20
4		20
5		20
Total		100

1. Consider the differential equation

$$3x^2(x+2)y'' + 7xy' - 2y = 0 \tag{1}$$

   (a) Classify the points $-\infty < x < \infty$ as ordinary points, regular singular points, or irregular singular points.

   (b) What form of expansion would you use around the point $x_0 = -2$? What is the minimal radius of convergence of this series?

   (c) Find two values of r such that there are solutions of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.

   (d) Use the series expansion in (c) to determine two independent solutions of (1). You only need to calculate the first three non-zero terms in each case.

[total 20 marks]

(Question 1 Continued)

(Question 1 Continued)

2. Consider the following diffusion initial-boundary value problem

$$\begin{aligned} u_t &= u_{xx},\ 0 < x < \pi,\ t > 0 \\ u(0,t) &= 0 = u(\pi,t) \\ u(x,0) &= x \end{aligned} \tag{2}$$

   (a) Determine the solution to (2) by separation of variables. [10 marks]

   (b) Briefly describe how you would use the method of finite differences to obtain an approximate solution to this boundary value problem that is accurate to $O(\Delta x^2, \Delta t)$ terms. Use the notation $u_n^k \simeq u(x_n, t_k)$ to represent the nodal values on the finite difference mesh. [6 marks]

   (c) Use the solution $u_n^k = G^k e^{in\theta}$ to derive a condition for the stability of this scheme. [4 marks]

[total 20 marks]

(Question 2 Continued)

(Question 2 Continued)

3. Solve the following initial boundary value problem for the wave equation subject to a periodic forcing with $\omega \notin \{1, 2, \ldots\}$:

$$\begin{aligned} u_{tt} &= u_{xx} + \sin \omega t \sin(3x)\,, \;\; 0 < x < \pi,\;\; t > 0 \\ u(0,t) &= 0 \text{ and } u(\pi, t) = 0, \quad t > 0 \\ u(x,0) &= \sin x, \qquad u_t(x,0) = 0, \quad 0 < x < \pi \end{aligned}$$

[total 20 marks]

(Question 3 Continued)

(Question 3 Continued)

4. Consider the eigenvalue problem

$$\begin{aligned} x^2y'' + xy' + \lambda y &= 0 \\ y'(1) &= 0 = y'(e^{\pi}) \end{aligned}$$

(a) Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions. [8 marks]

(b) Use the eigenfunctions in (a) to solve the following mixed boundary value problem for Laplace's equation on the semi-annular region:

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0, \quad 1 < r < e^{\pi}, \quad 0 < \theta < \pi \\ u(r,0) &= 0 \quad \text{and} \quad u(r,\pi) = f(r) \\ \frac{\partial u(1,\theta)}{\partial r} &= 0 \quad \text{and} \quad \frac{\partial u(e^{\pi},\theta)}{\partial r} = 0 \end{aligned}$$

[12 marks]

[total 20 marks]

(Question 4 Continued)

(Question 4 Continued)

5. We wish to determine how long a steel beam will take to lose its structural integrity when one end is subjected to a fire of increasing intensity. Consider the following one dimensional model in which the left boundary condition represents the heat flux due to the fire and the right boundary condition represents the heat lost to the environment. Solve the inhomogeneous heat conduction problem subject to time dependent boundary conditions:

$$\begin{aligned} u_t &= u_{xx} - x, \ 0 < x < 1, \ t > 0 \\ u_x(0,t) &= -t, \ \text{and} \ \frac{\partial u(1,t)}{\partial x} = -u(1,t) \\ u(x,0) &= x^2. \end{aligned}$$

(a) Determine a simple function $w(x,t)$ that satisfies the inhomogeneous boundary conditions. [4 marks]

(b) Now let $u(x,t) = w(x,t) + v(x,t)$ and determine the boundary value problem satisfied by $v(x,t)$. [4 marks]

(c) Now determine a steady-state solution $\omega(x)$ for the equation for $v(x,t)$. Let $v(x,t) = \omega(x) + \phi(x,t)$, and determine the boundary value problem satisfied by $\phi(x,t)$. [4 marks]

(d) Complete the solution to the problem by using separation of variables to solve the boundary value problem for $\phi(x,t)$. Determine the equation satisfied by the eigenvalues and illustrate the solutions graphically - you need not obtain an explicit expression for the eigenvalues. [8 marks]

[total 20 marks]

(Question 5 Continued)

(Question 5 Continued)

(Additional Page)