1. Consider the differential equation

$$Ly = 3x^2(x+2)y'' + 7xy' - 2y = 0 \qquad (1)$$

(a) Classify the points $-\infty < x < \infty$ as ordinary points, regular singular points, or irregular singular points.

(b) What form of expansion would you use around the point $x_0 = -2$? What is the minimal radius of convergence of this series?

(c) Find two values of r such that there are solutions of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.

(d) Use the series expansion in (c) to determine two independent solutions of (1). You only need to calculate the first three non-zero terms in each case.

[total 20 marks]

(a) $X=0$ AND $X=-2$ ARE SINGULAR POINTS, ALL OTHER POINTS $X \neq 0,-2$ ARE ORDINARY POINTS

$X=0$: $\lim_{X\to 0} X \cdot \frac{7X}{3X^2(X+2)} = \frac{7}{6} = p_0$, $\lim_{X\to 0} \frac{X^2(-2)}{3X^2(X+2)} = -\frac{1}{3} = q_0$, $|p_0|, |q_0| < \infty \Rightarrow X=0$ IS A REGULAR SINGULAR POINT

$X=-2$ $\lim_{X\to -2} \frac{\cancel{(X+2)}\, 7X}{3X^2\cancel{(X+2)}} = -\frac{7}{6} = p_0$, $\lim_{X\to -2} \frac{(X+2)^2(-2)}{3X^2(X+2)} = 0 = q_0$, $|p_0|, |q_0| < \infty \Rightarrow X=-2$ IS A REGULAR SINGULAR POINT

(b) SINCE $X_0=-2$ IS A RSP WE WOULD USE AN EXPANSION OF THE FORM $y = \sum_{n=0}^{\infty} a_n (X+2)^{n+r}$ WHOSE RADIUS OF CONVERGENCE $\rho \geq |-2-0| = 2$ THE DISTANCE TO THE NEAREST SINGULAR POINT WHICH IS AT $X=0$.

(c) FROM (a) THE INDICIAL EQ IS $r(r-1) + \frac{7}{6}r - \frac{1}{3} = 0 \Rightarrow 6r^2 + r - 2 = (3r+2)(2r-1) = 0 \Rightarrow r = -2/3,\ 1/2$

(d) $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, $y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$

$$Ly = 6x^2y'' + 3x^3y'' + 7xy' - 2y = 0$$

$$= \sum_{n=0}^{\infty} 6a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} 3a_n(n+r)(n+r-1)x^{n+r+1} + \sum_{n=0}^{\infty} 7a_n(n+r)x^{n+r} - \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$n=m$; $m=n+1 \Rightarrow n=m-1$, $n=0 \Rightarrow m=1$; $n=m$; $n=m$

$$= a_0\{6r(r-1)+7r-2\}x^r + \sum_{m=1}^{\infty} [a_m\{(m+r)[6(m+r-1)+7]-2\} + 3a_{m-1}(m+r-1)(m+r-2)]x^{m+r} = 0$$

$x^r]$ $6r^2+r-2 = (3r+2)(2r-1) = 0$ $r = -2/3,\ 1/2$

$x^{m+r}, m\geq 1]$ $a_m = \frac{-3a_{m-1}(m+r-1)(m+r-2)}{(m+r)[6(m+r)+1]-2}$ $m \geq 1$

$r = -2/3$: $a_m = \frac{-3a_{m-1}(m-5/3)(m-8/3)}{(m-2/3)[6m-3]-2} = \frac{-a_{m-1}(3m-5)(3m-8)/3}{(3m-2)(2m-1)-2} = \frac{-a_{m-1}(3m-5)(3m-8)}{3m(6m-7)}$

$a_1 = -\frac{a_0(-2)(-5)}{3(-1)} = \frac{10a_0}{3}$ $a_2 = -\frac{a_1(1)(-2)}{6(5)} = \frac{a_1}{15} = \frac{2a_0}{9}$

$\therefore$ $y(x) = x^{-2/3}\left[1 + \frac{10}{3}x + \frac{2x^2}{9} + \cdots\right]$

$r = 1/2$: $a_m = \frac{-3a_{m-1}(m-1/2)(m-3/2)}{(m+1/2)(6m+4)-2} = \frac{-3a_{m-1}(2m-1)(2m-3)/4}{(2m+1)(3m+2)-2} = \frac{-3a_{m-1}(2m-1)(2m-3)}{4m(6m+7)}$

$a_1 = \frac{-3a_0(1)(-1)}{4 \cdot 13} = \frac{3a_0}{52}$ $a_2 = \frac{-3a_1(3)(1)}{8 \cdot 19} = -\frac{27a_0}{52 \cdot 8 \cdot 19}$

$y(x) = x^{1/2}\left[1 + \frac{3x}{52} - \frac{27}{52 \cdot 8 \cdot 19}x^2 + \cdots\right]$

2. Consider the following diffusion initial-boundary value problem

$$\begin{aligned} u_t &= u_{xx},\ 0<x<\pi,\ t>0 \\ u(0,t) &= 0 = u(\pi,t) \\ u(x,0) &= x \end{aligned} \tag{2}$$

(a) Determine the solution to (2) by separation of variables. [10 marks]

(b) Briefly describe how you would use the method of finite differences to obtain an approximate solution to this boundary value problem that is accurate to $O(\Delta x^2, \Delta t)$ terms. Use the notation $u_n^k \simeq u(x_n, t_k)$ to represent the nodal values on the finite difference mesh. [6 marks]

(c) Use the solution $u_n^k = G^k e^{in\theta}$ to derive a condition for the stability of this scheme. [4 marks]

[total 20 marks]

(a) LET $u(x,t) = X(x)T(t) \Rightarrow X\dot{T} = X''T \overset{\div XT}{\Rightarrow} \frac{\dot{T}}{T(t)} = \frac{X''}{X(x)} = -\mu^2$

T] $\dot{T} = -\mu^2 T \Rightarrow T(t) = Ce^{-\mu^2 t}$

X] $X'' + \mu^2 X = 0 \Rightarrow X(x) = A\cos\mu x + B\sin\mu x \quad 0 = X(0) = A \Rightarrow A = 0$

$X(0) = 0 = X(\pi) \quad 0 = X(\pi) = B\sin\mu\pi = 0 \Rightarrow \mu_n = n \quad n = 1, 2, \ldots$ ARE THE EIGENVALUES

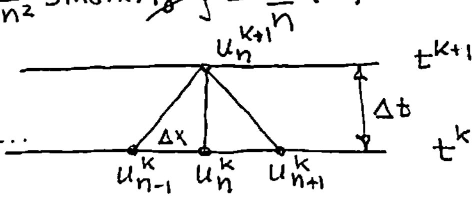
AND $X_n = \sin nx$ ARE THE EIGENFUNCTIONS

$\therefore u(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2 t} \sin nx$

$x = u(x,0) = \sum_{n=1}^{\infty} B_n \sin(nx) \Rightarrow B_n = \frac{2}{\pi}\int_0^\pi x\sin(nx)\,dx = \frac{2}{\pi}\left\{ \frac{-x\cos nx}{n}\Big|_0^\pi + \frac{1}{n}\int_0^\pi 1\cos(nx)\,dx \right\}$

$= \frac{2}{\pi}\left\{ \frac{-\pi\cos(n\pi)}{n} + \frac{1}{n^2}\cancel{\sin(nx)\Big|_0^\pi} \right\} = \frac{2}{n}(-1)^{n+1}$

$\therefore u(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} e^{-n^2 t} \sin(nx)$



b) $u(x_n \pm \Delta x, t) = u(x_n) \pm u'(x_n)\Delta x + u''(x_n)\frac{\Delta x^2}{2} \pm u'''(x_n)\frac{\Delta x^3}{6} + u''''(x_n)\frac{\Delta x^4}{24} + \cdots$

ADD

$\therefore u_{n+1} + u_{n-1} = 2u_n + u''(x_n)\Delta x + O(\Delta x^4)$

$\therefore u''(x_n) = \frac{u_{n+1} - 2u_n + u_{n-1}}{\Delta x^2} + O(\Delta x^2)$

$u(x_n, t^k + \Delta t) = u(x_n, t^k) + \Delta t\, \dot{u}(x_n, t) + \frac{\Delta t^2}{2}\ddot{u}(x_n, t) + \cdots \Rightarrow \dot{u}_n(t) = \frac{u_n^{k+1} - u_n^k}{\Delta t} + O(\Delta t)$

$\therefore u_t = u_{xx} \Rightarrow u_t \simeq \frac{u_n^{k+1} - u_n^k}{\Delta t} = u_{xx} = \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{\Delta x^2}$

$\therefore u_n^{k+1} = u_n^k + \frac{\Delta t}{\Delta x^2}(u_{n+1}^k - 2u_n^k + u_{n-1}^k) \quad + O(\Delta x^2, \Delta t)$

c) $u_n^k = G^k e^{in\theta} \Rightarrow G^{k+1}e^{in\theta} = G^k e^{in\theta} + \frac{\Delta t}{\Delta x^2}[e^{i\theta} - 2 + e^{-i\theta}]G^k e^{in\theta}$

$\therefore G = 1 + \frac{\Delta t}{\Delta x^2}[2\cos\theta - 2] = 1 - 4\frac{\Delta t}{\Delta x^2}\sin^2\frac{\theta}{2}$

FOR STABILITY $|G| < 1 \Rightarrow \left|1 - 4\frac{\Delta t}{\Delta x^2}\sin^2\frac{\theta}{2}\right| < 1 \Rightarrow -1 < 1 - 4\frac{\Delta t}{\Delta x^2}\sin^2\frac{\theta}{2} < 1$

$\therefore -2 < -\frac{4\Delta t}{\Delta x^2}\sin^2\frac{\theta}{2} \Rightarrow \frac{4\Delta t}{\Delta x^2}\sin^2\frac{\theta}{2} < 2$

SINCE $\sin^2\frac{\theta}{2} \leq 1$ THE CONDITION WILL BE SATISFIED PROVIDED $\boxed{\Delta t < \frac{\Delta x^2}{2}}$

3. Solve the following initial boundary value problem for the wave equation subject to a periodic forcing with $\omega \notin \{1,2,\ldots\}$:

$$u_{tt} = u_{xx} + \sin\omega t \sin(3x)\,,\ 0 < x < \pi,\ t > 0$$
$$u(0,t) = 0 \text{ and } u(\pi,t) = 0,\quad t > 0$$
$$u(x,0) = \sin x,\qquad u_t(x,0) = 0,\ 0 < x < \pi$$

[total 20 marks]

THE EIGENFUNCTIONS & EIGENVALUES ASSOCIATED WITH THE HOMOGENEOUS BC ARE $\mu_n = \frac{n\pi}{\pi} = n \quad n = 1,2,\ldots$ AND $X_n(x) = \sin(nx)$

NOW LET $S(x,t) = \sin\omega t \sin(3x) = \sum_{n=1}^{\infty} S_n(t)\sin(nx) \Rightarrow S_n(t) = \delta_{n3}\sin\omega t$

NOW EXPAND $u(x,t)$ IN EIGENFUNCTIONS $u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin nx$

$$u_{tt} = \sum_{n=1}^{\infty} \ddot{u}_n(t)\sin(nx) \qquad u_{xx} = \sum_{n=1}^{\infty} u_n(t)[-n^2]\sin(nx)$$

$$\therefore\ u_{tt} - u_{xx} - \sin\omega t\sin(nx) = \sum_{n=1}^{\infty}\{\ddot{u}_n + n^2 u_n - \sin\omega t\,\delta_{n3}\}\sin(nx) = 0$$

SINCE $\sin(nx)$ ARE L.I IT FOLLOWS THAT $\ddot{u}_n + n^2 u_n = \sin\omega t\,\delta_{n3} \qquad n = 1,2,\ldots$

LET US SOLVE THE HOMOGENEOUS EQ : $\ddot{u}_n^H + n^2 u_n^H = 0 \Rightarrow u_n^H = A_n\cos nt + B_n\sin nt$

NOW LOOK FOR A PARTICULAR SOLUTION OF THE FORM $u_n^P = C\sin\omega t \quad \dot{u}_n^P = C\omega\cos\omega t$

$$\therefore\ \ddot{u}_n = -C\omega^2\sin\omega t \Rightarrow \ddot{u}_n^P + n^2 u_n^P = C\{-\omega^2 + n^2\}\sin\omega t = \sin\omega t\,\delta_{n3}$$

$$\therefore\qquad C = \frac{\delta_{n3}}{n^2-\omega^2}$$

THUS THE GENERAL SOLN IS OF THE FORM $u_n(t) = A_n\cos nt + B_n\sin nt + \frac{\delta_{n3}}{n^2-\omega^2}\sin\omega t$

$$\therefore\ u(x,t) = \sum_{n=1}^{\infty}\left[\{A_n\cos nt + B_n\sin nt\} + \frac{\delta_{n3}}{n^2-\omega^2}\sin\omega t\right]\sin nx$$

$$u_t(x,t) = \sum_{n=1}^{\infty}\left[-A_n n\sin nt + B_n n\cos nt + \frac{\delta_{n3}}{n^2-\omega^2}\omega\cos\omega t\right]\sin nx$$

NOW

$$\sin x = u(x,0) = \sum_{n=1}^{\infty} A_n\sin nx \Rightarrow A_n = \delta_{n1}$$

$$0 = u_t(x,0) = \sum_{n=1}^{\infty}\left[B_n n + \frac{\delta_{n3}\omega}{n^2-\omega^2}\right]\sin nx \Rightarrow B_n = -\frac{\delta_{n3}\omega}{n(n^2-\omega^2)}$$

$$\therefore\ u(x,t) = \sum_{n=1}^{\infty}\left[\left[\delta_{n1}\cos nt - \frac{\delta_{n3}\omega}{n(n^2-\omega^2)}\sin nt\right] + \frac{\delta_{n3}}{n^2-\omega^2}\sin\omega t\right]\sin(nx)$$

$$= \cos t\sin x + \frac{1}{3^2-\omega^2}\left[\sin\omega t - \frac{\omega}{3}\sin 3t\right]\sin 3x.$$

4. Consider the eigenvalue problem

$$
\begin{aligned}
Ly = x^2y'' + xy' + \lambda y &= 0 \\
y'(1) &= 0 = y'(e^\pi)
\end{aligned}
$$

(a) Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions. [8 marks]

(b) Use the eigenfunctions in (a) to solve the following mixed boundary value problem for Laplace's equation on the semi-annular region:

$$
\begin{aligned}
u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0, \quad 1 < r < e^\pi, \quad 0 < \theta < \pi \\
u(r,0) &= 0 \quad \text{and} \quad u(r,\pi) = f(r) \\
\frac{\partial u(1,\theta)}{\partial r} &= 0 \quad \text{and} \quad \frac{\partial u(e^\pi,\theta)}{\partial r} = 0
\end{aligned}
$$

[12 marks]

[total 20 marks]

(a) $F = \frac{e^{\int \frac{x}{x^2}dx}}{x^2} = \frac{e^{\ln x}}{x^2} = \frac{1}{x} \Rightarrow \mathcal{L}y = FLy = xy'' + y' + \frac{\lambda}{x}y = (xy')' + \frac{\lambda}{x}y = 0$ WHICH IS IN A S-L FORM.

$\lambda \neq 0$ SINCE $Ly=0$ IS A CAUCHY EULER EQ LET IS LOOK FOR SOLUTIONS OF THE FORM $y = x^\gamma \Rightarrow \gamma(\gamma-1)+\gamma+\lambda = \gamma^2+\lambda = 0 \Rightarrow \gamma = \pm i\sqrt{\lambda} = \pm i\mu$ WHERE $\mu = \sqrt{\lambda} > 0$

THEN $y(x) = A\cos(\mu\ln x) + B\sin(\mu\ln x)$ $\quad y' = -\frac{A\mu}{x}\sin(\mu\ln x) + \frac{B\mu}{x}\cos(\mu\ln x)$

$0 = y'(1) = B\mu \overset{\mu>0}{\Rightarrow} B = 0 \qquad 0 = y'(e^\pi) = -\frac{A\mu}{e^\pi}\sin\mu(\ln(e^\pi)) = -\frac{A\mu}{e^\pi}\sin\mu\pi = 0 \Rightarrow \mu_n = n \quad n=1,2,\ldots$

THUS THE EIGENFUNCTIONS AND EIGENVALUES ARE: $\mu_n = n$ AND $X_n = \cos(n\ln x)$

$\underline{\lambda=0}$: $Ly = x^2y'' + xy' = 0 \quad y = x^\gamma \Rightarrow \gamma(\gamma-1)+\gamma = \gamma^2 = 0 \quad \gamma = 0,0$ IS A DOUBLE ROOT

$\therefore y(x) = A1 + B\ln x \qquad y' = B/x$

$0 = y'(1) = B/1 \Rightarrow B = 0 \Rightarrow y'(e^\pi) = 0$

THUS FOR $\lambda_0 = 0 \quad X_0 = 1$

b) LET $u(r,\theta) = R(r)\Theta(\theta) \Rightarrow \frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = -\mu^2 = \text{CONST}$

R] $\left.\begin{aligned} r^2R'' + rR' + \mu^2R = 0 \\ R'(1) = 0 = R'(e^\pi)\end{aligned}\right\} \overset{(a)}{\Rightarrow} \mu_n = n \quad n = 0,1,2,\ldots \quad X_n = \begin{cases} 1 & n=0 \\ \cos n\ln r & n\geq 1\end{cases}$

$\Theta]$ $n\neq 0$ $\Theta'' - n\Theta = 0 \qquad \Theta = A\cosh n\theta + B\sinh n\theta \qquad \Theta(0) = A = 0$

$\Theta(0) = 0 \qquad \therefore \Theta_n = B_n\sinh(n\theta)$

$n=0$: $\Theta_0'' = 0 \qquad \Theta_0 = B_0\theta + A_0 \qquad \Theta_0(0) = A_0 = 0 \qquad \therefore \Theta_0(\theta) = B_0\theta$

$\Theta_0(0) = 0$

$\therefore u(r,\theta) = B_0 1\theta + \sum_{n=1}^{\infty} B_n\sinh(n\theta)\cos(n\ln r)$

$x = \ln r \quad dx = \frac{dr}{r}$

$r=1 \; x=0 \quad r=e^\pi \; x=\pi$

$f(r) = u(r,\pi) = B_0\pi + \sum_{n=1}^{\infty} B_n\sinh(n\pi)\cos(n\ln r)$

$\therefore \int_1^{e^\pi}\frac{1}{r}f(r)\cos(m\ln r)dr = B_0\pi\underbrace{\int_1^{e^\pi}\frac{1}{r}\cos(m\ln r)dr}_{\delta_{m0}\cdot\pi} + \sum_{n=1}^{\infty}B_n\sinh(n\pi)\underbrace{\int_1^{e^\pi}\frac{1}{r}\cos n\ln r\cos m\ln r\,dr}_{\delta_{mn}\frac{\pi}{2}}$

$\therefore B_m = \frac{2}{\pi\sinh(n\pi)}\int_1^{e^\pi}\frac{1}{r}f(r)\cos(m\ln r)dr \qquad B_0 = \frac{1}{\pi^2}\int_1^{e^\pi}\frac{f(r)}{r}dr$

$u(r,\theta) = B_0\theta + \sum_{n=1}^{\infty} B_n\sinh(n\theta)\cos(n\ln r)$

5. We wish to determine how long a steel beam will take to lose its structural integrity when one end is subjected to a fire of increasing intensity. Consider the following one dimensional model in which the left boundary condition represents the heat flux due to the fire and the right boundary condition represents the heat lost to the environment. Solve the inhomogeneous heat conduction problem subject to time dependent boundary conditions:

$$\begin{aligned} u_t &= u_{xx} - x, \ 0 < x < 1, \ t > 0 \\ u_x(0,t) &= -t, \text{ and } \frac{\partial u(1,t)}{\partial x} = -u(1,t) \\ u(x,0) &= x^2 \end{aligned}$$

(a) Determine a simple function $w(x,t)$ that satisfies the inhomogeneous boundary conditions. [4 marks]

(b) Now let $u(x,t) = w(x,t) + v(x,t)$ and determine the boundary value problem satisfied by $v(x,t)$. [4 marks]

(c) Now determine a steady-state solution $\omega(x)$ for the equation for $v(x,t)$. Let $v(x,t) = \omega(x) + \phi(x,t)$, and determine the boundary value problem satisfied by $\phi(x,t)$. [4 marks]

(d) Complete the solution to the problem by using separation of variables to solve the boundary value problem for $\phi(x,t)$. Determine the equation satisfied by the eigenvalues and illustrate the solutions graphically - you need not obtain an explicit expression for the eigenvalues. [8 marks]

[total 20 marks]

(a) LET $W(x,t) = A(t)x + B(t)$, $W_x = A(t)$

$-t = W_x(0,t) = A(t)$; $W_x(1,t) = -t = -\{-t \cdot 1 + B(t)\} = -W(1,t) \Rightarrow B = 2t$ $\Big\}$ $W(x,t) = -tx + 2t$

(b) PDE: $U_t = W_t + V_t = (-x+2) + V_t = \cancel{W_{xx}} + V_{xx} - x = U_{xx} - x \Rightarrow V_t = V_{xx} - 2$

BC: $-t = U_x(0,t) = W_x(0,t) + V_x(0,t) = -t + V_x(0,t) \Rightarrow V_x(0,t) = 0$

$0 = U_x(1,t) + U(1,t) = \{W_x(1,t) + W(1,t)\} + \{V_x(1,t) + V(1,t)\} \Rightarrow V_x(1,t) = -V(1,t)$

IC: $x^2 = U(x,0) = W(x,0) + V(x,0) = 0 + V(x,0) \Rightarrow V(x,0) = x^2$

c) $\omega_t = 0 = \omega_{xx} - 2 \Rightarrow \omega_{xx} = 2$ $\omega_x = 2x + A \Rightarrow \omega = x^2 + Ax + B$

$0 = \omega_x(0) = 0 + A$, $\omega_x(1) = 2 \cdot 1 = -\omega(1) = -(1+B) \Rightarrow B = -3$ $\Big\}$ $\omega(x) = x^2 - 3$

PDE: $V_t = \omega_t + \phi_t = \{\omega_{xx} - 2\} + \phi_{xx} \Rightarrow \boxed{\phi_t = \phi_{xx}}$; BC ARE THE SAME; IC: $x^2 = V(x,0) = \{x^2 - 3\} + \phi(x,0) \Rightarrow \phi(x,0) = 3.$

D) $\phi(x,t) = X(x)T(t) \Rightarrow X\dot{T} = X''T \Rightarrow \frac{\dot{T}}{T(t)} = \frac{X''}{X(x)} = -\mu^2 = \text{CONST}$

T] $\dot{T} = -\mu^2 T \Rightarrow T(t) = Ce^{-\mu^2 t}$

X] $X'' + \mu^2 X = 0$, $X'(0) = 0$, $X'(1) = -X(1)$ $\Big\}$ $X = A\cos\mu x + B\sin\mu x$, $X' = -A\mu\sin\mu x + B\mu\cos\mu x$

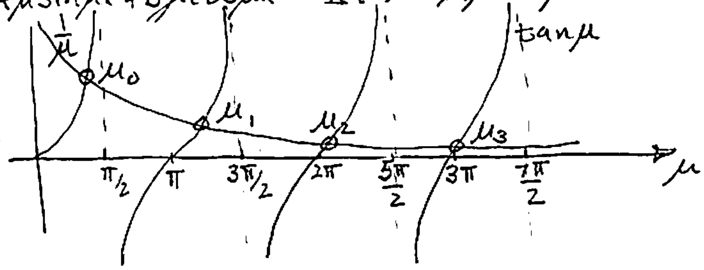
$0 = X'(0) = B\mu \xRightarrow{\mu \neq 0} B = 0$

$X'(1) = -A\mu\sin\mu = -X(1) = -A\cos\mu$, $A \neq 0$

$\therefore \frac{1}{\mu_n} = \tan\mu_n$ $n = 0, 1, 2..$

$X_n = \cos\mu_n x$

NOTE $\mu_n \sim n\pi$ AS $n \to \infty$



THUS $\phi(x,t) = \sum_{n=0}^{\infty} A_n e^{-\mu_n^2 t} \cos\mu_n x$ WHERE $\frac{1}{\mu_n} = \tan\mu_n$ $n=0,1,2\ldots$

NOW $3 = \phi(x,0) = \sum_{n=0}^{\infty} A_n \cos\mu_n x$

$$\int_0^1 3\cos\mu_m x\,dx = \sum_{n=1}^{\infty} A_n \int_0^1 \cos\mu_m x \cos\mu_n x\,dx$$

NOW BY S-L THEORY $\int_0^1 \cos\mu_m x \cos\mu_n x\,dx = 0$ IF $m \neq n$ AND IF $m=n$ WE HAVE

$$\int_0^1 \cos^2\mu_m x\,dx = \frac{1}{2}\int_0^1 (1+\cos 2\mu_m x)\,dx = \frac{1}{2}\left[1 + \frac{\sin 2\mu_m}{2\mu_m}\right] = \frac{1}{2}\left[1 + \frac{\cancel{2}\sin\mu_m \cos\mu_m}{\cancel{2}\mu_m}\right]$$

$$= \frac{1}{2}[1 + \sin^2\mu_m] \quad \text{SINCE} \quad \frac{\cos\mu_m}{\mu_m} = \sin\mu_m$$

$$\therefore \quad A_m = \frac{2\cdot 3}{1+\sin^2\mu_m}\int_0^1 \cos\mu_m x\,dx = \frac{6}{\mu_m(1+\sin^2\mu_m)} \sin\mu_m x\Big|_{\cancel{0}}^{1} = \frac{6\sin\mu_m}{\mu_m(1+\sin^2\mu_m)}$$

$$\therefore \quad u(x,t) = (-xt + 2t) + (x^2 - 3) + 6\sum_{m=0}^{\infty} \frac{\sin\mu_m}{\mu_m(1+\sin^2\mu_m)} e^{-\mu_m^2 t} \cos\mu_m x$$