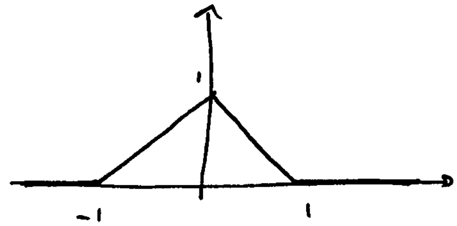
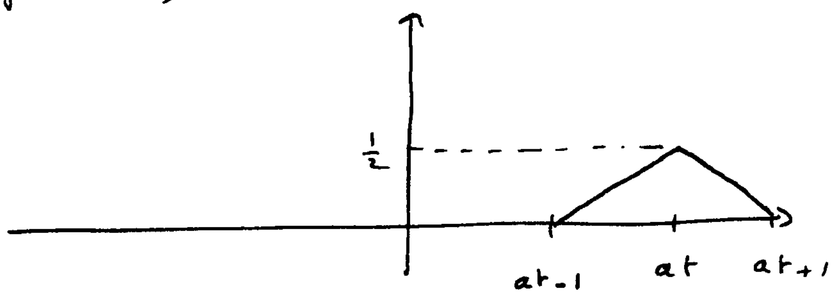
Problem 1 we have

$$u(x,t) = \frac{1}{2} f(x-at) + \frac{1}{2} f(x+at)$$

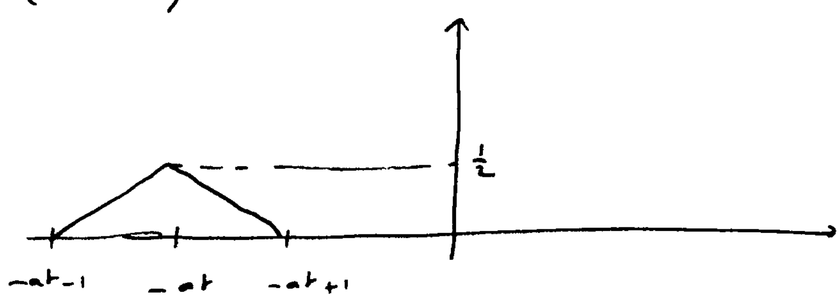
where f is given by



the function $\frac{1}{2} f(x-at)$ looks like:

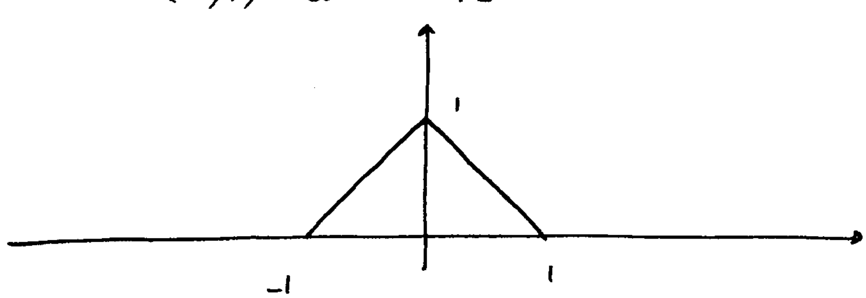


the function $\frac{1}{2} f(x+at)$ looks like



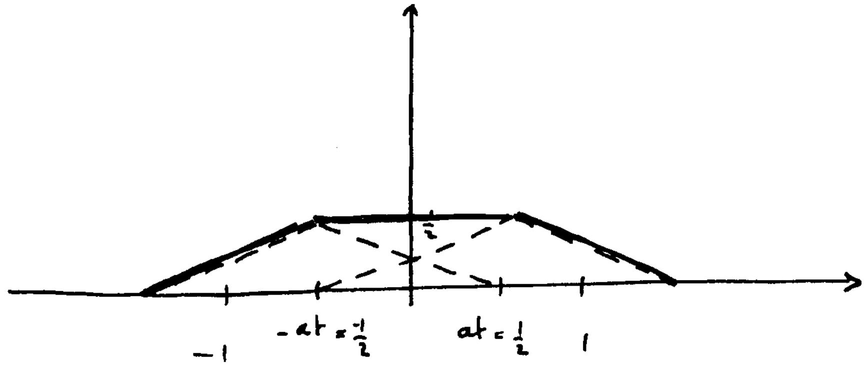
We deduce that $u(x,t)$ looks like:

for $t=0$

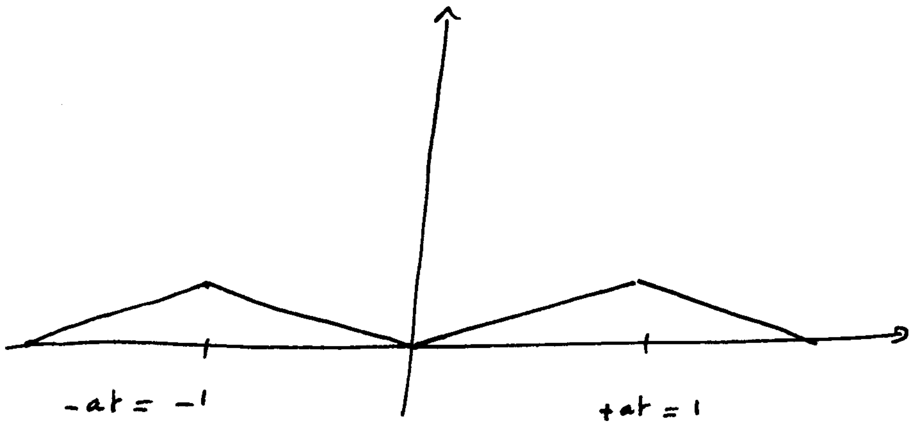


for $t = \frac{1}{2a}$

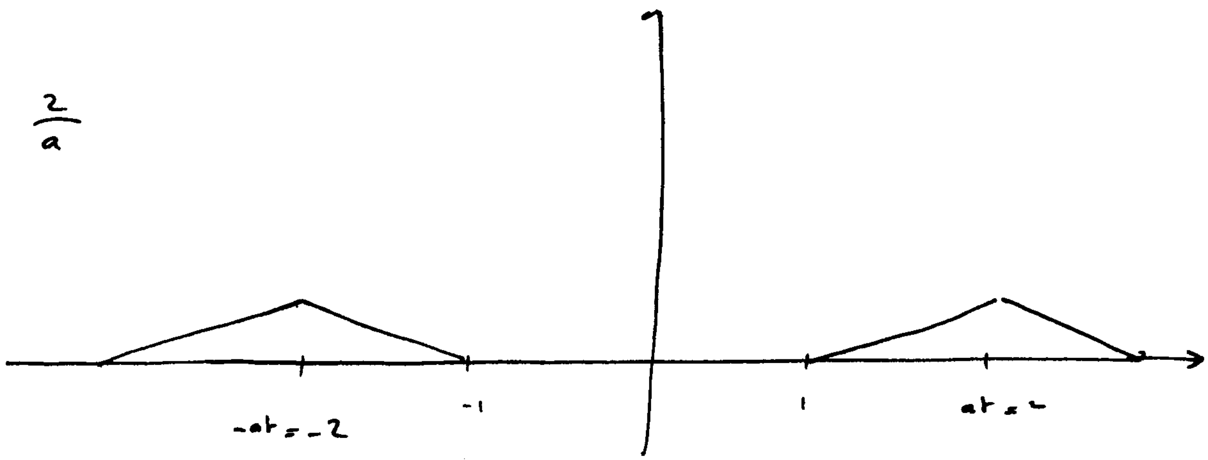
$(at = \frac{1}{2})$



for $t = \frac{1}{a}$



for $t = \frac{2}{a}$



Problem 2 the shape of the string is given by

$$u(x,t) = \frac{1}{2a} \int_{x-at}^{x+at} g(s)\, ds$$

with

$$g(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

So $u(x,t) = 0$ if $x + at \leq -1$ $\quad (x \leq -1 - at)$

and $u(x,t) = 0$ if $x - at \geq 1$ $\quad (x \geq 1 + at)$

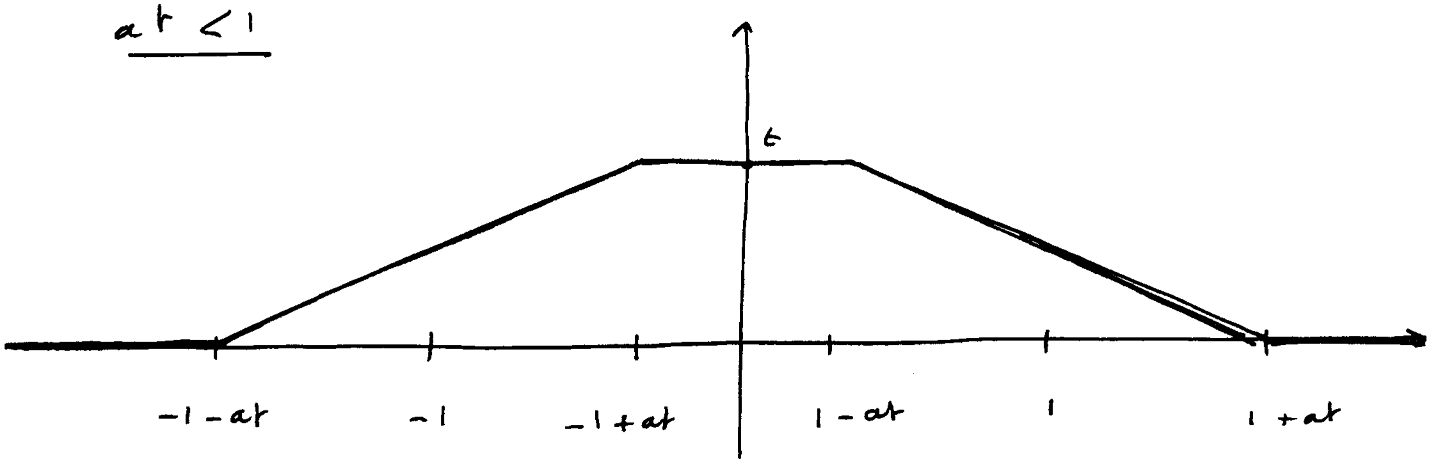
If $at < 1$ we get

$$u(x,t) = \begin{cases} \dfrac{x + at + 1}{2a} & \text{if } -1 - at \leq x \leq -1 + at \\ t & \text{if } -1 + at \leq x \leq 1 - at \\ \dfrac{1 - x + at}{2a} & \text{if } 1 - at \leq x \leq 1 + at \end{cases}$$

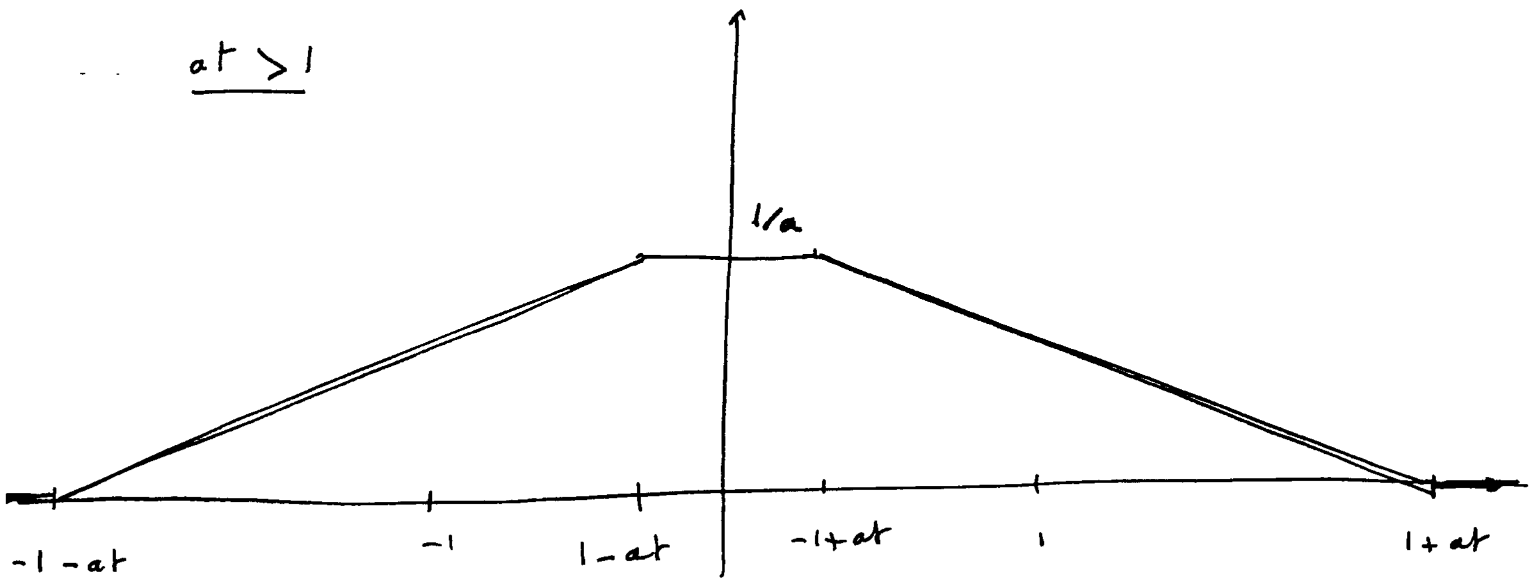
If $at > 1$ we get

$$u(x,t) = \begin{cases} \dfrac{x + at + 1}{2a} & \text{if } -1 - at \leq x \leq 1 - at \\ \dfrac{1}{a} & \text{if } 1 - at \leq x \leq -1 + at \\ \dfrac{1 - x + at}{2a} & \text{if } -1 + at \leq x \leq 1 + at \end{cases}$$

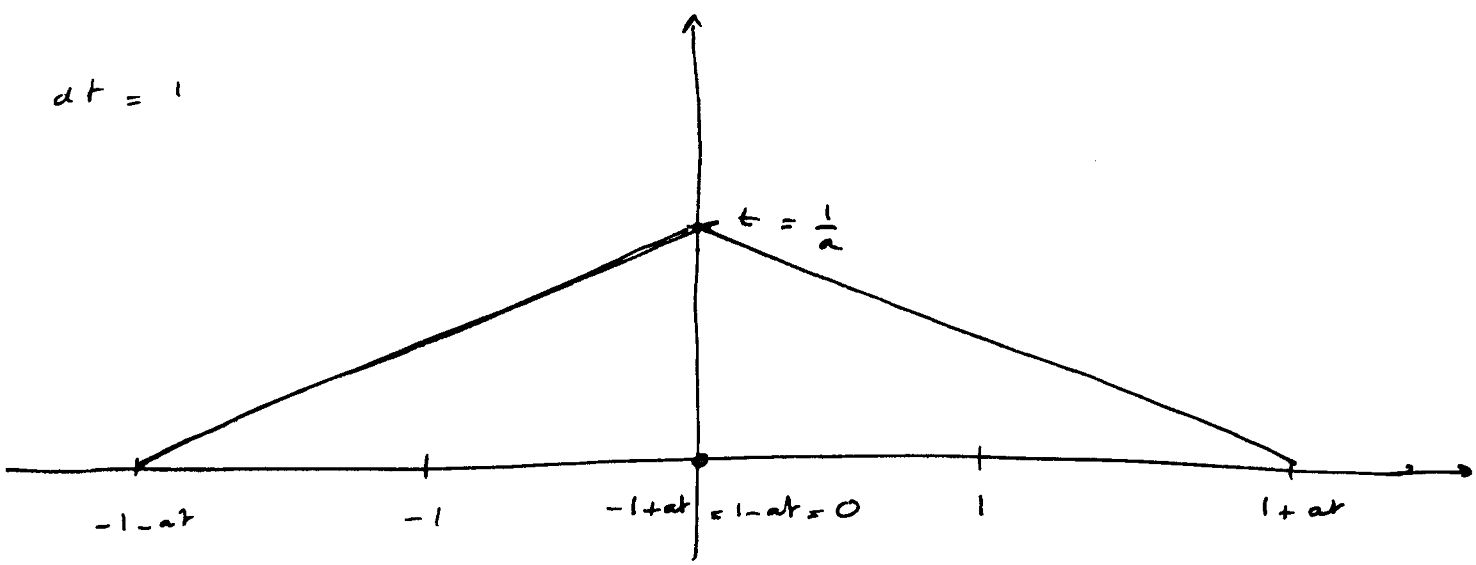
$\underline{at < 1}$



$\underline{at > 1}$



$at = 1$



Problem 3

a) G.S.:

$$u(x,t) = \sum_{n=1}^{\infty} \left[c_n \cos(n\pi a t) + d_n \sin(n\pi a t) \right] \sin(n\pi x)$$

$u(x,0) = 0$ so $c_n = 0$ for all n

$$u_t(x,t) = \sum_{n=1}^{\infty} n\pi a \, d_n \cos(n\pi a t) \sin(n\pi x)$$

So $u_t(x,0) = \sum_{n=1}^{\infty} n\pi a \, d_n \sin(n\pi x) = g(x)$

We deduce that $n\pi a \, d_n$ is the Fourier Coeff.

in the Fourier sine series of g:

$$n\pi a \, d_n = \frac{2}{1} \int_0^1 g(x) \sin(n\pi x)\, dx$$

$$= 2 \int_{1/2}^{1} \sin(n\pi x)\, dx$$

$$= \frac{2}{n\pi} \left[\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) \right]$$

So $d_n = \frac{2}{a(n\pi)^2} \left[\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) \right]$

and the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{a(n\pi)^2} \left[\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi) \right] \sin(n\pi a t) \sin(n\pi x)$$

b)

Solution: The general solution for the wave equation is

$$u(x,t) = \sum_{n=1}^{\infty} \big(c_n \cos(n\pi t) + d_n \sin(n\pi t)\big) \sin(n\pi x)$$

In particular, we have

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)$$

and

$$u_t(x,0) = \sum_{n=1}^{\infty} n\pi d_n \sin(n\pi x).$$

So we must take $c_1 = 2$, $c_3 = 1$, $3\pi d_3 = 1$ and all the other c_n and d_n equal to zero. We deduce

$$u(x,t) = 2\cos(\pi t)\sin(\pi x) + \cos(3\pi t)\sin(3\pi x) + \frac{1}{3\pi}\sin(3\pi t)\sin(3\pi x)$$

Problem 4 $u(x,t)$ satisfies $\begin{cases} u_{tt} = a^2 u_{xx} & 0 < x < L \quad t > \\ u(0,t) = 0 \quad u_x(L,t) = 0 & t \\ u(x,0) = f(x) \quad u_t(x,0) = 0 \end{cases}$

Since the B.C. are different from those considered in class, we need to go back to the Separation of variable methods:

we write $u(x,t) = X(x)T(t)$, then X and T must solve

$$X'' + \lambda X = 0 \quad , \quad X(0) = 0 \quad , \quad X'(L) = 0$$

$$T'' + a^2\lambda T = 0 \qquad T'(0) = 0$$

the Eigenvalue pb for X has a non trivial solution if

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2 \quad \text{and then} \quad X_n(x) = c \sin\left(\frac{(2n-1)\pi}{2L}x\right)^2$$

For a given λ_n, T must solve $T'' + a^2\lambda_n T = 0$, and so

$$T(t) = c_1 \cos\left(\frac{a(2n-1)\pi}{2L}t\right) + c_2 \sin\left(\frac{a(2n-1)\pi}{2L}t\right)$$

the Condition $T'(0) = 0$ implies $c_2 = 0$

So the fundamental solutions are

$$u_n(x,t) = \sin\left(\frac{(2n-1)\pi}{2L}x\right)\cos\left(\frac{a(2n-1)\pi}{2L}t\right)$$

As in class, we deduce that the displacement $u(x,t)$ is of the form $u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{(2n-1)\pi}{2L}x\right)\cos\left(\frac{a(2n-1)\pi}{2L}t\right)$

(c) the c_n must be such that

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{(2n-1)\pi}{2L}x\right) = f(x) = 2\sin\left(\frac{9\pi}{2L}x\right)$$

So $c_5 = 2$ and $c_n = 0$ for all $n \neq 5$:

$$u(x,t) = 2\sin\left(\frac{9\pi}{2L}x\right)\cos\left(a\frac{9\pi}{2L}t\right)$$

Question 5:

1. $$u_{tt} = c^2 u_{xx} + e^{-t}\sin 5x \qquad 0<x<\pi/2 \qquad (1)$$

BC: $u(0,t)=0 \qquad u_x(\frac{\pi}{2},t)=t$

IC: $u(x,0)=0 \qquad u_t(x,0)=x+\sin 3x$

LET US CONSTRUCT A FUNCTION $W(x,t)$ THAT SATISFIES THE INHOMOGENEOUS BC. ASSUME $W(x,t)=\alpha(t)+\beta(t)x \quad W_x=\beta(t)$

NOW $0=W(0,t)=\alpha(t) \quad \therefore t=W_x(\pi/2,t)=\beta(t) \Rightarrow W(x,t)=xt$

NOW LET $u(x,t)=W(x,t)+V(x,t)$ THEN SUBSTITUTING INTO (1) WE OBTAIN

$$u_{tt}=\cancel{W_{tt}}^{0}+V_{tt}=c^2(\cancel{W_{xx}}+V_{xx})+e^{-t}\sin 5x \Rightarrow V_{tt}=c^2V_{xx}+e^{-t}\sin 5x$$

BC: $0=u(0,t)=W(0,t)+V(0,t)=0+V(0,t) \Rightarrow V(0,t)=0$

$\cancel{t}=u_x(\pi/2,t)=W_x(\pi/2,t)+V_x(\pi/2,t)=\cancel{t}+V_x(\pi/2,t) \Rightarrow V_x(\pi/2,t)=0$ (2)

IC: $0=u(x,0)=W(x,0)+V(x,0)=0+V(x,0) \Rightarrow V(x,0)=0$

$\cancel{x}+\sin 3x=u_t(x,0)=W_t(x,0)+V_t(x,0)=\cancel{x}+V_t(x,0) \Rightarrow V_t(x,0)=\sin 3x.$

SINCE PROBLEM (2) FOR $V(x,t)$ HAS HOMOGENEOUS BC WE CAN DEFINE EIGENFUNCTIONS AND EIGENVALUES BY SEPARATING VARIABLES (EXCLUDING THE FORCING TERM $e^{-t}\sin 5x$)

THE APPROPRIATE EIGENVALUE PROBLEM IS: $X''+\lambda^2 X=0 \quad X(0)=0=X'(\pi/2)$

$\therefore X(x)=A\cos\lambda x+B\sin\lambda x \qquad X'=-A\lambda\sin\lambda x+B\lambda\cos\lambda x$

$X(0)=A=0 \qquad X'(\pi/2)=B\lambda\cos(\lambda\pi/2)=0 \Rightarrow \lambda_n\pi/2=(2n+1)\frac{\pi}{2} \Rightarrow \boxed{\lambda_n=(2n+1),\ n=0,1,\ldots \quad X_n=\sin\lambda_n x.}$

NOW ASSUME AN EIGENFUNCTION EXPANSION FOR $V(x,t)$

$$V(x,t)=\sum_{n=0}^{\infty}\hat{V}_n(t)\sin\lambda_n x \qquad V_{tt}=\sum_{n=0}^{\infty}\frac{d^2\hat{V}_n}{dt^2}\sin\lambda_n x \qquad V_{xx}=\sum_{n=0}^{\infty}(-\lambda_n^2)\hat{V}_n\sin\lambda_n x.$$

$$\therefore V_{tt}-c^2V_{xx}-e^{-t}\sin 5x=\sum_{n=0}^{\infty}\left\{\frac{d^2\hat{V}_n}{dt^2}+c^2\lambda_n^2\hat{V}_n-e^{-t}\delta_{n2}\right\}\sin\lambda_n x=0.$$

$$\frac{d^2\hat{V}_n}{dt^2}+c^2\lambda_n^2\hat{V}_n=e^{-t}\delta_{n2}$$

TO OBTAIN A PARTICULAR SOLUTION ASSUME $\hat{V}_n=Ae^{-t} \Rightarrow Ae^{-t}+c^2\lambda_n^2Ae^{-t}=e^{-t}\delta_{n2}$.

$\therefore A=\left(\frac{\delta_{n2}}{1+\lambda_n^2c^2}\right)$. SO THE GEN. SOL'N IS: $\hat{V}_n=\left(\frac{\delta_{n2}}{1+\lambda_n^2c^2}\right)+A_n\cos\lambda_n ct+B_n\sin\lambda_n ct)$

$$\therefore V(x,t)=\sum_{n=0}^{\infty}\left\{\left(\frac{\delta_{n2}e^{-t}}{1+\lambda_n^2c^2}\right)+A_n\cos\lambda_n ct+B_n\sin\lambda_n ct\right\}\sin\lambda_n x$$

$$0=V(x,0)=\sum_{n=0}^{\infty}\left[\left(\frac{\delta_{n2}}{1+\lambda_n^2c^2}\right)+A_n\right]\sin\lambda_n x \Rightarrow A_2=\frac{-1}{1+\lambda_2^2c^2} \qquad A_n=0 \quad n\neq 2$$

$$\sin(3x)=V_t(x,0)=\sum_{n=0}^{\infty}\left\{\left(-\frac{\delta_{n2}}{1+\lambda_n^2c^2}\right)+B_n\lambda_n c\right\}\sin\lambda_n x \Rightarrow B_1=\frac{1}{\lambda_1 c},\ B_2=\frac{1}{c\lambda_2(1+\lambda_2^2c^2)}$$

$B_n=0 \quad n\neq 1,2$

$$\therefore u(x,t)=xt+\frac{1}{1+\lambda_2^2c^2}\left[e^{-t}-\cos(\lambda_2ct)+\frac{\sin(\lambda_2ct)}{\lambda_2c}\right]\sin\lambda_2x+\frac{1}{\lambda_1c}\sin\lambda_1ct\sin\lambda_1x.$$