Math 257/316 PDE   Assignment 4

[Separation of variables]

1. Determine whether the method of separation of variables can be used to replace the following PDE's by a pair of ODE's. If so, find the equations.

   (a) $xu_{xx} + tu_t = 0$.   (b) $u_{xx} + u_{yy} = x$.   (c) $u_x + u_{xt} + u_t = 0$.

   (a) Suppose $u = X(x)T(t)$. The equation gives $xX''T + tXT' = 0$ which can be rewritten as $\frac{xX''}{X} = -\frac{tT''}{T}$. This separates into $\frac{xX''}{X} = k$ and $k = -\frac{tT''}{T}$.

   (b) Suppose $u = X(x)T(t)$. The equation gives $X''T + XT'' = x$. Division by XT fails to separate the variables and in fact the variables cannot be separated.

   (c) Suppose $u = X(x)T(t)$. The equation gives $X'T + X'T' + T'X = 0$. Division by XT gives $\frac{X'}{X} + \frac{X'}{X}\frac{T'}{T} + \frac{T'}{T} = 0$ which is the same as $\frac{X'}{X}\big(1 + \frac{T'}{T}\big) + \frac{T'}{T} = 0$ which we write as $\frac{X'}{X} = -\big(1 + \frac{T'}{T}\big)^{-1}\frac{T'}{T}$ and this separates to $\frac{X'}{X} = k$ and $-k\big(1 + \frac{T'}{T}\big) = \frac{T'}{T}$ which we can write as $-k\big(T + T'\big) = T'$.

2. Find all eigenvalues and corresponding eigenfunctions for the following problem

$$-y'' = \lambda y \quad (0 < x < 1), \quad y(0) = 0, \quad y'(1) = 0.$$

Case 1, $\lambda < 0$. We may write $\lambda = -\mu^2$ with $\mu > 0$ and hence $y'' = \mu^2 y$. The general solution is $y(x) = Ae^{\mu x} + Be^{-\mu x}$ and $y' = \mu Ae^{\mu x} - \mu Be^{-\mu x}$. The BC $0 = y(0)$ gives $0 = A + B$, thus $B = -A$. The BC $0 = y'(1)$ gives $0 = \mu(Ae^{\mu} - Be^{-\mu}) = \mu A(e^{\mu} + e^{-\mu})$. Thus $A = 0$ and there is no nonzero solution.

Case 2, $\lambda = 0$. The general solution is $y(x) = A + Bx$. The BC $0 = y(0)$ gives $A = 0$. The BC $0 = y'(1)$ gives $B = 0$. There is no nonzero solution.

Case 3, $\lambda > 0$. We may write $\lambda = \mu^2$ with $\mu > 0$ and hence $y'' = -\mu^2 y$. The general solution is $y(x) = A\cos\mu x + B\sin\mu x$ and $y' = -\mu A\sin\mu x + \mu B\cos\mu x$. The BC $0 = y(0)$ gives $0 = A + 0$. Thus $A = 0$. The BC $0 = y'(1)$ gives $0 = \mu B\cos\mu$, thus $\mu = \frac{(2n-1)\pi}{2}$, $n = 1, 2, 3, \ldots$. We have eigenfunctions

$$\phi_n(x) = \sin\frac{(2n-1)\pi x}{2}, \quad (n = 1, 2, 3, \ldots)$$

with corresponding eigenvalues $\lambda_n = \mu^2 = \dfrac{[(2n-1)\pi]^2}{4}$.

3. For each (real) constant k find all the non-zero solutions of the following boundary value problem

$$X'' = kX, \text{ for } x \in (0,1), \qquad X(0) = -X(1).$$

Case $k > 0$: Let $\mu = \sqrt{k}$. The general solution is any linear combination of $e^{\mu x}$ and $e^{-\mu x}$ with $\mu = \sqrt{k} > 0$. In particular, $\cosh \mu x$ and $\sinh \mu x$ are solutions and when one end of the interval is at $x = 0$ it can be more convenient to write the general solution as $a \cosh \mu x + b \sinh \mu x$. The boundary condition implies $a = -a \cosh \mu - b \sinh \mu$. Since $\mu > 0$, $\sinh \mu > 0$, so that we can solve for b in terms of a. After expressing b in terms of a we find that all the solutions are

$$X = a\big(\cosh \mu x + c \sinh \mu x\big), \text{ with } c = -\frac{1 + \cosh \mu}{\sinh \mu} \text{ and } \mu = \sqrt{k}$$

By the way, this can be simplified to $X \propto \sinh(\mu x) + \sinh(\mu(x-1))$.

Case $k = 0$: The general solution is $ax+b$. The boundary condition requires $b = -a-b$, so $b = -\frac{a}{2}$ and therefore the solution is any constant times $x - \frac{1}{2}$.

Case $k < 0$. Let $\mu = \sqrt{-k}$. The general solution is $a \cos \mu x + b \sin \mu x$. The boundary condition requires $a = -a \cos \mu - b \sin \mu$.

If $\sin \mu \neq 0$ this can be solved for b in terms of a and we find that all solutions are

$$a\big(\cos \mu x + c \sin \mu x\big), \text{ with } c = -\frac{1 + \cos \mu}{\sin \mu} \text{ and } \mu = \sqrt{-k}$$

which can be simplified to (any multiple of) $\sin(\mu(x-1)) + \sin(\mu x)$.

If $\sin \mu = 0$, then $\mu = n\pi$ with $n = 1, 2, 3, \ldots$. Then $a = -a \cos \mu - b \sin \mu$ simplifies to $a(1 + \cos \mu) = 0$. There are two ways to satisfy this: case (i) $1 + \cos \mu = 0$ and case (ii) $a = 0$.

Case (i): $n = 1, 3, 5, \ldots$ because the two equations $1 + \cos \mu = 0$ and $\sin \mu = 0$ imply $\mu = n\pi$ with $n = 1, 3, 5, \ldots$. The solution is $a \cos n\pi x + b \sin n\pi x$ with a, b arbitrary.

Case (ii) $n = 2, 4, 6, \ldots$. The solution is $b \sin \mu x$

Q4(a) $\quad u_t = \alpha^2 u_{xx} \qquad 0 < x < L$

BC: $u_x(0,t) = 0 = u(L,t)$

IC: $u(x,0) = f(x)$

SEPARATE VARIABLES: $u(x,t) = X(x)T(t) \Rightarrow X\dot{T} = \alpha^2 X''T$

$\boxed{\div \alpha^2 XT}$ $\quad \dfrac{\dot{T}(t)}{\alpha^2 T} = \dfrac{X''}{X} = -\lambda^2 = \text{CONST}$

$\boxed{\text{TIME EQ}}$ $\quad \dot{T}(t) = -\alpha^2\lambda^2 T \Rightarrow T(t) = De^{-\lambda^2\alpha^2 t}$

$\boxed{\text{SPACE EQ}}$ $\quad \underline{\lambda \neq 0}$: $\left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X'(0) = 0 = X(L). \end{array}\right\}$ EIGENVALUE PROBLEM

$X(x) = A\cos\lambda x + B\sin\lambda x$

$X'(x) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$

$X'(0) = B\lambda = 0 \Rightarrow B = 0$

$X(L) = A\cos(\lambda L) = 0 \qquad \lambda_k L = (2k+1)\pi/2 \quad k = 0, 1, \ldots$

$\underline{\lambda = 0}$: $X'' = 0 \Rightarrow X = A + Bx$, $X' = B$, $X'(0) = B = 0$, $X(L) = A = 0 \Rightarrow X \equiv 0$ TRIVIAL SOLN

EIGENVALUES ARE: $\lambda_k = \dfrac{(2k+1)\pi}{2L} \qquad k = 0, 1, \ldots$

EIGENFUNCTIONS ARE:

$$X_k(x) = \cos\left[\frac{(2k+1)\pi x}{2L}\right]$$

GENERAL SOLUTION: $\quad u(x,t) = \sum_{k=0}^{\infty} A_k e^{-\lambda_k^2\alpha^2 t}\cos(\lambda_k x)$

IC: $\quad f(x) = u(x,0) = \sum_{k=0}^{\infty} A_k \cos(\lambda_k x)$.

TO OBTAIN THE A_k WE PROJECT $f(x)$ ONTO $\cos(\lambda_k x)$

$$\langle f, \cos\lambda_j x\rangle = \int_0^L f(x)\cos\lambda_j x\,dx$$

$$= \sum_{k=0}^{\infty} A_k \int_0^L \cos\lambda_j x\cos\lambda_k x\,dx$$

NOW $\quad \displaystyle\int_0^L \cos\lambda_j x\cos\lambda_k x\,dx = \begin{cases} 0 & k \neq j \\ L/2 & k = j \end{cases}$

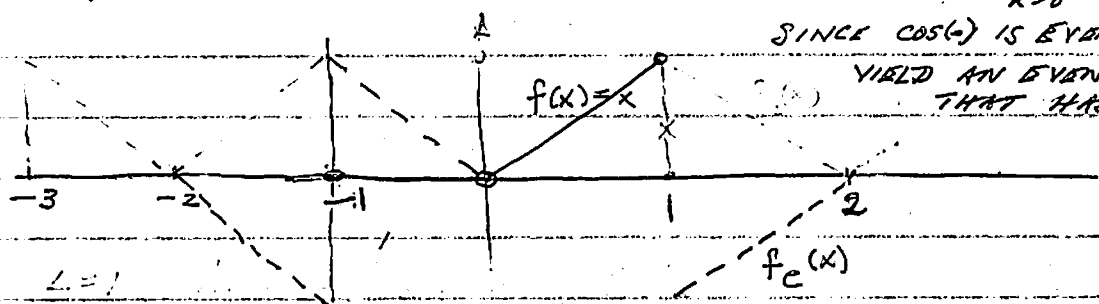
$$\therefore \quad A_k = \frac{2}{L}\int_0^L f(x)\cos\lambda_k x\,dx.$$

$$u(x,t) = \sum_{k=0}^{\infty} A_k e^{-\lambda_k^2\alpha^2 t}\cos(\lambda_k x)$$

Q4(b) $u(x,t) = \sum_{k=0}^{\infty} A_k e^{-\lambda_k^2 \alpha^2 t} \cos(\lambda_k x) \qquad \lambda_k = \frac{(2k+1)\pi}{2L}$

$f(x) = x \quad 0 < x < L = 1 \qquad u(x,0) = f(x) = \sum_{k=0}^{\infty} A_k \cos\left(\frac{(2k+1)\pi x}{2L}\right) = S(x)$

SINCE $\cos(\cdot)$ IS EVEN WE EXPECT $S(x)$ TO YIELD AN EVEN EXTENSION OF $f(x)$ THAT HAS A PERIOD $2L = 2$.



$L=1 \quad A_k = \frac{2}{1}\int_0^1 x\cos\left(\frac{(2k+1)\pi x}{2}\right)dx$

$= 2\left\{ \frac{x}{\lambda_k}\sin(\lambda_k x)\Big|_0^1 - \frac{1}{\lambda_k}\int_0^1 1.\sin(\lambda_k x)\,dx \right\}$

$= 2\left\{ \frac{1}{\lambda_k}\sin(\lambda_k 1) + \frac{1}{\lambda_k^2}\cos(\lambda_k x)\Big|_0^1 \right\}$

$= 2\left\{ \frac{2\times 1}{(2k+1)\pi}\sin\left(\frac{(2k+1)\pi}{2}\right) + \frac{1}{\lambda_k^2}\left[\cos\left(\frac{(2k+1)\pi}{2\times 1}\right) - 1\right] \right\}$

$= \frac{4}{\pi}\frac{(-1)^k}{2k+1} - \frac{8}{\pi^2(2k+1)^2}$

$= \frac{4}{\pi^2}\left[\frac{(-1)^k\pi(2k+1) - 2}{(2k+1)^2}\right]$

k	0	1	2	3	
2k+1	1	3	5	7	
$\sin\lambda_k$	1	−1	1	−1	$\to (-1)^k$
$\cos\lambda_k$	0	0	0	0	

$f(x) = \frac{4}{\pi^2}\sum_{k=0}^{\infty}\left\{\frac{\pi(-1)^k}{(2k+1)} - \frac{2}{(2k+1)^2}\right\}\cos\left[\frac{(2k+1)\pi x}{2}\right]$

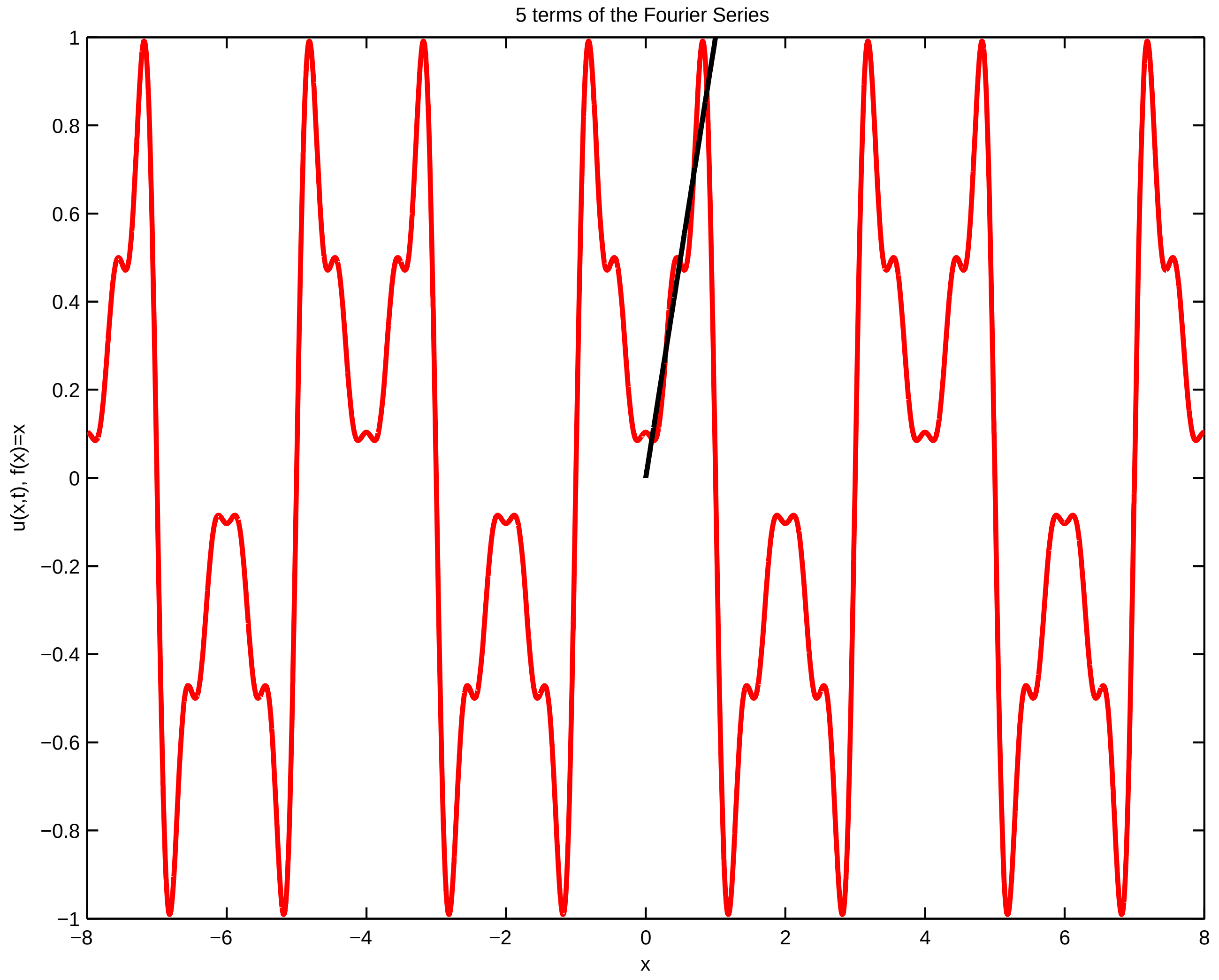
- NOTE THAT THE LARGEST PERIOD COMMON TO ALL THE EIGENFUNCTIONS IS $T = 4$ THIS FOLLOWS FROM THE STANDARD PERIOD CALCULATION

$$\cos\left(\frac{(2k+1)\pi(x+T)}{2}\right) = \cos\left(\frac{(2k+1)\pi x}{2}\right)$$

$$\Rightarrow \frac{(2k+1)\pi T}{2} = 2\pi \Rightarrow T_k = \frac{4}{(2k+1)}$$

THE LARGEST COMMON PERIOD IS ACHIEVED BY $T_0 = \frac{4}{1} = 4$

- THE NON-STANDARD EXTENSION $f_e(x)$ FOLLOWS FROM THE FACT THAT $\cos\left(\frac{(2k+1)\pi x_0}{2}\right) = 0$ AT $x_0 = \pm 1$ (THIS IS BECAUSE OF THE BOUNDARY CONDITION THAT WAS IMPOSED.



5 terms of the Fourier Series
u(x,t), f(x)=x
x
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−8
−6
−4
−2
0
2
4
6
8

5. Which of the following functions are periodic? For those that are, find the fundamental period.

$$\text{(a)}\ \sin x \cos x \quad \text{(b)}\ \sec x + \tan \sqrt{2}x \quad \text{(c)}\ \sin(x^2)$$

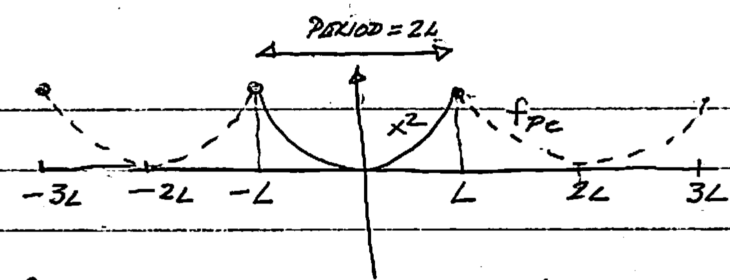
Solution:

(a) $\sin x \cos x = \dfrac{1}{2} \sin 2x$ is periodic with fundamental period π.

(b) The fundamental period of $\sec x$ is 2π and that of $\tan \sqrt{2}x$ is $\pi/\sqrt{2}$. If $\sec x + \tan \sqrt{2}x$ were periodic with fundamental period p then there would exist positive integers m, n such that $2m\pi = n\pi/\sqrt{2}$. But this is not possible as $\sqrt{2}$ is irrational.

(c) $\sin(x^2)$ is not periodic, for if it were the values of x where $\sin(x^2) = 0$ would be periodic, but these values are $x = \pm\sqrt{n\pi}, n = 0, 1, 2, \ldots$ (not a periodic sequence).

Q6: $f(x) = x^2$



$$a_0 = \frac{1}{L}\int_{-L}^{L} x^2\,dx \overset{f\text{ EVEN}}{=} \frac{2}{L}\int_0^L x^2\,dx = \frac{2}{L}\left.\frac{x^3}{3}\right|_0^L = \frac{2L^2}{3}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \overbrace{x^2\cos\left(\frac{n\pi x}{L}\right)}^{\text{EVEN}}dx \overset{f\text{ EVEN}}{=} \frac{2}{L}\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\left[x^2\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right)\right]_0^{L} \xrightarrow{} 0 - \frac{2L}{n\pi}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= -\frac{4}{n\pi}\left\{-x\left(\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \left(\frac{L}{n\pi}\right)\int_0^L 1.\cos\left(\frac{n\pi x}{L}\right)dx\right\}$$

$$= -\frac{4}{n\pi}\left\{-\frac{L^2}{(n\pi)}\cos(n\pi) + \left(\frac{L}{n\pi}\right)^2\sin\left(\frac{n\pi x}{L}\right)\Big|_0^{L} \xrightarrow{} 0\right\}$$

$$= +\frac{4L^2}{(n\pi)^2}(-1)^n$$

$$b_n = \frac{1}{L}\int_{-L}^{L}\overbrace{x^2\sin\left(\frac{n\pi x}{L}\right)}^{\text{ODD}}dx = 0$$

$$\therefore\quad f(x) = x^2 \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + \cancel{b_n}\sin\left(\frac{n\pi x}{L}\right) = S(x)$$

$$= \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{L}\right)$$

AT $x = L$: $$f(x) = L^2 = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n^2}$$

$$\therefore\quad \frac{2}{3}\left(\frac{\pi^2}{4}\right) = \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \cdots$$

7. (a) Show that $\cos x = \dfrac{8}{\pi}\sum_{n=1}^{\infty}\dfrac{n}{(2n)^2-1}\sin(2nx),\ 0<x<\pi.$

(b) The series in (a) converges for all x. What function does it converge to?

(c) Show that $\displaystyle\sum_{n=1,3,5,\ldots}\frac{n}{4n^2-1}(-1)^{(n-1)/2}=\frac{\pi}{8\sqrt{2}}.$

Solution:

(a) This is the Fourier sine series $F(x)$ for the function $f(x)=\cos x, 0<x<\pi:$

$$F(x)=\sum_{n=1}^{\infty}b_n\sin nx,\ \text{ where } b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx dx$$

Therefore

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi}\cos x\sin nx dx=\frac{2}{\pi}\int_0^{\pi}\frac{\sin(nx+x)+\sin(nx-x)}{2}dx\\
&= -\frac{1}{\pi}\frac{\cos(n+1)x}{n+1}\Big|_0^{\pi}-\frac{1}{\pi}\frac{\cos(n-1)x}{n-1}\Big|_0^{\pi}\\
&= -\frac{1}{\pi(n+1)}((-1)^{n+1}-1)-\frac{1}{\pi(n-1)}((-1)^{n-1}-1)\\
&= \begin{cases} 0 & \text{if } n \text{ is odd}\\ \dfrac{4n}{\pi(n^2-1)} & \text{if } n \text{ is even}\end{cases}
\end{aligned}$$

So we have $F(x)=\displaystyle\sum_{n=1}^{\infty}\frac{8n}{\pi(4n^2-1)}\sin 2nx=\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin 2nx.$

(b) The Fourier series $F(x)$ converges to the 2π periodic function defined by

$$g(x)=\begin{cases}\cos x & 0<x<\pi\\ 0 & x=0,\pm\pi\\ -\cos x & -\pi<x<0\end{cases}$$

(c) To derive this identity we put $x=\pi/4$ in the Fourier series in (a):

$$\frac{1}{\sqrt{2}}=\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin(n\pi/2)=\frac{8}{\pi}\sum_{n=1,3,5,\ldots}(-1)^{(n-1)/2}\frac{n}{4n^2-1},$$

which is equivalent to the identity in (c).