Math 257/316 Assignment 5

Due Friday October 30th in class

Problem 1: Sketch the odd, even, and full periodic extensions on $[3L, 3L]$ of

(a) e^x, with $L = 1$

(b) $4 - x^2$, with $L = 2$

(c) $g(x) = \begin{cases} 1 + x, & x < 0 \\ x/2, & x \geq 0 \end{cases}$, with $L = 1$

Problem 2: Chemical diffusion through a thin layer is governed by the equation

$$\frac{\partial C}{\partial t} = k\frac{\partial^2 C}{\partial x^2} - LC$$

where $C(x,t)$ is the concentration in moles/cm^3, the diffusivity k is a positive constant with units cm^2/sec, and $L > 0$ is a consumption rate with units sec^{-1}. Assume boundary conditions are

$$C(0,t) = C(a,t) = 0, \ t > 0,$$

and the initial concentration is given by

$$C(x,0) = f(x), \ 0 < x < a.$$

(a) Use the method of separation of variables to solve for the concentration $C(x,t)$.

(b) What happens to the concentration as $t \to \infty$?

(c) What is the concentration $C(x,t)$ if the initial condition is $C(x,0) = \cos(\pi x/a)$?

Hint: It may be useful to know that

$$\int_0^a \sin(n\pi x/a)\cos(\pi x/a)\,dx = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{2an}{\pi(n^2-1)}, & \text{if } n \text{ is even} \end{cases}$$

Problem 3: Find the Fourier Sine series of period 2π of the following function. Sketch the graph of the function to which the series converges (sketch at least three periods).

$$f(x) = \begin{cases} 1, & 0 \leq x \leq \pi/2 \\ 0, & \pi/2 < x \leq \pi \end{cases}$$

Math 257-316, Ass. 5 Due (IN CLASS) Wed 10 th Oct.

- The first midterm exam is on Mon., Oct. 22 th - practice exams on web site.

1. [**Orthogonality**] Show that the following functions are orthogonal on the prescribed intervals:

   (a) $f_1(x) = x$ and $f_2(x) = x^2$ on $[-2, 2]$.

   (b) The Legendre Polynomials $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$ on $[-1, 1]$.

   Use this orthogonality to expand $f(x) = \begin{cases} -1 \text{ for } -1 < x < 0 \\ 1 \text{ for } 0 < x < 1 \end{cases}$ as follows $f(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x)$.

2. [**Periodic functions**] Which of the following functions are periodic? For those that are, find the fundamental period: (a) $3 \sin x - 4 \sin^3 x$ (b) $\cot x + \sec 3^{1/2} x$ (c) $\cos(x^3)$.

3. [**Fourier series**] Consider the function $f(x) = \begin{cases} x^2 \text{ for } 0 < x < \pi \\ 0 \text{ for } \pi < x < 2\pi \end{cases}$ defined on the interval $0 < x < 2\pi$.

   (a) Sketch 3 periods of the periodic extension of f of period 2π. Determine the Fourier series representation of f. This series converges for all x. Indicate on your sketch the function to which the series converges? Use this series to show that:

   (i) $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ and (ii) $\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$.

   (b) Sketch the even and odd periodic extensions of f that are of period 4π. Determine the Fourier series associated with the even periodic extension.