Math 257/316 Assignment 7

**Supplemental Examples: Not to be handed in - solutions are posted.**

**Problem 1: (A bar subject to a time dependent boundary condition also Eg 22.2 in Lecture 22).** Consider a bar of length $L$ and thermal diffusivity $\alpha^2$ having initial temperature distribution of 0.Suppose that the right end of the bar is maintained at a temperature of 0 while the temperature of the left end of the bar varies according to $At$. Thus

$$\begin{aligned} u_t &= \alpha^2 u_{xx}, \quad 0 < x < 1,\ t > 0 \\ \text{BC: } & u(0,t) = At,\ u(1,t) = 0 \\ \text{IC: } & u(x,0) = 0 \end{aligned} \tag{1}$$

a) Identify an appropriate particular solution to the heat equation that can be used to reduce the problem with inhomogeneous boundary conditions to one with homogeneous boundary conditions.

b) Use the method of eigenfunction expansions to determine $u(x,t)$.

c) Numerics: Solve the initial-boundary value problem (1) numerically using finite differences. Now code the corresponding Fourier Series solution obtained in part (b) of this question and evaluate $u(x,t)$ at the time horizon $t = 0.05$ and using, $L = 1$, $A = 2$, and $\alpha = 1$. Plot the graph comparing the two solutions at $t = 0.05$.Use $\Delta x = 0.05$ and $\Delta t = 0.0005$ for the finite difference solution.(the answer is given in the spreadsheet: `HeatFS_TDBC.xls`).

**Problem 2: (A bar subject to an external heat source).** Consider a bar of length 1 and thermal diffusivity $\alpha^2$ having initial temperature distribution given by $u(x,0) = \sin(\pi x)$. Suppose that both ends of the bar are maintained at a temperature of 0 and that a time independent heat source $\sin(3\pi x)$ is applied across the length of the bar. Thus

$$\begin{aligned} u_t &= \alpha^2 u_{xx} + \sin(3\pi x), \quad 0 < x < 1,\ t > 0 \\ \text{BC: } & u(0,t) = 0 = u(1,t) \\ \text{IC: } & u(x,0) = \sin(\pi x) \end{aligned}$$

a) Determine the steady-state temperature in the bar.

b) Find the temperature $u(x,t)$.

**Problem 3 (A heat conducting bar subject to a time varying boundary condition):**

Consider the following boundary value problem that has a time-dependent, Dirichlet boundary condition:

$$\begin{aligned} u_t &= u_{xx}, \qquad 0 < x < \pi,\ t > 0 \\ \text{BC} : u(0,t) &= 0, \qquad u(\pi,t) = t^2 \\ \text{IC} : u(x,0) &= 0 \end{aligned}$$

a) Solve the problem by the method of eigenfunction expansion.

b) Verify that

$$w(x,t) = \frac{1}{\pi}t^2 x + \frac{1}{3\pi}(x^2 - \pi^2)tx + \frac{1}{180\pi}(3x^4 - 10\pi^2 x^2 + 7\pi^4)x \; ,$$

satisfies the PDE (i.e. $w_t = w_{xx}$) and the boundary conditions.

c) Let $v(x,t) = u(x,t) - w(x,t)$ and solve by separation of variables. It may be useful to know that:

$$\int_0^{\pi} (3x^5 - 10\pi^2 x^3 + 7\pi^4 x)\sin(nx)\; dx = -\frac{360\pi(-1)^n}{n^5} \; .$$

d) Is your answer in (c) the same as your answer in (a)? Why or why not?