Math 257/316 Ass 8 Solutions

Q1

(a) $0 = c^2 u^s_{xx} - g \Rightarrow u^s_{xx} = \frac{g}{c^2}; \quad u^s_x = \frac{g}{c^2}x + A, \; u^s = \frac{g}{2c^2}x^2 + Ax + B$

$0 = u^s(0) = B; \quad u^s(1) = \frac{g}{2c^2} + A = 0 \quad A = -g/2c^2$

$\therefore \quad \boxed{u^s(x) = \frac{g}{2c^2}(x^2 - x)}$

(b) LET $u(x,t) = u^s(x) + v(x,t)$

$\therefore \quad u_{tt} = \cancel{u^s_{tt}} + v_{tt} = c^2(u^s_{xx} + v_{xx}) - g = c^2 v_{xx} + [\cancel{c^2 u^s_{xx} - g}] \Rightarrow v_{tt} = c^2 v_{xx}$

BC $\begin{cases} 0 = u^s(0) + v(0,t) = 0 + v(0,t) & \Rightarrow v(0,t) = 0 \\ 0 = u^s(1) + v(1,t) = 0 + v(1,t) & \Rightarrow v(1,t) = 0 \end{cases}$

IC: $\sin \pi x = u(x,0) = u^s(x) + v(x,0) \Rightarrow v(x,0) = \sin(\pi x) - \frac{g}{2c^2}(x^2 - x)$

$0 = u_t(x,0) = \cancel{u^s_t(x)} + v_t(x,0) \Rightarrow v_t(x,0) = 0$

LET $v(x,t) = X(x)T(t) \Rightarrow \frac{\ddot{T}}{c^2 T} = \frac{X''}{X} = -\lambda^2$

$X]$ $\left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X(1) \end{array}\right\}$ $\quad \lambda_n = \frac{n\pi}{1} \; n = 1,2.. \quad X_n = \sin(n\pi x)$

$T]$ $\ddot{T} + \lambda^2 c^2 T = 0 \qquad T = e^{rt} \Rightarrow T = A\cos\lambda ct + B\sin\lambda ct$

$\therefore \quad v(x,t) = \sum_{n=1}^{\infty} [A_n \cos\lambda_n ct + B_n \sin\lambda_n ct] \sin\lambda_n x$

$v_t(x,t) = \sum_{n=1}^{\infty} [-A_n\lambda_n c \sin\lambda_n ct + B_n\lambda_n c\cos\lambda_n ct] \sin\lambda_n x.$

$\sin(\pi x) - \frac{g}{2c^2}(x^2 - x) = v(x,0) = \sum_{n=1}^{\infty} A_n \sin\lambda_n x \Rightarrow A_n = \frac{2}{1}\int_0^1 \left\{\sin(\pi x) - \frac{g}{2c^2}(x^2 - x)\right\} \sin n\pi x \, dx$

$\therefore \quad A_n = \delta_{n1} - \frac{g}{c^2}\int_0^1 (x^2 - x)\sin(n\pi x)\,dx = \delta_{n1} - \frac{2g}{c^2}\left[\frac{\cos n\pi - 1}{n^3\pi^3}\right]$

$0 = v_t(x,0) = \sum_{n=1}^{\infty} (B_n \lambda_n c \cdot 1) \sin\lambda_n x \Rightarrow B_n = 0$

$\therefore \quad u(x,t) = \frac{g}{2c^2}(x^2 - x) + \frac{2g}{c^2}\sum_{n=1}^{\infty} \frac{[1 - \cos(n\pi)]}{n^3\pi^3} \cos(n\pi ct)\sin(n\pi x) + \cos(\pi ct)\sin(\pi x)$

Q2

[5 marks]
[total 20 marks]

a) LET $u(x,t) = X(x)T(t) \Rightarrow X\ddot{T} = X''T - \gamma XT$

$\div XT]$ $\quad \dfrac{\ddot{T}}{T} + \gamma = \dfrac{X''}{X} = -\lambda^2$

$T]$ $\quad \ddot{T} + (\gamma+\lambda^2)T = 0 \Rightarrow T(t) = A\cos\mu t + B\sin\mu t$ WHERE $\mu = \sqrt{\gamma+\lambda^2}$

$X]$ $\quad \left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X(1) \end{array}\right\} \Rightarrow X = A\cos\lambda x + B\sin\lambda x$

$X(0) = A = 0 \quad X(1) = B\sin\lambda = 0 \Rightarrow \lambda_n = n\pi \quad n = 1, 2, \ldots; \; X_n = \sin(n\pi x)$

$\therefore \; u(x,t) = \sum_{n=1}^{\infty} [A_n\cos\mu_n t + B_n \sin\mu_n t] \sin(\lambda_n x)$

$0 = u(x,0) = \sum_{n=1}^{\infty} A_n \sin(\lambda_n x) \Rightarrow A_n = 0 \;\forall\, n.$

$u_t(x,t) = \sum_{n=1}^{\infty} B_n\mu_n\cos\mu_n t \sin(\lambda_n x)$

$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} B_n\mu_n \sin\lambda_n x \Rightarrow \mu_n B_n = 2\int_0^1 g(x)\sin(n\pi x)\,dx.$

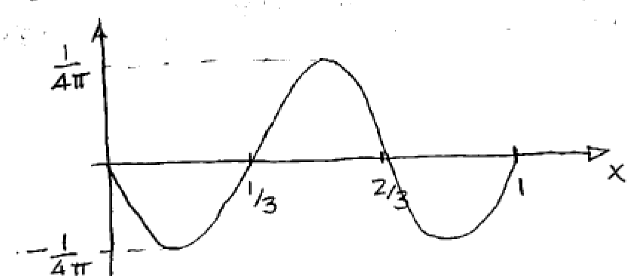
$\therefore \; u(x,t) = \sum_{n=1}^{\infty} B_n \sin\mu_n t \sin(\lambda_n x)$ WHERE $B_n = \dfrac{2}{\mu_n}\int_0^1 g(x)\sin(n\pi x)\,dx$

b) $\gamma = 7\pi^2 \quad g(x) = \sin 3\pi x \Rightarrow B_n = \dfrac{2}{\mu_n}\int_0^1 \sin(3\pi x)\sin(n\pi x)\,dx = \dfrac{2}{\mu_n}\delta_{n3}\dfrac{1}{2} = \dfrac{\delta_{n3}}{\mu_3}$

$\therefore \; u(x,t) = \dfrac{1}{\mu_3}\sin\mu_3 t \sin(3\pi x) \qquad \mu_3 = \sqrt{7\pi^2 + 9\pi^2} = 4\pi$

$= \dfrac{1}{4\pi}\sin(3\pi x)\sin(4\pi t)$

$u(x, \tfrac{3}{8}) = \dfrac{1}{4\pi}\sin(3\pi x)\sin\left(\dfrac{3\pi}{2}\right) = -\dfrac{1}{4\pi}\sin(3\pi x)$ HAS A WAVELENGTH OF $2/3$



Q3 PDE $u_{tt} = u_{xx} + e^{-t}\sin x + 1$ $\quad 0 < x < \pi/2$

BC: $u(0,t) = t^2/2 \qquad u_x(\frac{\pi}{2},t) = t$ $\quad (1)$

IC: $u(x,0) = \sin 3x \qquad u_t(x,0) = x$

SOL'N: LET $W(x,t) = A(t)x + B(t)$ $\quad W_x = A$, $t^2/2 = W(0,t) = B(t)$, $t = W_x(\frac{\pi}{2},t) = A(t) \Rightarrow \boxed{W = xt + t^2/2}$

LET $u(x,t) = W(x,t) + V(x,t)$ AND SUBSTITUTE INTO (1) TO FIND THE IBVP FOR V:

PDE: $u_{tt} = (xt + t^2/2)_{tt} + V_{tt} = 1 + V_{tt} = (xt + t^2/2)_{xx} + V_{xx} + e^{-t}\sin x + 1 = u_{xx} + e^{-t}\sin x + 1$

$\therefore \; V_{tt} = V_{xx} + e^{-t}\sin x$

BC: $\cancel{t^2/2} = u(0,t) = W(0,t) + V(0,t) = \cancel{t^2/2} + V(0,t) \Rightarrow V(0,t) = 0$

$t = u_x(\pi/2,t) = W_x(\pi/2,t) + V_x(\pi/2,t) = t + V_x(\pi/2,t) \Rightarrow V_x(\pi/2,t) = 0$

IC: $\sin 3x = u(x,0) = W(x,0) + V(x,0) = 0 + V(x,0) \Rightarrow V(x,0) = \sin 3x$

$\cancel{x} = u_t(x,0) = W_t(x,0) + V_t(x,0) = \cancel{x} + V_t(x,0) \Rightarrow V_t(x,0) = 0$

IBVP FOR V

THE EIGENVALUES & EIGENFCNS ARE: $\lambda_n = -\left[\frac{(2n-1)\pi}{2(\pi/2)}\right]^2 = -(2n-1)^2 = -\mu_n^2$, $X_n = \sin\mu_n x \quad n = 1, 2 \ldots$

LET $S(x,t) = e^{-t}\sin x = \sum_{n=1}^{\infty} S_n(t)\sin\mu_n x \Rightarrow S_n(t) = e^{-t}\delta_{n1}$

LET $V(x,t) = \sum_{n=1}^{\infty} V_n(t)\sin\mu_n x \qquad V_{tt} = \sum_{n=1}^{\infty} \ddot{V}_n(t)\sin\mu_n x \qquad V_{xx} = \sum_{n=1}^{\infty} V_n(t)\{-\mu_n^2\}\sin\mu_n x$

$0 = V_{tt} - V_{xx} - e^{-t}\sin x = \sum_{n=1}^{\infty}\{\ddot{V}_n + \mu_n^2 V_n - e^{-t}\delta_{n1}\}\sin\mu_n x$

$\ddot{V}_n + \mu_n^2 V_n = e^{-t}\delta_{n1}$

HOMOG EQ: $\ddot{V}_n^H + \mu_n^2 V_n^H = 0$, $V_n^H = e^{rt} \Rightarrow r^2 + \mu_n^2 = 0$, $r = \pm\mu_n i$, $V_n^H = A_n\cos\mu_n t + B_n\sin\mu_n t$

PART. SOLN: LET $V_n^P = De^{-t}$ $\quad \ddot{V}_n^P + \mu_n^2 V_n^P = De^{-t} + \mu_n^2 De^{-t} = e^{-t}\delta_{n1} \Rightarrow D = \frac{\delta_{n1}e^{-t}}{1+\mu_n^2}$

GENERAL SOLN: $V_n(t) = A_n\cos\mu_n t + B_n\sin\mu_n t + \frac{\delta_{n1}}{1+\mu_n^2}e^{-t}$

$\therefore \; V(x,t) = \sum_{n=1}^{\infty}\left\{A_n\cos\mu_n t + B_n\sin\mu_n t + \frac{\delta_{n1}}{1+\mu_n^2}e^{-t}\right\}\sin\mu_n x$

$V_t(x,t) = \sum_{n=1}^{\infty}\left\{-A_n\mu_n\sin\mu_n t + B_n\mu_n\cos\mu_n t - \frac{\delta_{n1}}{1+\mu_n^2}e^{-t}\right\}\sin\mu_n x$

$\sin 3x = V(x,0) = \sum_{n=1}^{\infty}\left[A_n + \frac{\delta_{n1}}{1+\mu_n^2}\right]\sin\mu_n x \Rightarrow A_n = \delta_{n2} - \frac{\delta_{n1}}{1+\mu_n^2}$

$0 = V_t(x,0) = \sum_{n=1}^{\infty}\left[B_n\mu_n - \frac{\delta_{n1}}{1+\mu_n^2}\right]\sin\mu_n x \Rightarrow B_n = \frac{\delta_{n1}}{1+\mu_n^2}$

$V(x,t) = \sum_{n=1}^{\infty}\left\{\left(\delta_{n2} - \frac{\delta_{n1}}{1+\mu_n^2}\right)\cos\mu_n t + \frac{\delta_{n1}}{1+\mu_n^2}\sin\mu_n t + \frac{\delta_{n1}}{1+\mu_n^2}e^{-t}\right\}\sin\mu_n x$

$= \cos 3t\sin 3t + \frac{1}{2}[\sin t + e^{-t} - \cos t]\sin x$

$u(x,t) = xt + \frac{t^2}{2} + \cos 3t\sin 3t + \frac{1}{2}[\sin t + e^{-t} - \cos t]\sin x$

3. Solve the telegraph equation with $0 < \gamma < 1$ subject to an exponentially decaying forcing function:

$$u_{tt} + 2\gamma u_t = u_{xx} + e^{-2t}\cos(5x), \ 0 < x < \pi/2, \ t > 0$$
$$u_x(0,t) = 0 \text{ and } u(\pi/2,t) = 0, \quad t > 0$$
$$u(x,0) = 0, \quad u_t(x,0) = \cos(3x), \quad 0 < x < \pi/2$$

THE EIGENVALUE PROBLEM ASSOCIATED WITH THESE HOMOGENEOUS BC [total 20 marks]

IS $\left.\begin{array}{l} X'' + \mu^2 X = 0 \\ X'(0) = 0 = X(\pi/2) \end{array}\right\} \Rightarrow \mu_n = \frac{(2n-1)\pi}{2(\pi/2)} = (2n-1) \quad n = 1, 2, \ldots$ ARE THE EIGENVALUES

$X_n = \cos(2n-1)x$ ARE THE EIGENFUNCTIONS

LET $S(x,t) = e^{-2t}\cos 5x = \sum_{n=1}^{\infty} S_n(t)\cos(2n-1)x \Rightarrow S_n(t) = e^{-2t}\delta_{n3}$

NOW ASSUME $u(x,t) = \sum_{n=1}^{\infty} u_n(t)\cos\mu_n x \qquad u_{xx} = \sum_{n=1}^{\infty} u_n\{-\mu_n^2\}\cos\mu_n x$

$0 = u_{tt} + 2\gamma u_t - u_{xx} - e^{-2t}\cos 5x = \sum_{n=1}^{\infty}\{\ddot{u}_n + 2\gamma\dot{u}_n + \mu_n^2 u_n - e^{-2t}\delta_{n3}\}\cos\mu_n x$

SINCE THE $\cos\mu_n x$ ARE LINEARLY INDEPENDENT IT FOLLOWS THAT u_n SATISFIES THE ODE

$\ddot{u}_n + 2\gamma\dot{u}_n + \mu_n^2 u_n = e^{-2t}\delta_{n3}$

CONSIDER THE HOMOGENEOUS EQ $\ddot{u}_n^H + 2\gamma\dot{u}_n^H + \mu_n^2 u_n^H = 0$

LET $u_n = e^{rt} \Rightarrow r^2 + 2\gamma r + \mu_n^2 = 0 \quad r = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\mu_n^2}}{2} = -\gamma \pm \sqrt{\gamma^2 - \mu_n^2} = -\gamma \pm i\nu_n$

WHERE $\nu_n = \sqrt{\mu_n^2 - \gamma^2} \in \mathbb{R}$ SINCE $\gamma < 1 \le \mu_n$

THUS $u_n^H = [A_n\cos\nu_n t + B_n\sin\nu_n t]e^{-\gamma t}$

SINCE THE FORCING TERM IS NOT A SOLUTION TO THE HOMOGENEOUS EQ GUESS $u_n^P = Ce^{-2t}$

$\ddot{u}_n^P + 2\gamma\dot{u}_n^P + \mu_n^2 u_n^P = 4Ce^{-2t} - 4\gamma Ce^{-2t} + \mu_n^2 Ce^{-2t} = C[4 - 4\gamma + \mu_n^2]e^{-2t} = \delta_{n3}e^{-2t}$

$\therefore \quad C_n = \delta_{n3}/[4 - 4\gamma + \mu_n^2] = \delta_{n3}/\beta_n$ WHERE $\beta_n = 4 - 4\gamma + \mu_n^2$

$\therefore \quad u_n(t) = [A_n\cos\nu_n t + B_n\sin\nu_n t]e^{-\gamma t} + \frac{\delta_{n3}}{4 - 4\gamma + \mu_n^2}e^{-2t}$

$\therefore \quad u(x,t) = \sum_{n=1}^{\infty}\{e^{-\gamma t}[A_n\cos\nu_n t + B_n\sin\nu_n t] + C_n e^{-2t}\}\cos\mu_n x$

$u_t(x,t) = \sum_{n=1}^{\infty}\{-\gamma e^{-\gamma t}[A_n\cos\nu_n t + B_n\sin\nu_n t] + e^{-\gamma t}[-A_n\nu_n\sin\nu_n t + B_n\nu_n\cos\nu_n t] - 2C_n e^{-2t}\}\cos\mu_n x$

$0 = \sum_{n=1}^{\infty}(A_n + C_n)\cos\mu_n x \Rightarrow \quad A_n = -C_n = -\delta_{n3}/[4 - 4\gamma + \mu_n^2] = -\frac{\delta_{n3}}{\beta_n}$

$\cos 3x = \sum_{n=1}^{\infty}\delta_{n2}\cos(2n-1)x = \sum_{n=1}^{\infty}(-\gamma A_n + B_n\nu_n - 2C_n)\cos\mu_n x$

$\therefore \quad B_n = (\delta_{n2} + \gamma A_n + 2C_n)/\nu_n = \frac{\delta_{n2}}{\nu_n} - \frac{\gamma\delta_{n3}}{(4 - 4\gamma + \mu_n^2)\nu_n} + \frac{2\delta_{n3}}{(4 - 4\gamma + \mu_n^2)\nu_n}$

$\therefore \ u(x,t) = \sum_{n=1}^{\infty}\left\{e^{-\gamma t}\left[-\frac{\delta_{n3}}{\beta_n}\cos\nu_n t + \left\{\frac{\delta_{n2}}{\nu_n} - \frac{\gamma\delta_{n3}}{\beta_n\nu_n} + \frac{2\delta_{n3}}{\beta_n\nu_n}\right\}\sin\nu_n t\right] + \frac{\delta_{n3}}{\beta_n}e^{-2t}\right\}\cos\mu_n x$

$= \left\{\frac{e^{-\gamma t}}{29 - 4\gamma}\left[-\cos\nu_3 t + \frac{2-\gamma}{\nu_3}\sin\nu_3 t\right] + \frac{e^{-2t}}{29 - 4\gamma}\right\}\cos 5x + \frac{e^{-\gamma t}}{\nu_2}\sin\nu_2 t\cos 3x$