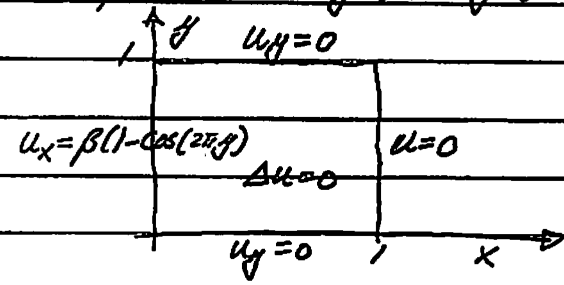
MATH 257/316 ASSIGNMENT 9 SOLUTIONS

Q1: $\Delta u = 0 \quad 0 \le x, y < 1 \quad u_x(0,y) = \beta(1-\cos 2\pi y), \; u(1,y) = 0 = u_y(x,0) = u_y(x,1).$



LET $u(x,y) = X(x)Y(y)$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda^2$$

$$\left.\begin{array}{l} Y'' + \lambda^2 Y = 0 \\ Y'(0) = 0 = Y'(1) \end{array}\right\} \Rightarrow \lambda_n = n\pi \quad n = 0,1,2\ldots \qquad Y_n \in \{1, \cos n\pi y\}$$

$\underline{\lambda_0 = 0:}$ $X_0'' = 0 \quad X_0 = A_0 x + B_0 \quad X_0(1) = A_0 + B_0 \Rightarrow B_0 = -A_0, X_0 = A_0(x-1)$

$\underline{\lambda_n > 0:}$ $X_n'' - \lambda_n^2 X_n = 0 \quad X_n = A_n \sinh \lambda_n (x-1)$

$$u(x,y) = A_0(x-1) + \sum_{n=1}^{\infty} A_n \sinh \lambda_n (x-1) \cos(\lambda_n y)$$

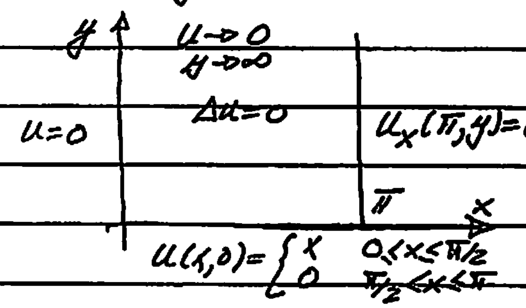
$$u_x(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \lambda_n \cosh \lambda_n (x-1) \cos(\lambda_n y)$$

$$\beta(1-\cos(2\pi y)) = u_x(0,y) = A_0 + \sum_{n=1}^{\infty} \{A_n \lambda_n \cosh \lambda_n\} \cos(n\pi y)$$

SINCE $\{1, \cos(n\pi y)\}$ ARE INDEPENDENT WE CAN EQUATE COEFFICIENTS TO OBTAIN: $A_0 = \beta \quad A_1 = 0 \quad A_2 \lambda_2 \cosh(2\pi) = -\beta \quad A_n = 0 \quad n \ge 3$

$$\therefore \quad u(x,y) = \beta(x-1) - \frac{\beta \sinh(2\pi(x-1))}{(2\pi)\cosh(2\pi)} \cos(2\pi y)$$

Q2: $\Delta u = 0 \quad 0 < x < \pi, \; 0 < y$



LET $u(x,y) = X(x)Y(y) \Rightarrow \dfrac{X''}{X(x)} = -\dfrac{Y''(y)}{Y(y)} = -\lambda^2$

$$\left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X(0) = 0 = X'(\pi) \end{array}\right\} \quad \begin{array}{l} X = A\cos\lambda x + B\sin\lambda x \\ X' = -A\lambda\sin\lambda x + B\lambda\cos\lambda x \end{array}$$

$X(0) = A = 0 \quad X'(\pi) = B\lambda\cos\lambda\pi = 0 \quad \cancel{\pi}\lambda_n = \dfrac{(2n+1)\cancel{\pi}}{2} \quad n = 0,1,\ldots$

$\therefore \lambda_n = \left(\dfrac{2n+1}{2}\right) \quad n = 0,1,2.. \quad X_n = \sin\lambda_n x$

$Y_n'' - \lambda_n^2 Y_n = 0 \quad Y_n = A_n e^{\lambda_n y} + B_n e^{-\lambda_n y} \quad Y_n \xrightarrow{y\to\infty} 0 \Rightarrow A_n = 0.$

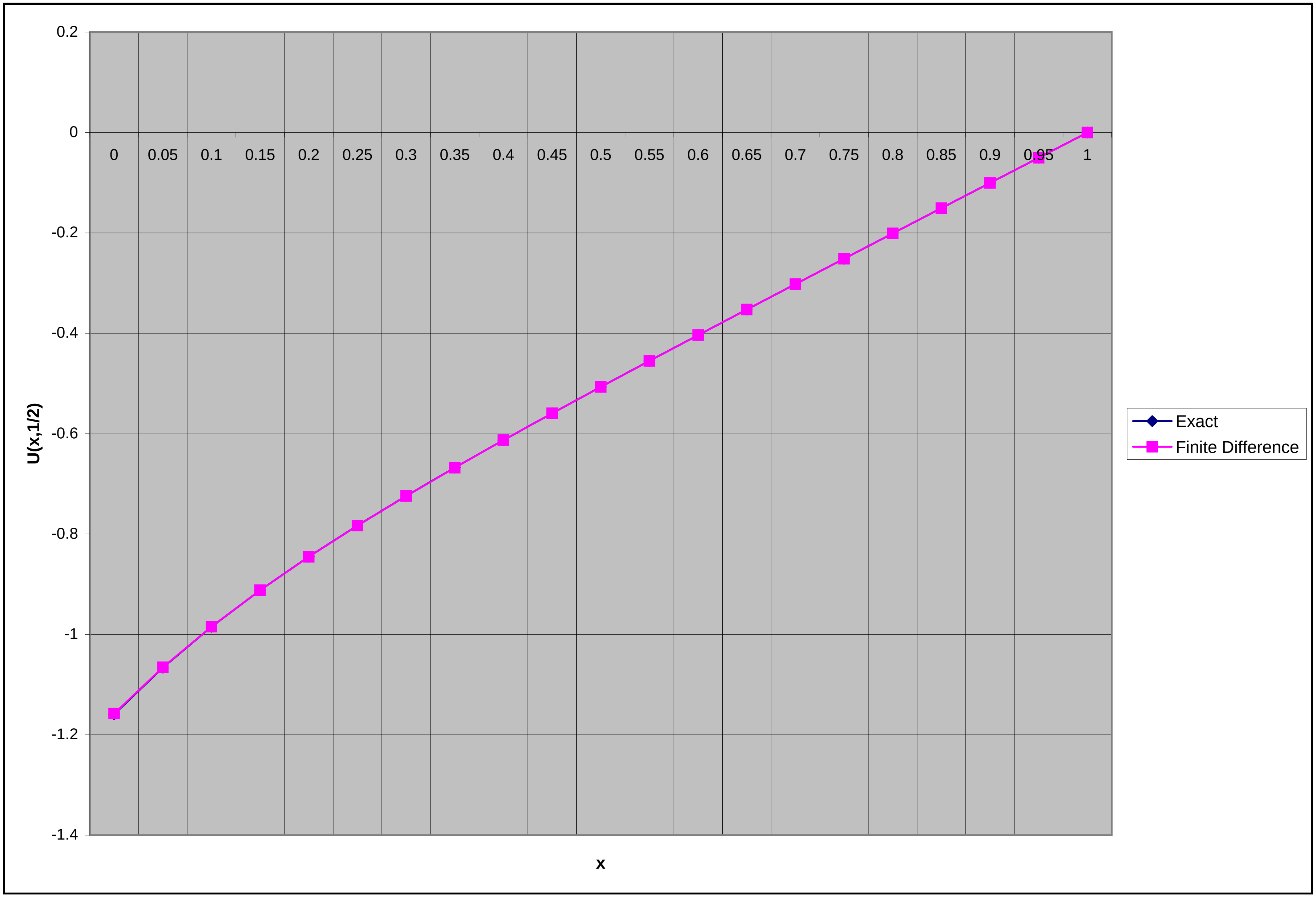
$$u(x,y) = \sum_{n=0}^{\infty} B_n e^{-\lambda_n y} \sin\lambda_n x$$

$$u(x,0) = \begin{cases} x & 0<x<\pi/2 \\ 0 & \pi/2<x<\pi \end{cases} = u(x,0) = \sum_{n=0}^{\infty} B_n \sin\lambda_n x$$

$$B_n = \frac{2}{1}\int_0^1 u(x,0)\,dx = 2\int_0^{\pi/2} x\sin\lambda_n x\,dx = 2\left\{ \frac{-x\cos\lambda_n x}{\lambda_n}\Big|_0^{\pi/2} + \frac{1}{\lambda_n}\int_0^{\pi/2} \cos\lambda_n x\,dx \right\}$$

$$= 2\left\{ \frac{-(\pi/2)\cos\left(\left(\frac{2n+1}{2}\right)\frac{\pi}{2}\right)}{\lambda_n} + \frac{1}{\lambda_n^2}\sin\left(\left(\frac{2n+1}{2}\right)x\right)\Big|_0^{\pi/2}\right.$$

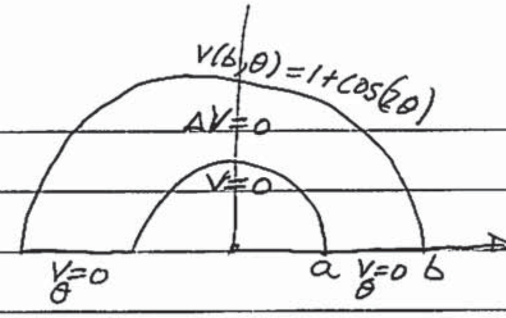
$$= 2\left\{ -\cos[(2n+1)\pi/4]/(2n+1) + \frac{4}{(2n+1)^2}\sin[(2n+1)\pi/4] \right\}$$



U(x,1/2)
x
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55
0.6
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
Exact
Finite Difference

Y	-0.844935	-0.849838	-0.840041	-0.819481	-0.791039	-0.756826	-0.718386	-0.676852	-0.633052	-0.587592	-0.540919	-0.493356
1	-0.834382	-0.84211	-0.834382	-0.815337	-0.788005	-0.754604	-0.716759	-0.67566	-0.632179	-0.586954	-0.540451	-0.493014
0.95	-0.844935	-0.849838	-0.840041	-0.819481	-0.791039	-0.756826	-0.718386	-0.676852	-0.633052	-0.587592	-0.540919	-0.493356
0.9	-0.875562	-0.872264	-0.856463	-0.831506	-0.799845	-0.763274	-0.723107	-0.680309	-0.635583	-0.589445	-0.542275	-0.494348
0.85	-0.923263	-0.907195	-0.882042	-0.850236	-0.81356	-0.773317	-0.730461	-0.685694	-0.639525	-0.592332	-0.544387	-0.495894
0.8	-0.98337	-0.951209	-0.914272	-0.873838	-0.830842	-0.785972	-0.739728	-0.692479	-0.644493	-0.595968	-0.547049	-0.497841
0.75	-1.05	-1	-0.95	-0.9	-0.85	-0.8	-0.75	-0.7	-0.65	-0.6	-0.55	-0.500000
0.7	-1.116629	-1.048791	-0.985728	-0.926162	-0.869157	-0.814028	-0.760272	-0.707521	-0.655507	-0.604031	-0.552951	-0.502159
0.65	-1.176737	-1.092805	-1.017958	-0.949763	-0.88644	-0.826683	-0.769538	-0.714306	-0.660475	-0.607668	-0.555612	-0.504106
0.6	-1.224438	-1.127735	-1.043536	-0.968493	-0.900155	-0.836726	-0.776892	-0.719691	-0.664417	-0.610554	-0.557725	-0.505651
0.55	-1.255064	-1.150162	-1.059959	-0.980519	-0.908961	-0.843174	-0.781614	-0.723148	-0.666948	-0.612407	-0.559081	-0.506644
0.5	-1.265617	-1.15789	-1.065617	-0.984663	-0.911995	-0.845396	-0.783241	-0.724339	-0.66782	-0.613046	-0.559548	**-0.506986**
0.45	-1.255064	-1.150162	-1.059959	-0.980519	-0.908961	-0.843174	-0.781614	-0.723148	-0.666948	-0.612407	-0.559081	-0.506644
0.4	-1.224438	-1.127735	-1.043536	-0.968493	-0.900155	-0.836726	-0.776892	-0.719691	-0.664417	-0.610554	-0.557725	-0.505651
0.35	-1.176737	-1.092805	-1.017958	-0.949763	-0.88644	-0.826683	-0.769538	-0.714306	-0.660475	-0.607668	-0.555612	-0.504106
0.3	-1.116629	-1.048791	-0.985728	-0.926162	-0.869157	-0.814028	-0.760272	-0.707521	-0.655507	-0.604031	-0.552951	-0.502159
0.25	-1.05	-1	-0.95	-0.9	-0.85	-0.8	-0.75	-0.7	-0.65	-0.6	-0.55	-0.500000
0.2	-0.98337	-0.951209	-0.914272	-0.873838	-0.830842	-0.785972	-0.739728	-0.692479	-0.644493	-0.595968	-0.547049	-0.497841
0.15	-0.923263	-0.907195	-0.882042	-0.850236	-0.81356	-0.773317	-0.730461	-0.685694	-0.639525	-0.592332	-0.544387	-0.495894
0.1	-0.875562	-0.872264	-0.856463	-0.831506	-0.799845	-0.763274	-0.723107	-0.680309	-0.635583	-0.589445	-0.542275	-0.494348
0.05	-0.844935	-0.849838	-0.840041	-0.819481	-0.791039	-0.756826	-0.718386	-0.676852	-0.633052	-0.587592	-0.540919	-0.493356
0	-0.834382	-0.84211	-0.834382	-0.815337	-0.788005	-0.754604	-0.716759	-0.67566	-0.632179	-0.586954	-0.540451	-0.493014
	-0.844935	-0.849838	-0.840041	-0.819481	-0.791039	-0.756826	-0.718386	-0.676852	-0.633052	-0.587592	-0.540919	-0.493356
		0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.500000

-0.445143	-0.396455	-0.347419	-0.298131	-0.24866	-0.199058	-0.149364	-0.099608	-0.049813	
-0.444893	-0.396272	-0.347287	-0.298035	-0.248592	-0.19901	-0.149332	-0.099588	-0.049803	0
-0.445143	-0.396455	-0.347419	-0.298131	-0.24866	-0.199058	-0.149364	-0.099608	-0.049813	0
-0.445869	-0.396984	-0.347805	-0.29841	-0.248861	-0.199199	-0.149459	-0.099666	-0.049841	0
-0.446998	-0.397809	-0.348405	-0.298845	-0.249172	-0.199418	-0.149607	-0.099758	-0.049884	0
-0.448422	-0.398848	-0.349161	-0.299393	-0.249565	-0.199694	-0.149793	-0.099873	-0.049939	0
-0.45	-0.4	-0.35	-0.3	-0.25	-0.2	-0.15	-0.1	-0.05	0
-0.451578	-0.401152	-0.350838	-0.300607	-0.250435	-0.200306	-0.150207	-0.100127	-0.050061	0
-0.453002	-0.402191	-0.351595	-0.301155	-0.250828	-0.200582	-0.150393	-0.100242	-0.050116	0
-0.454131	-0.403016	-0.352195	-0.301589	-0.251139	-0.200801	-0.150541	-0.100334	-0.050159	0
-0.454857	-0.403545	-0.35258	-0.301869	-0.251339	-0.200941	-0.150636	-0.100392	-0.050187	0
-0.455107	-0.403728	-0.352713	-0.301965	-0.251408	-0.20099	-0.150668	-0.100412	-0.050197	0
-0.454857	-0.403545	-0.35258	-0.301869	-0.251339	-0.200941	-0.150636	-0.100392	-0.050187	0
-0.454131	-0.403016	-0.352195	-0.301589	-0.251139	-0.200801	-0.150541	-0.100334	-0.050159	0
-0.453002	-0.402191	-0.351595	-0.301155	-0.250828	-0.200582	-0.150393	-0.100242	-0.050116	0
-0.451578	-0.401152	-0.350838	-0.300607	-0.250435	-0.200306	-0.150207	-0.100127	-0.050061	0
-0.45	-0.4	-0.35	-0.3	-0.25	-0.2	-0.15	-0.1	-0.05	0
-0.448422	-0.398848	-0.349161	-0.299393	-0.249565	-0.199694	-0.149793	-0.099873	-0.049939	0
-0.446998	-0.397809	-0.348405	-0.298845	-0.249172	-0.199418	-0.149607	-0.099758	-0.049884	0
-0.445868	-0.396984	-0.347805	-0.29841	-0.248861	-0.199199	-0.149459	-0.099666	-0.049841	0
-0.445143	-0.396455	-0.347419	-0.298131	-0.24866	-0.199058	-0.149364	-0.099608	-0.049813	0
-0.444893	-0.396272	-0.347287	-0.298035	-0.248592	-0.19901	-0.149332	-0.099588	-0.049803	0
-0.445143	-0.396455	-0.347419	-0.298131	-0.24866	-0.199058	-0.149364	-0.099608	-0.049813	
0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9	0.95	1



Q4: $v_{rr}+\frac{1}{r}v_r+\frac{1}{r^2}v_{\theta\theta}=0$

LET $v(r,\theta)=R(r)\Theta(\theta)$

$$R''(r)\Theta+\frac{1}{r}R'(r)\Theta+\frac{1}{r^2}R(r)\Theta''(\theta)=0$$

$\times\frac{r^2}{R\Theta}$: $\dfrac{r^2R''(r)+rR'(r)}{R(r)}=-\dfrac{\Theta''(\theta)}{\Theta(\theta)}=\lambda^2$

$$\left.\begin{array}{l}\Theta''+\lambda^2\Theta=0\\ \Theta'(0)=0=\Theta'(\pi)\end{array}\right\}\quad \begin{array}{l}\lambda_n=n\quad n=0,1,2.\\ \Theta_n\in\{1,\cos(n\theta)\}\end{array}$$

$\lambda_n>0$: $r^2R_n''+rR_n'-\lambda_n^2R_n=0$ LET $R_n=r^\gamma\Rightarrow r^\gamma\{\gamma(\gamma-1)+\gamma-\lambda_n^2\}=0\Rightarrow\gamma=\pm n$

$$R_n(r)=A_nr^n+B_nr^{-n}$$

$\lambda_n=0$: $0=rR_0''+R_0'=(rR_0')'\Rightarrow R_0=A_0\log r+B_0$

$$\therefore\quad v(r,\theta)=\{A_0\log r+B_0\}1+\sum_{n=1}^{\infty}\{A_nr^n+B_nr^{-n}\}\cos(n\theta)$$

$$0=v(a,\theta)=(A_0\log a+B_0)+\sum_{n=1}^{\infty}(A_na^n+B_na^{-n})\cos(n\theta)$$

$$\therefore\ B_0=-A_0\log a\qquad B_n=-A_na^{2n}$$

$$\therefore\quad v(r,\theta)=A_0\log(r/a)+\sum_{n=1}^{\infty}A_n(r^n-a^{2n}r^{-n})\cos n\theta$$

$$1+\cos2\theta=v(b,\theta)=A_0\log\left(\frac{b}{a}\right)+\sum_{n=1}^{\infty}A_n(b^n-a^{2n}b^{-n})\cos n\theta$$

EQUATING COEFFICIENTS $A_0=[\log(b/a)]^{-1}$, $A_1=0$, $A_2(b^2-a^4b^{-2})=1$, $A_n=0\ n\geq3$

$$\therefore\quad v(r,\theta)=\frac{\log(r/a)}{\log(b/a)}+\frac{\left[\left(\frac{r}{a}\right)^2-\left(\frac{a}{r}\right)^2\right]}{\left[\left(\frac{b}{a}\right)^2-\left(\frac{a}{b}\right)^2\right]}\cos2\theta$$

Problem 4 Using Separation of variable method, we get

$$v(r,\theta) = R(r)\,\Theta(\theta) \quad \text{with}$$

$$\begin{cases} r^2 R'' + rR' - \lambda R = 0 \\ \Theta'' + \lambda \Theta = 0 \end{cases}$$

with B.C. $\begin{cases} R(b) = 0 \\ \Theta(0) = 0 \qquad \Theta(\alpha) = 0 \end{cases}$

the eigenvalue pb for Θ has a non trivial solution if

$$\lambda = \lambda_n = \left(\frac{n\pi}{\alpha}\right)^2 \qquad n = 1, 2, \ldots$$

and then $\Theta_n(\theta) = \sin\left(\frac{n\pi}{\alpha}\theta\right)$

the Eq. for R becomes

$$r^2 R'' + rR' - \left(\frac{n\pi}{\alpha}\right)^2 R = 0$$

so $R(r) = c_1\, r^{-\frac{n\pi}{\alpha}} + c_2\, r^{\frac{n\pi}{\alpha}}$

R must satisfy $R(b) = c_1\, b^{-\frac{n\pi}{\alpha}} + c_2\, b^{\frac{n\pi}{\alpha}} = 0$

so $c_1 = -c_2\, b^{2\frac{n\pi}{\alpha}}$

and thus $R(r) = c_2\left[r^{\frac{n\pi}{\alpha}} - b^{2\frac{n\pi}{\alpha}}\, r^{-\frac{n\pi}{\alpha}} \right]$

We deduce the following fundamental solutions:

$$v_n(r,\theta) = \left(r^{\frac{n\pi}{\alpha}} - b^{2\frac{n\pi}{\alpha}} r^{-\frac{n\pi}{\alpha}} \right) \sin\left(\frac{n\pi}{\alpha}\theta\right)$$

Finally, the solution of the BVP is

$$v(r,\theta) = \sum_{n=1}^{\infty} c_n \left(r^{\frac{n\pi}{\alpha}} - b^{2\frac{n\pi}{\alpha}} r^{-\frac{n\pi}{\alpha}} \right) \sin\left(\frac{n\pi}{\alpha}\theta\right)$$

where the c_n are such that

$$v(a,\theta) = \sum_{n=1}^{\infty} c_n \left(a^{\frac{n\pi}{\alpha}} - b^{2\frac{n\pi}{\alpha}} a^{-\frac{n\pi}{\alpha}} \right) \sin\left(\frac{n\pi}{\alpha}\theta\right) = f(\theta)$$

So $$c_n = \frac{1}{a^{\frac{n\pi}{\alpha}} - b^{2\frac{n\pi}{\alpha}} a^{-\frac{n\pi}{\alpha}}} \frac{2}{\alpha} \int_0^{\alpha} f(\theta) \sin\left(\frac{n\pi}{\alpha}\theta\right) d\theta$$

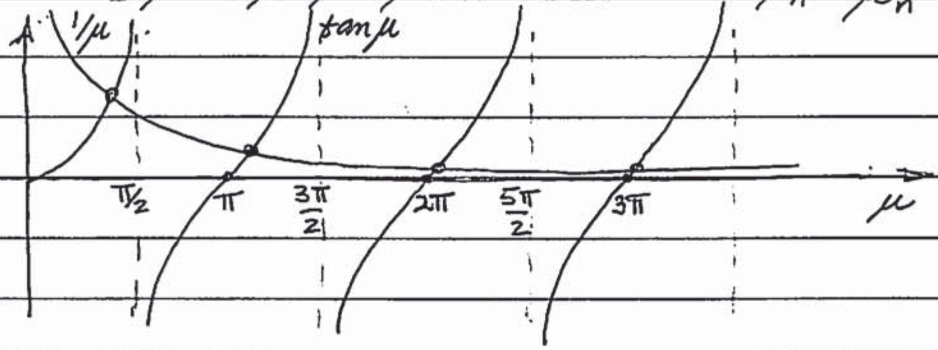
Q5: (a) $y'' + \lambda y = 0 \quad y'(0) - y(0) = 0 \quad y'(1) = 0. \qquad$ LET $\lambda = \mu^2$

$y = A\cos\mu x + B\sin\mu x \quad \Rightarrow \quad y = B[\mu\cos\mu x + \sin\mu x]$

$y' = -A\mu\sin\mu x + B\mu\cos\mu x \quad \Rightarrow \quad y' = B[-\mu^2\sin\mu x + \mu\cos\mu x]$

$y'(0) = B\mu = y(0) = A \quad \Rightarrow \quad A = \mu B$

$y'(1) = B[-\mu^2\sin\mu + \mu\cos\mu] = 0 \quad \Rightarrow \quad \tan\mu_n = \frac{1}{\mu_n}$



$\lambda = 0$ IS NOT AN EIGENVALUE

$\lambda_n = \mu_n^2 \sim [(n-1)\pi]^2 \qquad n >> 1.$

$y_n(x) = \mu_n\cos\mu_n x + \sin\mu_n x = \dfrac{\cos\mu_n\cos\mu_n x + \sin\mu_n\sin\mu x}{\sin\mu_n} = \dfrac{\cos\mu_n(x-1)}{\sin\mu_n}$

(b) $y'' + \lambda y = 0 \quad y'(0) - y(0) = 0 \quad y(1) + y'(1) = 0 \qquad$ LET $\lambda = \mu^2$

$y = A\cos\mu x + B\sin\mu x \quad \Rightarrow \quad y = B[\mu\cos\mu x + \sin\mu x]$

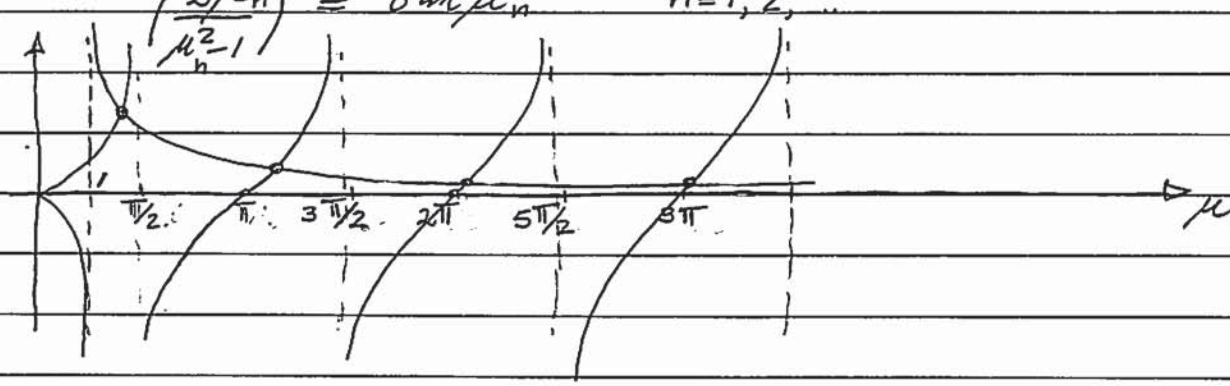
$y' = -A\mu\sin\mu x + B\mu\cos\mu x \quad \Rightarrow \quad y' = B[-\mu^2\sin\mu x + \mu\cos\mu x]$

$y'(0) = B\mu = y(0) = A \quad \Rightarrow \quad A = \mu B$

$y'(1) = B[-\mu^2\sin\mu + \mu\cos\mu] = -y(1) = -B[\mu\cos\mu + \sin\mu]$

$B[\,2\mu\cos\mu + (1-\mu^2)\sin\mu\,] = 0$

$\left(\dfrac{2\mu_n}{\mu_n^2 - 1}\right) = \tan\mu_n \qquad n = 1, 2, \ldots$



- $\mu = 0$ IS NOT AN EIGENVALUE
- $\lambda_n = \mu_n^2 \sim [(n-1)\pi]^2$ AS $n \to \infty$
- $y_n = \mu_n\cos\mu_n x + \sin\mu_n x$

PROBLEM 5(c) $Ly = -(xy')' = \frac{1}{x}\lambda y \qquad 0 < a < x < b$

$$y(a) = 0 = y(b)$$

LET US EXPAND THE DIFFERENTIAL EQ TO THE FORM

$$x^2 y'' + x y' + \lambda y = 0$$

WHICH IS A CAUCHY-EULER EQ, SO CONSIDER A SOLUTION OF THE FORM:

$$y = x^r \Rightarrow \quad r(r-1) + r + \lambda = r^2 + \lambda = 0$$

$$r = \pm i\sqrt{\lambda} = \pm i\mu \qquad \text{WHERE} \quad \lambda = \mu^2$$

$$y = C_1 x^{i\mu} + C_2 x^{-i\mu} \qquad\qquad x^{i\mu} = e^{i\mu \ln x}$$

$$= A\cos(\mu \ln x) + B\sin(\mu \ln x)$$

NOW $\quad y(a) = A\cos(\mu \ln a) + B\sin\mu \ln a = 0 \qquad A = -B\tan(\mu \ln a)$

$$y(x) = B\left[\sin(\mu \ln x) - \tan(\mu \ln a)\cos\mu \ln x\right]$$

$$= B\left[\frac{\sin(\mu \ln x)\cos(\mu \ln a) - \cos(\mu \ln x)\sin(\mu \ln a)}{\cos(\mu \ln a)}\right]$$

$$= \frac{B}{\cos(\mu \ln a)} \sin\left[\mu \ln\left(\frac{x}{a}\right)\right]$$

$$y(b) = \frac{B}{\cos\mu \ln a} \sin\left[\mu \ln\left(\frac{b}{a}\right)\right] = 0 \Rightarrow \mu_n = \frac{n\pi}{\ln\left(\frac{b}{a}\right)} \qquad n = 1, 2, \ldots$$

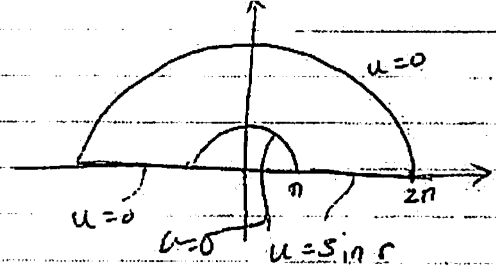
AND THE CORRESPONDING EIGENFUNCTIONS ARE

$$y_n(x) = \sin\left(\mu_n \ln\left(\frac{x}{a}\right)\right) = \sin\left(\frac{n\pi \ln\left(\frac{x}{a}\right)}{\ln\left(\frac{b}{a}\right)}\right) \qquad n = 1, 2, \ldots$$

NOTE: EIGENFUNCTIONS ARE ONLY UNIQUE UP TO AN ARBITRARY CONSTANT, WHICH WE CHOOSE TO BE 1 IN THIS CASE.

PROBLEM 6

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad \pi < r < 2\pi \\ u(r,0) = \sin(r), \quad u(r,\pi) = 0 \\ u(\pi,\theta) = u(2\pi,\theta) = 0 \end{cases}$$



Use separation of variables: $u(r,\theta) = R(r)\Theta(\theta)$

Obtain $\begin{cases} r^2R'' + rR' - \lambda R = 0, & R(\pi) = R(2\pi) = 0 \\ \Theta'' + \lambda\Theta = 0, & \Theta(\pi) = 0 \end{cases}$

Since we have 2 homog. BCs in r, look at R eq. first.

Let $R = r^\gamma \Rightarrow$ char eq. $\gamma(\gamma-1) + \gamma - \lambda = 0$ or $\gamma^2 = \lambda$.

Case $\lambda = -\mu^2 < 0$: $\gamma^2 = -\mu^2$ so $\gamma = i\mu$ and $R(r) = C_1\cos(\mu\ln(r)) + C_2\sin(\mu\ln(r))$

$R(\pi) = 0 \Rightarrow C_1\cos(\mu\ln(\pi)) + C_2\sin(\mu\ln(\pi)) \Rightarrow C_1 = -C_2\tan(\mu\ln(\pi))$

so $R(r) = C_2[-\tan(\mu\ln\pi)\cos(\mu\ln(r)) + \sin(\mu\ln(r))]$

$= \frac{C_2}{\cos(\mu\ln\pi)}[-\sin(\mu\ln\pi)\cos(\mu\ln r) + \cos(\mu\ln\pi)\sin(\mu\ln r)]$

C_μ

$= C_\mu\sin(\mu\ln r - \mu\ln\pi)$ so $R(r) = C_\mu\sin\left[\mu\ln\left(\frac{r}{\pi}\right)\right]$

$R(2\pi) = 0 \Rightarrow C_\mu\sin[\mu\ln(2)] = 0 \Rightarrow \mu\ln(2) = n\pi \Rightarrow \mu = \frac{n\pi}{\ln(2)}$

Thus for $\lambda < 0$, eigenvalues $\boxed{\lambda_n = -\left(\frac{n\pi}{\ln 2}\right)^2}$

eigenfunctions $\boxed{R_n = C_n\sin\left[\frac{n\pi\ln(r/\pi)}{\ln(2)}\right]}$

Case $\lambda = 0$: $\gamma^2 = 0$ so $\gamma = 0$ and $R(r) = C_1 + C_2\ln(r)$

$R(\pi) = 0 = C_1 + C_2\ln(\pi) \Rightarrow C_1 = -C_2\ln(\pi)$

and $R(r) = C_1[-\ln(\pi) + \ln(r)] = C_1\ln(r/\pi)$

$R(2\pi) = 0 = C_1\ln\left(\frac{2\pi}{\pi}\right) = C_1\ln(2) \Rightarrow C_1 = 0$ $\therefore$ no eigenfunction/zero eigenvalue.

Case $\lambda > 0$: $\lambda = \mu^2 > 0 \Rightarrow \gamma^2 = \mu^2$, $\gamma = \pm\mu$ and $R(r) = C_1 r^{\mu} + C_2 r^{-\mu}$

$R(\pi) = 0 \Rightarrow C_1 \pi^{\mu} + C_2 \pi^{-\mu}$ so $C_2 = -C_1 \pi^{2\mu}$

and then $R(r) = C_1 r^{\mu} - C_1 \pi^{2\mu} r^{-\mu}$

$= C_1 \left[\left(\frac{r}{\pi}\right)^{\mu} + \left(\frac{r}{\pi}\right)^{-\mu} \right]$

$R(2\pi) = 0 \Rightarrow C_1 [2^{\mu} + 2^{-\mu}] = 0 \Rightarrow C_1 = 0$

$\therefore$ no eigenfunctions/eigenvalues for $\lambda > 0$

Thus we have $\boxed{\lambda_n = -\left(\frac{n\pi}{\ln(2)}\right)^2, \quad R_n(r) = \sin\left[\frac{n\pi \ln(r/\pi)}{\ln(2)}\right]}$

Now for the Θ-eq: $\begin{cases} \Theta_n'' - \left(\frac{n\pi}{\ln(2)}\right)^2 \Theta = 0 \\ \Theta(\pi) = 0 \end{cases}$

obtain $\Theta(\theta) = C_1 \sinh\left(\frac{n\pi\theta}{\ln(2)}\right) + C_2 \cosh\left(\frac{n\pi\theta}{\ln(2)}\right)$

$\Theta(\pi) = 0 \Rightarrow$ $\Theta_n = \sinh\left(\frac{n\pi}{\ln(2)}(\theta - \pi)\right)$

Then by superposition,

$$u(r,\theta) = \sum_{n=1}^{\infty} C_n \sin\left[\frac{n\pi \ln(r/\pi)}{\ln(2)}\right] \sinh\left(\frac{n\pi(\theta-\pi)}{\ln(2)}\right)$$

Now, $u(r,0) = \sin(r) = \sum_{n=1}^{\infty} -C_n \sinh\left(\frac{n\pi^2}{\ln 2}\right) \cdot \sin\left(\frac{n\pi \ln(r/\pi)}{\ln(2)}\right)$

a sine series! so $-C_n \sinh\left(\frac{n\pi^2}{\ln 2}\right) = \frac{2}{\pi}\int_{\pi}^{2\pi} \sin r \sin\left(\frac{n\pi \ln(r/\pi)}{\ln(2)}\right) dr.$

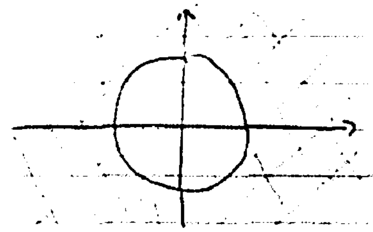
or $C_n = \frac{-2}{\pi \sinh(n\pi^2/\ln 2)} \int_{\pi}^{2\pi} \sin\left(\frac{n\pi \ln(r/\pi)}{\ln(2)}\right) \sin(r)\, dr.$

Thus, $u(r,\theta) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi \ln(r/\pi)}{\ln(2)}\right) \sinh\left(\frac{n\pi(\theta-\pi)}{\ln(2)}\right)$

where $C_n = \frac{-2}{\pi \sinh(n\pi^2/\ln(2))} \int_{\pi}^{2\pi} \sin(r) \sin\left(\frac{n\pi \ln(r/\pi)}{\ln(2)}\right) dr$

Problem 7: $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$

$u_r(1,\theta) = f(\theta)$, $|u| < \infty$ as $r \to \infty$



Use separation of variables $u(r,\theta) = R(r)\Theta(\theta)$

Obtain $\begin{cases} r^2 R'' + rR' - \lambda R = 0, & R \text{ bdd as } r \to \infty \\ \Theta'' + \lambda \Theta = 0, & \text{periodic BCs. } \Theta(\theta + 2\pi) = \Theta(\theta) \end{cases}$

Θ-eq: since BCs are periodic, sol'n has to be, so $\lambda \geq 0$.

$\lambda = 0$: $\Theta'' = 0 \Rightarrow \Theta = C_1\theta + C_2$.
since $\Theta(2\pi + \theta) = \Theta(\theta)$, $C_1 = 0$ $\Rightarrow \begin{cases} \lambda_0 = 0 \\ \Theta_0 = 1 \end{cases}$

$\lambda = \mu^2 > 0$. $\Theta'' + \mu^2 \Theta = 0 \Rightarrow \Theta(\theta) = C_1 \cos(\mu\theta) + C_2 \sin(\mu\theta)$.
but $\Theta(2\pi + \theta) = \Theta(\theta)$; use $\theta = -\pi$ to find C_1, C_2.

$\Theta(\pi) = \Theta(-\pi)$

$\Rightarrow C_1 \cos(\mu\pi) + C_2 \sin(\mu\pi) = C_1 \cos(-\mu\pi) + C_2 \sin(-\mu\pi)$

$\Rightarrow 2C_2 \sin(\mu\pi) = 0$ as $\sin(-\varphi) = -\sin\varphi$, $\cos(-\varphi) = \cos\varphi$

$\sin\varphi$ has roots at $\varphi = n\pi$, $n = 1, 2, \ldots$

so $\mu\pi = n\pi$

$\mu = n$

Thus $\lambda_n = +n^2$

$\Theta_n = a_n \cos(n\theta) + b_n \sin(n\theta)$

R-eq: $r^2 R'' + rR' - n^2 R = 0$ as $\lambda_n = n^2$

let $R = r^\gamma \Rightarrow$ char. eq. $\gamma(\gamma - 1) + \gamma - n^2 = 0$

$\gamma = \pm n$.

So $R(r) = C_1 r^n + C_2 r^{-n}$.

but R bounded as $r \to \infty \Rightarrow C_1 = 0$ So $R_n = r^{-n}$

Then by superposition

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^{-n}\{A_n \cos(n\theta) + B_n \sin(n\theta)\}$$

Find constants using BC $u_r(1,\theta) = f(\theta)$:

$$u_r(1,\theta) = \sum_{n=1}^{\infty} \{-nA_n \cos(n\theta) - nB_n \sin(n\theta)\}$$

complete Fourier series, with missing $n=0$ term!

so for there to be a solution, $\int_0^{2\pi} f(\theta)\,d\theta = 0$.

Assuming that's the case, $A_n = \frac{-1}{n\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta$

$B_n = \frac{-1}{n\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta$

Thus, assuming $\int_0^{2\pi} f(\theta)\,d\theta = 0$,

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^{-n}\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right]$$

where A_0 is <u>arbitrary</u>, $A_n = \frac{-1}{n\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta$

$B_n = \frac{-1}{n\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta.$

Question 8:

(a) $\mathcal{L}\phi = -(x^{-1}\phi')' = \lambda x^{-3}\phi \qquad 1 < x < 2$

EXPAND: $\quad x^2\phi'' - x\phi' + \lambda\phi = 0 \qquad \phi(1) = 0 = \phi(2) \qquad (3)$

THIS IS AN EULER EQ SO LET $\phi(x) = x^\gamma \Rightarrow \gamma(\gamma-1) - \gamma + \lambda = 0$

$$\gamma = \frac{+2 \pm \sqrt{4-4\lambda}}{2} = 1 \pm \sqrt{1-\lambda}$$

FOR NONTRIVIAL SOLUTIONS WE REQUIRE $\lambda > 1 \Rightarrow \gamma = 1 \pm i\sqrt{\lambda - 1} = 1 \pm i\beta$.

$$\therefore \quad \phi(x) = c_1 x^{1+i\beta} + c_2 x^{1-i\beta}$$

$$= x\left[A\cos(\beta \ln x) + B\sin(\beta \ln x)\right]$$

$$\phi(1) = A = 0 \qquad \phi(2) = 2B\sin(\beta \ln 2) = 0 \Rightarrow \beta_n = \frac{n\pi}{\ln 2} \quad n = 1, 2, \ldots$$

$$\therefore \lambda_n = 1 + \beta_n^2 = 1 + \left(\frac{n\pi}{\ln 2}\right)^2 \quad n = 1, 2, \ldots$$

THE EIGENFUNCTIONS ARE $\quad \phi_n(x) = x\sin(\beta_n \ln x)$

(b) ASSUME $\quad u(x,t) = X(x)T(t) \Rightarrow \dfrac{\dot{T}}{T(t)} = \dfrac{x^2 X'' - xX'}{X(x)} = -\lambda$ CONST

$$\therefore \dot{T}(t) = -\lambda T \Rightarrow T(t) = ce^{-\lambda t}$$

$$\left.\begin{array}{l} x^2X'' - xX' + \lambda X = 0 \\ X(1) = 0 = X(2) \end{array}\right\} \Rightarrow \lambda_n = 1 + \left(\frac{n\pi}{\ln 2}\right)^2 \quad n = 1, 2, \ldots \quad X_n = \phi_n = x\sin(\beta_n \ln x)$$

$$\therefore \quad u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t}\phi_n(x) = \sum_{n=1}^{\infty} c_n e^{-\left[1 + \left(\frac{n\pi}{\ln 2}\right)^2\right]t} x\sin\left(\frac{n\pi \ln x}{\ln 2}\right)$$

$$x = u(x,0) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

SINCE (2) IS A S-L PROBLEM WITH WEIGHT FUNCTION $\sigma(x) = x^{-3}$ WE PROJECT x ONTO $\phi_n(x)$:

$$\int_1^2 (x^{-3})\, x\, \phi_m(x)\,dx = \sum_{n=1}^{\infty} c_n \int_1^2 (x^{-3})\,\phi_m(x)\,\phi_n(x)\,dx$$

$$\int_1^2 \frac{x^2}{x^3}\sin\left(\frac{m\pi \ln x}{\ln 2}\right)dx = c_m \int_1^2 \frac{x^2}{x^3}\sin^2\left(\frac{m\pi}{\ln 2}\ln x\right)dx \quad \text{BY ORTHOGONALITY OF } \phi_m$$

$u = \ln x$, $du = \dfrac{dx}{x}$

$$\int_0^{\ln 2} \sin\left(\frac{m\pi u}{\ln 2}\right)du = \frac{c_m}{2}\int_0^{\ln 2}\left\{1 - \cos\left(\frac{2m\pi u}{\ln 2}\right)\right\}du = \frac{c_m}{2}\left[u - \frac{\sin\left(\frac{2m\pi u}{\ln 2}\right)}{(2m\pi)/\ln 2}\right]_0^{\ln 2}$$

$$\therefore \quad c_m = \frac{2}{\ln 2}\left[\frac{-\cos\left(\frac{m\pi u}{\ln 2}\right)}{(m\pi)/\ln 2}\right]_0^{\ln 2} = \frac{2\left[1 - (-1)^m\right]}{(m\pi)}$$