Math 257/316 Assignment 9 Supplemental Problems

Supplemental problems not to be handed in - solutions are provided.

Problem 1: Solve the following BVP in the rectangle $\{(x, y)\,;\, 0 \le x \le 1\,,\ 0 \le y \le 1\}$:

$$\begin{cases} u_{xx} + u_{yy} = 0 & 0 \le x \le 1\,,\ 0 \le y \le 1 \\ u_x(0, y) = \beta(1 - \cos(2\pi y)) \quad 0 \le y \le 1, & u(1, y) = 0 \quad 0 \le y \le 1 \\ u_y(x, 0) = 0 \quad 0 \le x \le 1, & u_y(x, 1) = 0 \quad 0 \le x \le 1 \end{cases}$$

(EXCEL Part) The sample spreadsheet `Laplace0.xls` solves Laplace's equation for the steady state temperature in a rectangular plate subject to Dirichelt boundary conditions. Now adapt the spread sheet to implement the mixed boundary conditions in Problem 1 (above). Use the iterate feature to ensure that the solution has reached a steady state. On a separate sheet evaluate the series solution and compare the result to the numerical solution by plotting both solutions $u(x, \frac{1}{2})$ along the line $0 \le x \le 1$ on the same graph.

1. Set the constant $\beta = 1$ and provide the value of $u(\frac{1}{2}, \frac{1}{2})$ to 6 digits.

2. Now set $\beta = \dfrac{\text{your UBC student \#}}{10^6}$ and provide the value of $u(\frac{1}{2}, \frac{1}{2})$ to 6 digits.

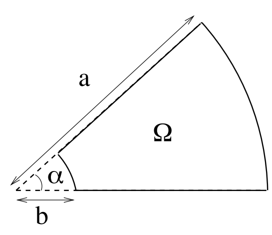
Problem 2: Find the solution of Laplace's equation in the semi-infinite strip $\{(x, y) : 0 \le x \le \pi\,,\ y \ge 0\}$ satisfying the following boundary conditions:

$$\begin{cases} u(0, y) = 0, \qquad u_x(\pi, y) = 0 \quad \text{for all } y \ge 0 \\ u(x, 0) = \begin{cases} x & \text{for all } 0 \le x \le \pi/2 \\ 0 & \text{for all } \pi/2 < x \le \pi \end{cases} \\ \lim_{y \to +\infty} u(x, y) = 0 \end{cases}$$

Problem 3: A piece of aluminium foil occupies a semi-annular region $0 < a \le r \le b$ and $0 \le \theta \le \pi$. The faces and the flat ends are insulated, while the inner hoop is maintained at a temperature of 0 degrees and the outer hoop at a temperature $1 + \cos(2\theta)$. Determine the steady state temperature by solving the following BVP in Ω:

$$\begin{cases} v_{rr} + \dfrac{1}{r}v_r + \dfrac{1}{r^2}v_{\theta\theta} = 0 & \text{in } \Omega \\ v_\theta(r, 0) = 0 \quad \text{for } a < r < b, & v_\theta(r, \pi) = 0 \quad \text{for } a < r < b \\ v(a, \theta) = 0 \quad \text{for } 0 < \theta < \pi, & v(b, \theta) = 1 + \cos(2\theta) \quad \text{for } 0 < \theta < \pi \end{cases}$$

Problem 4: Consider a domain Ω obtained by taking a circular sector with angle α and radius a and cutting out a smaller circular sector of radius b:



Find the solution of the following BVP in Ω:

$$\begin{cases} v_{rr} + \dfrac{1}{r}v_r + \dfrac{1}{r^2}v_{\theta\theta} = 0 & \text{in } \Omega \\ v(r, 0) = 0 \quad \text{for } b < r < a, & v(r, \alpha) = 0 \quad \text{for } b < r < a \\ v(b, \theta) = 0 \quad \text{for } 0 < \theta < \alpha, & v(a, \theta) = f(\theta) \quad \text{for } 0 < \theta < \alpha \end{cases}$$

Problem 5: For each of the following eigenvalue problems, determine the form of the eigenfunctions and the nonzero eigenvalues or the equation satisfied by the eigenvalues. For (a) and (b) determine whether $\lambda = 0$ is an eigenvalue and estimate λ_n for large values of n.

a) $y'' + \lambda y = 0 \qquad y'(0) - y(0) = 0 \quad y'(1) = 0$

b) $y'' + \lambda y = 0 \qquad y'(0) - y(0) = 0 \quad y(1) + y'(1) = 0$

c) $(xy')' + \frac{1}{x}\lambda y = 0, \qquad 0 < a < x < b,\ y(a) = 0 = y(b)$

Problem 6: Find a solution to the following Dirichlet problem in the half annulus:

$$\begin{aligned} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= 0,\ \pi < r < 2\pi,\ 0 < \theta < \pi, \\ u(r,0) = \sin(r),\ u(r,\pi) &= 0,\ \pi \le r \le 2\pi \\ u(\pi,\theta) = u(2\pi,\theta) &= 0,\ 0 \le \theta \le \pi. \end{aligned}$$

Problem 7: Find a solution to the following Neumann problem for an exterior domain:

$$\begin{aligned} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= 0,\ r > 1,\ 0 < \theta < 2\pi, \\ \frac{\partial u}{\partial r}(1,\theta) &= f(\theta),\ 0 \le \theta \le 2\pi \\ u(r,\theta)\ \text{remains bounded as } r \to \infty. \end{aligned}$$

Problem 8: Consider the following boundary value problem:

$$-\left(x^{-1}\phi'\right)' = \lambda x^{-3}\phi, \qquad 1 < x < 2 \tag{1}$$

$$\phi(1) = 0 \quad \text{and} \quad \phi(2) = 0 \tag{2}$$

(a) Determine the eigenvalues and eigenfunctions associated with the Euler equation (1) and the boundary conditions (2).

(b) Use these eigenvalues and eigenfunctions to solve the heat conduction problem:

$$\begin{aligned} u_t &= x^2 u_{xx} - x u_x,\ 1 < x < 2,\ t > 0 \\ u(1,t) &= 0 \text{ and } u(2,t) = 0 \\ u(x,0) &= x, \qquad 1 < x < 2 \end{aligned}$$