M257/316 MIDTERM 1 SECTION 101 OCT 2017 SOLUTIONS

1. $Ly = 2x^2 y'' + 3x y' + (2x-1) y = 0$

a) $0 < x < \infty$ ARE ORDINARY POINTS

$x=0$ IS A SINGULAR POINT $\lim_{x\to 0} x \frac{(3x)}{2x^2} = \frac{3}{2} = p_0 < \infty$ & $\lim_{x\to 0} \frac{x^2(2x-1)}{2x^2} = \frac{-1}{2} = q_0$ $|q_0|, |p_0| < \infty \Rightarrow$ RSP

CONSIDER $L_0 y = x^2 y'' + \frac{3}{2} x y' - \frac{y}{2} = 0$ C-E $\Rightarrow y = x^r \Rightarrow r(r-1) + \frac{3}{2} r - \frac{1}{2} = 0$ OR $2r^2 + r - 1 = (2r-1)(r+1) = 0$

THUS $r = 1/2$ AND $r = -1$ ARE THE ROOTS OF THE INDICIAL EQ

b) ABOUT $x_0 = -1$, WHICH IS AN ORDINARY POINT ASSUME $y(x) = \sum_{n=0}^{\infty} a_n (x+1)^n$

THE DISTANCE FROM $x_0 = -1$ TO $x = 0$ THE NEAREST SINGULAR POINT IS $|-1-0| = 1$ SO $\rho \geq 1$.

c) ABOUT THE RSP $x=0$ ASSUME $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, $y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$

$$Ly = 2x^2 y'' + 3x y' - y + 2x y = 0$$

$$= \sum_{n=0}^{\infty} 2a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} 3a_n (n+r) x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} 2a_n x^{n+r+1}$$

$m = n$ $\quad m = n$ $\quad m = n$ $\quad m = n+1 \Rightarrow n = m-1$ $\quad$ ~~$n=0 \Rightarrow m=1$~~

$$= a_0 \{2r(r-1) + 3r - 1\} x^r + \sum_{m=1}^{\infty} \left(a_m [(m+r)\{2(m+r-1)+3\} - 1] + 2a_{m-1} \right) x^{m+r} = 0$$

$x^r]$ $2r^2 + r - 1 = (2r-1)(r+1) = 0$ $r = 1/2, -1$ AS BEFORE

$x^{m+r}_{m\geq 1}]$
$$a_m = \frac{-2a_{m-1}}{(m+r)\{2(m+r)+1\} - 1}$$

$r = 1/2 \Rightarrow$
$$a_m = \frac{-2a_{m-1}}{(m+1/2)\{2(m+1)\} - 1} = \frac{-2a_{m-1}}{(2m+1)(m+1) - 1} = \frac{-2a_{m-1}}{2m^2 + 3m + 1 - 1} = \frac{-2a_{m-1}}{m(2m+3)}$$

$$a_1 = \frac{-2a_0}{1.5} = \frac{-2a_0}{5} \qquad a_2 = \frac{-\cancel{2}a_1}{\cancel{2}.7} = +\frac{2a_0}{35}$$

$$y(x) = a_0 x^{1/2} [1 - 2x/5 + \frac{2x^2}{35} - \dots]$$

$r = -1 \Rightarrow$
$$a_m = \frac{-2a_{m-1}}{(m-1)(2m-1) - 1} = \frac{-2a_{m-1}}{2m^2 - 3m + \cancel{1} - \cancel{1}} = \frac{-2a_{m-1}}{m(2m-3)}$$

$$a_1 = \frac{-2a_0}{1.(-1)} = 2a_0 \qquad a_2 = \frac{-2a_1}{2(1)} = -2a_0 =$$

$$y(x) = a_0 x^{-1} [1 + 2x - 2x^2 + \dots]$$

SINCE THERE ARE NO OTHER SINGULARITIES FOR FINITE x THE RADIUS OF CONVERGENCE IS INFINITE

2. $u_t = u_{xx} + \gamma^2 u \qquad 0 < x < \pi/2 \quad t > 0$

$u(0,t) = 0 = u_x(\pi/2, t)$

$u(x,0) = \sin 3x.$

LET $u(x,t) = X(x)\,T(t) \Rightarrow u_t = X\dot{T} = u_{xx} + \gamma^2 u = X''T + \gamma^2 XT$

$\div XT$ AND REARRANGE: $\dfrac{\dot{T}(t)}{T(t)} - \gamma^2 = \dfrac{X''(x)}{X(x)} = \text{CONST} = \lambda$ SAY

T] $\dot{T} = (\lambda + \gamma^2)T \qquad T(t) = C\,e^{(\lambda+\gamma^2)t}$

X] $X'' = \lambda X, \quad X(0) = 0 = X'(\pi/2)$

$\lambda > 0$: SAY $\lambda = \mu^2 \Rightarrow X'' - \mu^2 X = 0 \qquad X = A\cosh\mu x + B\sinh\mu x$

$X' = A\mu\sinh\mu x + B\mu\cosh\mu x$

$0 = X(0) = A \qquad 0 = X'(\pi/2) = B\mu\cosh\mu\pi/2 \overset{\mu\neq 0}{\Rightarrow} B = 0 \Rightarrow X \equiv 0$ TRIVIAL SOL'N

$\lambda = 0$: $X'' = 0 \qquad X' = A \qquad X = Ax + B$

$0 = X(0) = B, \; 0 = X'(\pi/2) = A \qquad \Rightarrow X \equiv 0$ THE TRIVIAL SOL'N

$\lambda < 0$: SAY $\lambda = -\mu^2 \Rightarrow X'' + \mu^2 X = 0 \qquad X = A\cos\mu x + B\sin\mu x$

$X' = -A\mu\sin\mu x + B\mu\cos\mu x$

$0 = X(0) = A$

$0 = X'(\pi/2) = B\mu\cos\mu\frac{\pi}{2} \overset{\mu\neq 0}{\Rightarrow} \mu_n\pi/2 = (2n-1)\pi/2 \quad n = 1, 2, \ldots$

THUS $\lambda_n = -\mu_n^2 = (2n-1)^2 \quad n = 1, 2, \ldots$ ARE THE EIGENVALUES AND

$X_n(x) = \sin(2n-1)x$ THE CORRESPONDING EIGENFUNTIONS

THUS $u(x,t) = \sum_{n=1}^{\infty} b_n\, e^{-(2n-1)^2 t}\, e^{\gamma^2 t} \sin(2n-1)x$

NOW $\sin 3x = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(2n-1)x$

SINCE THE X_n ARE LINEARLY INDEPENDENT WE MAY EQUATE COEFFICIENTS

TO OBTAIN $b_n = \begin{cases} 0 & n \neq 2 \\ 1 & n = 2 \end{cases}$

$\therefore \quad u(x,t) = e^{-9t}\, e^{\gamma^2 t} \sin 3x$