MATH 257/316 MIDTERM 1 SECTION 101 OCT 2018

1 $\quad Ly = 9x^2y'' + (x+2)y = 0$

a) $X > 0$ ARE ALL ORDINARY POINTS

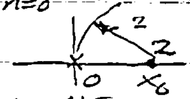
$X = 0$ IS A SINGULAR POINT

$\lim_{x\to 0} x\frac{0}{9x^2} = 0 = p_0 \quad \lim_{x\to 0} x^2\frac{(x+2)}{9x^2} = \frac{2}{9} = q_0$ SINCE $|p_0| \,\&\, |q_0| < \infty$ $x_0 = 0$ IS A RSP.

INDICIAL EQ: $r(r-1) + 0\cdot r + 2/9 = 0$ OR $9r^2 - 9r + 2 = (3r-1)(3r-2) = 0 \Rightarrow r = 2/3$ OR $1/3$.

b) SINCE $x_0 = 2$ IS AN ORDINARY POINT ASSUME A TAYLOR SERIES $y(x) = \sum_{n=0}^{\infty} a_n (x-2)^n$

SINCE THE NEAREST SP IS AT $x_0 = 0$ THE RADIUS OF CONVG. $\rho \geq 2$.



c) LET $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ $\quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$ $\quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$

$$Ly = 9x^2 y'' + 2y + xy = 0$$

$$= \sum_{n=0}^{\infty} 9a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} 2a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$m = n \qquad m = n \qquad m = n+1 \Rightarrow n = m-1 \quad n = 0 \Rightarrow m = 1.$

$$= \sum_{m=0}^{\infty} [9(m+r)(m+r-1)+2] a_m x^{m+r} + \sum_{m=1}^{\infty} a_{m-1} x^{m+r} = 0$$

$$= \{9r(r-1)+2\} a_0 x^r + \sum_{m=1}^{\infty} \{[9(m+r)(m+r-1)+2] a_m + a_{m-1}\} x^{m+r} = 0$$

$x^r]$ $\quad 9r^2 - 9r + 2 = (3r-1)(3r-2) = 0 \quad r = 2/3$ OR $1/3$ AS BEFORE

$x^{m+r}]$ $\quad a_m = \dfrac{-a_{m-1}}{9(m+r)(m+r-1)+2} \qquad m \geq 1$

$r = 2/3$: $\quad a_m = \dfrac{-a_{m-1}}{9(m+2/3)(m-1/3)+2} = \dfrac{-a_{m-1}}{(3m+2)(3m-1)+2} = \dfrac{-a_{m-1}}{9m^2+3m-\cancel{2}+\cancel{2}} = \dfrac{-a_{m-1}}{3m(3m+1)}$

$a_1 = \dfrac{-a_0}{3.1.4} = \dfrac{-a_0}{12} \qquad a_2 = \dfrac{-a_1}{3.2.(7)} = \dfrac{-a_1}{42} = +\dfrac{a_0}{504}$

$$y_1(x) = a_0 x^{2/3}[1 - x/12 + x^2/504 - \cdots]$$

$r = 1/3$: $\quad a_m = \dfrac{-a_{m-1}}{9(m+1/3)(m-2/3)+2} = \dfrac{-a_{m-1}}{(3m+1)(3m-2)+2} = \dfrac{-a_{m-1}}{9m^2-3m-\cancel{2}+\cancel{2}} = \dfrac{-a_{m-1}}{3m(3m-1)}$

$a_1 = \dfrac{-a_0}{3.1.2} = \dfrac{-a_0}{6} \qquad a_2 = \dfrac{-a_1}{3.2(5)} = \dfrac{-a_1}{30} = +\dfrac{a_0}{180}$

$$y_2(x) = a_0 x^{1/3}[1 - x/6 + x^2/180 - \cdots]$$

SINCE THERE ARE NO OTHER SINGULAR POINTS $|x_0| < \infty$ THE RADIUS OF CONVERGENCE ρ IS INFINITE. - ie THE SERIES CONVERGES FOR ALL FINITE X.

Q2. $u_t = u_{xx} - \gamma u \qquad -\pi < x < \pi \quad t > 0$

$u(-\pi,t) = u(\pi,t) \quad u_x(-\pi,t) = u_x(\pi,t)$

$u(x,0) = \cos 2x$.

LET $u(x,t) = X(x)T(t)$

$u_t = X(x)\dot{T}(t) = X''(x)T(t) - \gamma X(x)T(t) = u_{xx} - \gamma u$

$\div XT$ AND REARRANGE: $\frac{\dot{T}(t)}{T(t)} + \gamma = \frac{X''(x)}{X(x)} = \text{CONST} = \lambda$ SAY

T] $\dot{T} = (\lambda - \gamma)T \Rightarrow T(t) = Ce^{(\lambda-\gamma)t}$

X] i) $\lambda = \mu^2 > 0$: $X'' - \mu^2 X = 0 \qquad X(-\pi) = X(\pi) \qquad X'(-\pi) = X'(\pi)$

$X = A\cosh\mu x + B\sinh\mu x \qquad X' = A\mu\sinh\mu x + B\mu\cosh\mu x$

$X(-\pi) = A\cosh\mu\pi - B\sinh\mu\pi = X(\pi) = A\cosh\mu\pi + B\sinh\mu\pi \Rightarrow 2B\sinh\mu\pi = 0 \overset{\mu\neq 0}{\Rightarrow} B = 0$

$X'(-\pi) = -A\mu\sinh\mu\pi \ldots = X'(\pi) = A\mu\sinh\mu\pi \Rightarrow 2A\mu\sinh\mu\pi = 0 \overset{\mu\neq 0}{\Rightarrow} A = 0$

THUS WE ONLY GET THE TRIVIAL SOL'N

ii) $\lambda = 0 \quad X'' = 0 \quad X' = B \quad X = Bx + A1$

$X(-\pi) = -B\pi + A = X(\pi) = B\pi + A \Rightarrow B = 0 \qquad X'(-\pi) = B = 0 = X'(\pi)$ ✓

THUS $\lambda_0 = 0$ IS AN EIGENVALUE & $X_0 = 1$ IS THE EIGENFCN.

iii) $\lambda = -\mu^2 < 0$: $X'' + \mu^2 X = 0 \qquad X(-\pi) = X(\pi) \quad X'(-\pi) = X'(\pi)$

$X = A\cos\mu x + B\sin\mu x \qquad X' = -A\mu\sin\mu x + B\mu\cos\mu x$

$A \& B \neq 0$

$$\left.\begin{aligned} X(-\pi) &= A\cos\mu\pi - B\sin\mu\pi = X(\pi) = A\cos\mu\pi + B\sin\mu\pi \Rightarrow 2B\sin\mu\pi = 0 \\ X'(-\pi) &= A\mu\sin\mu\pi + B\mu\cos\mu\pi = X'(\pi) = -A\mu\sin\mu\pi + B\mu\cos\mu\pi \Rightarrow 2A\sin\mu\pi = 0 \end{aligned}\right\} \Rightarrow \mu_n\pi = n\pi, \ \mu_n = n, \ n = 1, 2, \ldots$$

EIGENVALUES ARE $\lambda_n = -n^2 \quad n = 1, 2, \ldots$ & EIGENFCNS $X_n \in \{\sin(nx), \cos nx\}$

$$\therefore u(x,t) = A_0 1 + \sum_{n=1}^{\infty} \{A_n\cos(nx) + B_n\sin nx\} e^{-(n^2+\gamma)t}$$

$$\cos 2x = u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n\cos(nx) + B_n\sin(nx)$$

SINCE 1, $\cos(nx)$ & $\sin(nx)$ ARE LINEARLY INDEPENDENT $A_n = \begin{cases} 0 & n \neq 2 \\ 1 & n = 2 \end{cases}$

AND $B_n = 0$.

$\therefore \quad u(x,t) = \cos(2x)\, e^{-(4+\gamma)t}$