MATH 257/316 MIDTERM 1 SECTION 102 OCT 2013

1. $Ly = 2xy'' - y' + 2y = 0 \qquad P = 2x \quad Q = -1 \quad R = 2.$

a) $X > 0$ ARE ALL ORDINARY POINTS & $X = 0$ IS A SINGULAR POINT

$X_0 = 0. \ \lim_{x\to 0} x\left(\frac{-1}{2x}\right) = -\frac{1}{2} = p_0$ FINITE, $\lim_{x\to 0} x^2\left(\frac{2}{2x}\right) = 0 \Rightarrow X = 0$ IS A REGULAR SP.

INDICIAL EQ $\quad r(r-1) - \frac{1}{2}r = 0 \quad 2r^2 - 3r = r(2r-3) = 0 \quad r = 0, 3/2$

b) SINCE $x_0 = -1$ IS AN ORDINARY POINT WE ASSUME

$$y(x) = \sum_{n=0}^{\infty} a_n (x+1)^n$$

THE NEAREST SINGULAR POINT IS $X = 0$ SO $\rho \geq |-1-0| = 1$

c) SINCE $X = 0$ IS A RSP LET $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, $y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$

$Ly = 2xy''$

$$= \sum_{n=0}^{\infty} 2a_n (n+r)(n+r-1) x^{n+r-1} - \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} + \sum_{n=0}^{\infty} 2a_n x^{n+r}$$

$m = n \qquad\qquad m = n \qquad\qquad m-1 = n \Rightarrow n = 0 \Rightarrow m = 1$

$$= \sum_{m=0}^{\infty} a_m (m+r)[2(m+r-1) - 1] x^{m+r-1} + \sum_{m=1}^{\infty} 2a_{m-1} x^{m+r-1}$$

$$= a_0 r(2r-3) x^{r-1} + \sum_{m=1}^{\infty} \{a_m (m+r)[2(m+r) - 3] + 2a_{m-1}\} x^{m+r-1} = 0$$

$x^{r-1} > a_0 r(2r-3) = 0 \quad r = 0, 3/2.$ (SAME AS ABOVE)

$x^{m+r-1} > \quad a_m = \dfrac{-2a_{m-1}}{(m+r)[2(m+r)-3]} \qquad m \geq 1.$

$\underline{r = 3/2}:$ $\quad a_m = \dfrac{-2a_{m-1}}{(m+3/2)(2m+\cancel{3}-\cancel{3})} = \dfrac{-2a_{m-1}}{m(2m+3)}$

$a_1 = \dfrac{-2a_0}{1.5} \qquad a_2 = \dfrac{-\cancel{2}a_1}{\cancel{2}(7)} = \dfrac{+a_0}{35}$

$y_1(x) = a_0 x^{3/2}[1 - x/5 + x^2/35 - \cdots]$

$\underline{r = 0}:$ $\quad a_m = \dfrac{-2a_{m-1}}{m(2m-3)}$

$a_1 = \dfrac{-2a_0}{1.(-1)} = 2a_0 \qquad a_2 = \dfrac{-2a_1}{2(4-3)} = -2a_0$

$y_2(x) = a_0 x^0 [1 + 2x - 2x^2 + \cdots]$

Q2. $u_t = u_{xx} \quad 0<x<\pi, \; t>0$

BC $\frac{\partial u(0,t)}{\partial x} = 0 = u(\pi,t)$

IC: $u(x,0) = 1$

LET $u(x,t) = X(x)T(t)$

$\therefore \quad \frac{\dot{T}(t)}{T(t)} = \frac{X''}{X} = -\lambda^2$

T EQ] $\dot{T} = -\lambda^2 T \Rightarrow T = Ce^{-\lambda^2 t}$.

X EQ] $\lambda \neq 0 \quad X'' + \lambda^2 X = 0, \quad X'(0) = 0 = X(\pi)$

$X = A\cos\lambda x + B\sin\lambda x \qquad X' = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$

$0 = X'(0) = B.\lambda.1 \overset{\lambda\neq 0}{\Rightarrow} B = 0$

$0 = X(\pi) = A\cos\lambda\pi \Rightarrow \lambda_n \cancel{\pi} = (2n+1)\cancel{\pi}/2, \; n=0,1,\dots$

$\therefore$ THE EIGENVALUES ARE $\lambda_n = \frac{(2n+1)}{2}$, $n = 0,1,2,\dots$ AND THE CORRESPONDING EIGENFUNCTIONS ARE $X_n = \cos\left(\frac{(2n+1)x}{2}\right)$

$\lambda = 0$: $X'' = 0 \Rightarrow X' = A, \; X = Ax + B$

$0 = X'(0) = A, \; 0 = X(\pi) = B \Rightarrow X \equiv 0$ ONLY THE TRIVIAL SOLUTION.

THUS $\quad u(x,t) = \sum_{n=0}^{\infty} a_n \cos[(2n+1)x/2]\, e^{-\left(\frac{2n+1}{2}\right)^2 t}$

IC: $\quad 1 = u(x,0) = \sum_{n=0}^{\infty} a_n \cos[(2n+1)x/2].1$

WHERE $\quad a_n = \frac{2}{\pi}\int_0^{\pi} 1.\cos[(2n+1)x/2]\,dx$

$= \frac{2}{\pi} \left.\frac{\sin[(2n+1)x/2]}{(2n+1)/2}\right]_0^{\pi}$

n	0	1	2	3
$\sin\left(\frac{(2n+1)\pi}{2}\right)$	1	-1	1	-1

$= \frac{4}{\pi} \frac{\sin[(2n+1)\pi/2]}{(2n+1)}$

$= \frac{4}{\pi} \frac{(-1)^n}{(2n+1)} \quad n \geq 0.$

$\therefore \; u(x,t) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin[(2n+1)\pi/2]}{(2n+1)} \cos\left(\frac{(2n+1)x}{2}\right) e^{-\left(\frac{2n+1}{2}\right)^2 t}$

$= \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos\left(\frac{(2n+1)x}{2}\right) e^{-\left(\frac{2n+1}{2}\right)^2 t}.$