MATH 257/316 MIDTERM 1 SECTION 102

1. $Ly = 6x^2 y'' + 7xy' + (x-1)y = 0$

a) $0 < x < \infty$ ARE ALL ORDINARY POINTS

$x = 0$ IS A SINGULAR POINT

$\lim_{x\to 0} x \frac{7x}{6x^2} = \frac{7}{6} = p_0$ $\quad \lim_{x\to 0} x^2 \frac{(x-1)}{6x^2} = -\frac{1}{6} = q_0$ $\quad |p_0| \& |q_0| < \infty$ IMPLY $x = 0$ IS A REGULAR S.P.

INDICIAL EQ: $r(r-1) + \frac{7}{6}r - \frac{1}{6} = 0 \Rightarrow 6r^2 + r - 1 = (3r-1)(2r+1) = 0,\ r = -\frac{1}{2}, +\frac{1}{3}$

b) WE WOULD ASSUME $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$. SINCE $x=1$ IS AN ORDINARY POINT. THE RADIUS OF CONVERGENCE ρ IS AT LEAST AS LARGE AS THE DIST TO THE NEAREST SINGULAR POINT I.E. $\rho \geq |0-1| = 1$

c) $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ $\quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$ $\quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$

$$Ly = 6x^2 y'' + 7xy' - y + xy$$

$$= \underbrace{\sum_{n=0}^{\infty} 6a_n (n+r)(n+r-1) x^{n+r}}_{m=n} + \underbrace{\sum_{n=0}^{\infty} 7a_n (n+r) x^{n+r}}_{m=n} - \underbrace{\sum_{n=0}^{\infty} a_n x^{n+r}}_{m=n} + \underbrace{\sum_{n=0}^{\infty} a_n x^{n+r+1}}_{m=n+1\ \ n=m-1}$$

$n=0 \Rightarrow m=1$

$$= \sum_{m=0}^{\infty} \left[(m+r)[6(m+r-1)+7] - 1\right] a_m x^{m+r} + \sum_{m=1}^{\infty} a_{m-1} x^{m+r}$$

$$= (6r^2 + r - 1) a_0 x^r + \sum_{m=1}^{\infty} \left[\{(m+r)[6(m+r)+1] - 1\} a_m + a_{m-1}\right] x^{m+r} = 0$$

$x^r]$ $\quad 6r^2 + r - 1 = (3r-1)(2r+1) = 0 \quad r = -\frac{1}{2}, \frac{1}{3}$ AS ABOVE

$x^{m+r}_{m\geq 1}]$ $\quad a_m = \dfrac{-a_{m-1}}{(m+r)[6(m+r)+1]-1}$

$\underline{r = 1/3}$: $a_m = \dfrac{-a_{m-1}}{(m+1/3)(6m+3)-1} = \dfrac{-a_{m-1}}{(3m+1)(2m+1)-1} = \dfrac{-a_{m-1}}{6m^2+5m+\cancel{1}-\cancel{1}} = \dfrac{-a_{m-1}}{m(6m+5)}$

$a_1 = \dfrac{-a_0}{11} \quad a_2 = \dfrac{-a_1}{2.17} = \dfrac{+a_0}{22.17}, \ldots$

$$y_1(x) = x^{1/3}\left[1 - x/11 + x^2/22.17 - \cdots\right]$$

$\underline{r = -1/2}$: $a_m = \dfrac{-a_{m-1}}{(m-1/2)[6m-2]-1} = \dfrac{-a_{m-1}}{(2m-1)(3m-1)-1} = \dfrac{-a_{m-1}}{6m^2-5m+\cancel{1}-\cancel{1}} = \dfrac{-a_{m-1}}{m(6m-5)}$

$a_1 = \dfrac{-a_0}{1} \quad a_2 = \dfrac{-a_1}{2.7} = \dfrac{a_0}{14}, \ldots$

$$y_2(x) = x^{-1/2}\left[1 - x + \frac{x^2}{14} - \cdots\right]$$

2. $u_t = u_{xx} \qquad -\pi < x < \pi$

BC : $u(-\pi,t) = u(\pi,t) \quad u_x(-\pi,t) = u_x(\pi,t)$

IC: $u(x,0) = x$

LET $u(x,t) = X(x)T(t)$

$X(x)\dot{T}(t) = X''(x)T(t)$

$\dot{T}/T(t) = X''(x)/X(x) = -\lambda^2$

T] $\dot{T} = -\lambda^2 T \Rightarrow T(t) = C e^{-\lambda^2 t}$

X] $\lambda \neq 0$: $\left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ X(-\pi) = X(\pi) \quad X'(-\pi) = X'(\pi) \end{array}\right\} \begin{array}{l} X(x) = A\cos\lambda x + B\sin\lambda x \\ X'(x) = -A\lambda \sin\lambda x + B\lambda\cos\lambda x \end{array}$

$X(-\pi) = A\cancel{\cos(\lambda\pi)} - B\sin(\lambda\pi) = X(\pi) = A\cancel{\cos(\lambda\pi)} + B\sin(\lambda\pi)$

$\Rightarrow \quad 2B\sin(\lambda\pi) = 0$

$X'(-\pi) = A\lambda\sin\lambda\pi + B\lambda\cancel{\cos\lambda\pi} = X'(\pi) = -A\lambda\sin\lambda\pi + B\lambda\cancel{\cos\lambda\pi}$

$\therefore \quad 2A\lambda\sin\lambda\pi = 0$

SINCE $\lambda \neq 0$ & A & B ARE NOT BOTH ZERO WE HAVE $\lambda\pi = n\pi \quad n = 1, 2 \ldots$

$\therefore \lambda_n = n \quad n = 1, 2 \ldots \qquad X_n \in \{\cos nx, \sin nx\}$

$\lambda = 0$: $\quad X'' = 0 \quad X' = B_0 \quad X = B_0 x + A_0$

$X'(-\pi) = B_0 = X'(\pi) \qquad X(-\pi) = -B_0\pi + \cancel{A_0} = X(\pi) = B_0\pi + \cancel{A_0} \Rightarrow B_0 = 0$

$\therefore \quad X_0 = 1$ IS AN EIGENFUNCTION

$\therefore \quad \lambda_0 = 0$ IS AN EIGENFUNCTION.

$\therefore \; u(x,t) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] e^{-n^2 t}$

$x = u(x,0) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$

$a_n = 0$ SINCE x IS ODD

$b_n = \dfrac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx = \dfrac{2}{\pi}\left[\left.\dfrac{-x\cos nx}{n}\right|_0^{\pi} + \dfrac{1}{n}\int_0^{\pi} 1 \cos nx\,dx\right]$

$= \dfrac{2}{\cancel{\pi}}\cancel{\pi}\dfrac{(-1)^{n+1}}{n} + \dfrac{2}{\pi n^2}\left.\sin nx\right|_0^{\pi} = 2\dfrac{(-1)^{n+1}}{n}$

$\therefore \; u(x,t) = 2\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n} \sin(nx)\, e^{-n^2 t}.$