M257/316 MIDTERM 1 SECTION 102 OCT 2017 SOLUTIONS

1. $Ly = 2x^2y'' - x(x-1)y' - y = 0$

a) • $0 < x < \infty$ ARE ALL ORDINARY POINTS

• $x = 0$ IS A SINGULAR POINT $\lim_{x\to 0} x\frac{[-x(x-1)]}{2x^2} = +\frac{1}{2} = p_0$; $\lim_{x\to 0} x^2\frac{(-1)}{2x^2} = -\frac{1}{2} = q_0$, $|p_0|, |q_0| < \infty \Rightarrow x=0$ IS A RSP

C-E $L_0 y = x^2y'' + \frac{x}{2}y' - \frac{y}{2} = 0 \Rightarrow y = x^r \Rightarrow r(r-1) + \frac{r}{2} - \frac{1}{2} = 0 \quad 2r^2 - r - 1 = (2r+1)(r-1) = 0 \Rightarrow r = \frac{-1}{2}, 1$

b) SINCE $x_0 = 2$ IS AN ORDINARY POINT ASSUME $y(x) = \sum_{n=0}^{\infty} a_n (x-2)^n$

SINCE THE DISTANCE BETWEEN $x_0 = 2$ AND THE NEAREST SINGULAR POINT AT $x=0$ IS

$|2-0| = 2$ THE MINIMAL RADIUS OF CONVERGENCE IS $\rho \geq 2$.

c) SINCE $x=0$ IS A RSP ASSUME $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, $y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$

$$Ly = 2x^2y'' + xy' - x^2y' - y = 0$$

$$= \underbrace{\sum_{n=0}^{\infty} 2a_n(n+r)(n+r-1)x^{n+r}}_{n=m} + \underbrace{\sum_{n=0}^{\infty} a_n(n+r)x^{n+r}}_{n=m} - \underbrace{\sum_{n=0}^{\infty} a_n(n+r)x^{n+r+1}}_{m=n+1,\ n=m-1,\ n=0 \Rightarrow m=1} - \underbrace{\sum_{n=0}^{\infty} a_n x^{n+r}}_{n=m} = 0$$

$$= a_0[2r(r-1) + r - 1]x^r + \sum_{m=1}^{\infty}\left(a_m[(m+r)\{2(m+r-1)+1\} - 1] - a_{m-1}(m+r-1)\right)x^{m+r} = 0$$

$x^r]$ $\quad 2r^2 - r - 1 = (2r+1)(r-1) = 0 \quad r = -1/2, 1$

$x^{m+r}_{m\geq 1}]$
$$a_m = \frac{a_{m-1}(m+r-1)}{(m+r)(2(m+r)-1) - 1}$$

$r = -1/2$: $a_m = \frac{(m-3/2)a_{m-1}}{(m-1/2)[2(m-1)]-1} = \frac{(m-3/2)a_{m-1}}{(2m-1)(m-1)-1} = \frac{(m-3/2)a_{m-1}}{2m^2-3m+\cancel{1}-\cancel{1}} = \frac{\cancel{(2m-3)}a_{m-1}}{2m\cancel{(2m-3)}}$

$$a_1 = \frac{a_0}{2} \qquad a_2 = \frac{a_1}{4} = \frac{a_0}{8}$$

$$y(x) = a_0 x^{-1/2}[1 + x/2 + x^2/8 + \cdots]$$

$r = 1$: $a_m = \frac{a_{m-1}(m+1-1)}{(m+1)[2(m+1)-1]-1} = \frac{a_{m-1}m}{(m+1)(2m+1)-1} = \frac{m a_{m-1}}{2m^2+3m+\cancel{1}-\cancel{1}} = \frac{\cancel{m} a_{m-1}}{\cancel{m}(2m+3)} = \frac{a_{m-1}}{(2m+3)}$

$$a_1 = \frac{a_0}{5} \qquad a_2 = \frac{a_1}{7} = \frac{a_0}{35}$$

$$y(x) = a_0 x[1 + x/5 + x^2/35 + \cdots]$$

SINCE ALL OTHER POINTS $|x_0| < \infty$ ARE ORDINARY POINTS, THE RADIUS OF CONVERGENCE = IS INFINITE.

2. $u_t = u_{xx} + \gamma^2 u \qquad 0 < x < \pi/2, \ t > 0$

$u_x(0,t) = 0 = u(\pi/2, t)$

$u(x,0) = \cos 5x$.

LET $u(x,t) = X(x)T(t)$

$\Rightarrow \quad u_t = X\dot{T} = u_{xx} + \gamma^2 u = X''T + \gamma^2 XT$

$\div XT$ AND REARRANGE $\quad \dfrac{\dot{T}(t)}{T(t)} - \gamma^2 = \dfrac{X''(x)}{X(x)} = \text{CONSTANT} = \lambda$

$T] \quad \dot{T} = (\lambda + \gamma^2)T \Rightarrow T(t) = C e^{(\lambda+\gamma^2)t}$

$X] \quad X'' = \lambda X \qquad X'(0) = 0 = X(\pi/2)$.

$\lambda > 0$, $\lambda = \mu^2$ SAY $\Rightarrow \left. \begin{array}{l} X'' - \mu^2 X = 0 \\ X'(0) = 0 = X(\pi/2) \end{array} \right\} \begin{array}{l} X = A\cosh\mu x + B\sinh\mu x \\ X' = A\mu\sinh\mu x + B\mu\cosh\mu x. \end{array}$

$\left. \begin{array}{l} 0 = X'(0) = B\mu \overset{\mu \neq 0}{\Rightarrow} B = 0 \\ 0 = X(\pi/2) = A\cosh(\mu\pi/2) \Rightarrow A = 0 \end{array} \right\} \ X \equiv 0$ THE TRIVIAL SOL'N

$\underline{\lambda = 0}$: $X'' = 0 \quad X' = A \quad X = Ax + B$

$0 = X'(0) = A, \ 0 = X(\pi/2) = B \Rightarrow X \equiv 0$ THE TRIVIAL SOL'N

$\underline{\lambda < 0}$, $\lambda = -\mu^2$ SAY $\Rightarrow \left. \begin{array}{l} X'' + \mu^2 X = 0 \\ X'(0) = 0 = X(\pi/2) \end{array} \right\} \begin{array}{l} X = A\cos\mu x + B\sin\mu x \\ X' = -A\mu\sin\mu x + B\mu\cos\mu x \end{array}$

$0 = X'(0) = B\mu$

$0 = X(\pi/2) = A\cos\mu\pi/2 \Rightarrow \mu_n \pi/2 = (2n-1)\pi/2 \quad n = 1, 2, \ldots$ YIELD THE EIGENVALUES

$\therefore \ \lambda_n = -\mu_n^2 = (2n-1)^2 \ n = 1, 2, \ldots$ & $X_n(x) = \cos(2n-1)x$ ARE THE EIGENFUNCTIONS

THUS THE GENERAL SOLUTION IS OF THE FORM

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(2n-1)^2 t} e^{\gamma^2 t} \cos(2n-1)x$$

NOW $\quad \cos 5x = u(x,0) = \sum_{n=1}^{\infty} a_n \cos(2n-1)x$

SINCE $\cos(2n-1)x$ ARE LINEARLY INDEPENDENT FUNCTIONS WE CAN EQUATE COEFFICIENTS TO OBTAIN

$$a_n = \begin{cases} 0 & n \neq 3 \\ 1 & n = 3 \end{cases}$$

$$\therefore \ u(x,t) = e^{-25t} e^{+\gamma^2 t} \cos 5x$$