MATH 257/316 MIDTERM 1 SECTION 102 OCT 2018

$$Ly = 9x^2 y'' + 3x^2 y' + 2y = 0$$

1.a) $x > 0$ ARE ORDINARY POINTS

$x = 0$ IS A SINGULAR POINT

$\lim_{x\to 0} \frac{x\,(3x^2)}{9x^2} = 0 = p_0 \quad \lim_{x\to 0} x^2 \frac{(2)}{9x^2} = \frac{2}{9} = q_0$ SINCE $|p_0|$ & $|q_0| < \infty$ $x_0 = 0$ IS A RSP.

INDICIAL EQ: $r(r-1) + 0r + \frac{2}{9} = 0 \quad 9r^2 - 9r + 2 = (3r-2)(3r-1) = 0 \quad r = 2/3, 1/3.$

b) SINCE $x_0 = 3$ IS AN ORDINARY POINT ASSUME A TAYLOR SERIES $y(x) = \sum_{n=0}^{\infty} a_n (x-3)^n$

SINCE THE NEAREST S.P. IS $x_0 = 0$ THE RADIUS OF CONVERGENCE $\rho \geq 3$.

c) LET $y = \sum_{n=0}^{\infty} a_n x^{n+r} \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} \quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$

$$Ly = 9x^2 y'' + 3x^2 y' + 2y = 0$$

$$= \sum_{n=0}^{\infty} 9a_n (n+r)(n+r-1) x^{n+r} + \sum_{n=0}^{\infty} 3a_n (n+r) x^{n+r+1} + \sum_{n=0}^{\infty} 2a_n x^{n+r} = 0$$

$m = n \qquad m = n+1,\ n = m-1,\ n = 0 \Rightarrow m = 1 \qquad m = n$

$$= \sum_{m=0}^{\infty} [9(m+r)(m+r-1) + 2] a_m x^{m+r} + \sum_{m=1}^{\infty} 3a_{m-1} (m+r-1) x^{m+r} = 0$$

$$= [9r(r-1) + 2] a_0 x^r + \sum_{m=1}^{\infty} \{[9(m+r)(m+r-1)+2] a_m + 3(m+r-1) a_{m-1}\} x^{m+r} = 0$$

$x^r]$ $\quad 9r^2 - 9r + 2 = (3r-1)(3r-2) = 0 \quad r = 1/3, 2/3$ AS BEFORE

$x^{m+r}]$ $\quad a_m = \frac{-3(m+r-1) a_{m-1}}{9(m+r)(m+r-1) + 2} \quad m \geq 1.$

$r = 2/3$: $a_m = \frac{-3(m-1/3) a_{m-1}}{9(m+2/3)(m-1/3)+2} = \frac{-(3m-1) a_{m-1}}{9m^2 + 3m - \cancel{2} + \cancel{2}} = \frac{-(3m-1) a_{m-1}}{3m(3m+1)}$

$a_1 = \frac{-2}{3.1.4} a_0 = -\frac{a_0}{6} \qquad a_2 = \frac{-(5)}{3.2(7)} a_1 = \frac{+5a_0}{6.42}$

$y_1(x) = a_0 x^{2/3} [1 - x/6 + x^2/6.42 - \cdots]$

$r = 1/3$: $a_m = \frac{-3(m-2/3) a_{m-1}}{9(m+1/3)(m-2/3)+2} = \frac{-(3m-2) a_{m-1}}{(3m+1)(3m-2)+2} = \frac{-(3m-2) a_{m-1}}{9m^2 - 3m - \cancel{2} + \cancel{2}} = \frac{-(3m-2) a_{m-1}}{3m(3m-1)}$

$a_1 = \frac{-1}{3.1.2} a_0 = -\frac{a_0}{6} \qquad a_2 = \frac{-4}{3.2.5} a_1 = \frac{-2a_1}{15} = \frac{+a_0}{45}$

$\therefore\ y_2(x) = a_0 x^{1/3} [1 - x/6 + x^2/45 - \cdots]$

SINCE THERE ARE NO OTHER SINGULAR POINTS $|x_0| < \infty$ THE RADIUS OF CONVERGENCE IS INFINITE.

Q2. $u_t = u_{xx} - \gamma u \qquad 0 < x < \pi/2 \qquad t > 0$

BC: $u(0,t) = 0 \qquad u_x(\pi/2, t) = 0$

IC: $u(x,0) = \sin 3x$

LET $u(x,t) = X(x)T(t)$

$u_t = X(x)\dot{T}(t) = X''(x)T(t) - \gamma X(x)T(t) = u_{xx} - \gamma u$

$\div XT$ AND REARRANGE: $\dfrac{\dot{T}(t)}{T(t)} + \gamma = \dfrac{X''(x)}{X(x)} = \text{CONST} = \lambda$ SAY

T] $\dot{T} = (\lambda - \gamma)T \Rightarrow T(t) = c e^{(\lambda-\gamma)t}$

X] i) $\lambda = \mu^2 > 0$: $X'' - \mu^2 X = 0 \qquad X(0) = 0 = X'(\pi/2)$

$X = A\cosh\mu x + B\sinh\mu x \qquad X'(x) = A\mu\sinh\mu x + B\mu\cosh\mu x$

$0 = X(0) = A \qquad 0 = X'(\pi/2) = B\mu\cosh\mu\pi/2 \Rightarrow B = 0 \Rightarrow X \equiv 0$ TRIVIAL

ii) $\lambda = 0$: $X'' = 0 \Rightarrow X' = B \Rightarrow X = A + Bx$

$0 = X(0) = A \quad 0 = X'(\pi/2) = B \Rightarrow X \equiv 0$ TRIVIAL.

iii) $\lambda = -\mu^2 < 0$: $X'' + \mu^2 X = 0 \qquad X(0) = 0 = X'(\pi/2)$

$X = A\cos\mu x + B\sin\mu x \qquad X' = -A\mu\sin\mu x + B\mu\cos\mu x$

$0 = X(0) = A \qquad 0 = X'(\pi/2) = B\mu\cos(\mu\frac{\pi}{2}) \Rightarrow \mu_n\frac{\pi}{2} = (2n-1)\pi/2 \quad n = 1, 2 \ldots$

EIGENVALUES: $\lambda_n = -(2n-1)^2 \quad n = 1, 2, \ldots$ EIGENFCNS: $X_n = \sin(2n-1)x$

$\therefore u(x,t) = \sum_{n=1}^{\infty} b_n \sin(2n-1)x \, e^{-[(2n-1)^2 + \gamma]t}$

APPLY IC:

$u(x,0) = \sin 3x = \sum_{n=1}^{\infty} b_n \sin(2n-1)x \Rightarrow b_n = \begin{cases} 0 & n \neq 2 \\ 1 & n = 2 \end{cases}$ since $\sin(2n-1)x$ ARE LINEARLY INDEPENDENT

$\therefore u(x,t) = \sin 3x \, e^{-(9+\gamma)t}$