MATH 257/316 MIDTERM 1 SECTION 202 FEB 2012

Q1. $Ly = 9x^2(1-x)y'' + 9xy' - y = 0$

(a) ALL $x \notin \{0,1\}$ ARE ORDINARY POINTS

$x=0$ & $x=1$ ARE SINGULAR POINTS

$x=0$: $\lim_{x\to 0} \dfrac{x \cdot 9x}{9x^2(1-x)} = 1 = p_0 < \infty$; $\lim_{x\to 0} \dfrac{x^2(-1)}{9x^2(1-x)} = -\dfrac{1}{9} = q_0 < \infty \Rightarrow x=0$ IS A RSP.

$x=1$: $\lim_{x\to 1} \dfrac{(x-1)\,9x}{9x^2(1-x)} = -1 = p_0 < \infty$; $\lim_{x\to 1} \dfrac{(x-1)^2(-1)}{9x^2(1-x)} = 0 = q_0 < \infty \Rightarrow x=1$ IS A RSP.

(b) SINCE $x=-1$ IS AN ORDINARY POINT WE ASSUME $y(x) = \sum_{n=0}^{\infty} c_n (x+1)^n$

THE CLOSEST S.P. TO $x=-1$ IS AT $x=0$ THUS $\rho \geq 1$.

(c) SINCE $x=1$ IS A RSP WE ASSUME $y(x) = \sum_{n=0}^{\infty} c_n (x-1)^{n+r}$

FROM PART (a) THE INDICIAL EQ IS $r(r-1) - r = r^2 - 2r = r(r-2) = 0$

THUS $y_1(x) = c_0(x-1)^0 + c_1(x-1) + \cdots$ AND $y_2(x) = d_0(x-1)^2 + c_2(x-1)^3 + \cdots$

ARE THE FORM OF THE SERIES AS $x \to 1$.

(d) LET $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$, $y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}$, $y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}$

$$\therefore Ly = 9x^2 y'' - 9x^3 y'' + 9xy' - y = 0$$

$$= \underbrace{\sum_{n=0}^{\infty} 9c_n (n+r)(n+r-1) x^{n+r}}_{\boxed{n=m}} - \underbrace{\sum_{n=0}^{\infty} 9c_n (n+r)(n+r-1) x^{n+r+1}}_{n+1=m \Rightarrow \boxed{n=m-1},\ n=0 \Rightarrow m=1} + \underbrace{\sum_{n=0}^{\infty} 9c_n (n+r) x^{n+r}}_{\boxed{n=m}} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r}}_{\boxed{n=m}} = 0$$

$$= \sum_{m=0}^{\infty} c_m [9(m+r)(m+r-1) + 9(m+r) - 1] x^{m+r} - \sum_{m=1}^{\infty} 9c_{m-1}(m+r-1)(m+r-2) x^{m+r} = 0$$

$$= c_0[9r(r-1) + 9r - 1]x^r + \sum_{m=1}^{\infty} \{c_m[9(m+r)^2 - 1] - 9(m+r-1)(m+r-2)c_{m-1}\} x^{m+r} = 0$$

$x^r >$ $\quad 9r^2 - 1 = 0 \quad r = \pm 1/3$ INDICIAL EQ.

$x^{m+r}, m \geq 1 >$ $\quad c_m[9(m+r)^2 - 1] - 9(m+r-1)(m+r-2)c_{m-1} = 0$

$$c_m = \frac{9(m+r-1)(m+r-2)}{9(m+r)^2 - 1} c_{m-1} \qquad \text{RECURSION.}$$

NOTE THAT THIS INDICIAL EQ AGREES WITH THAT OBTAINED USING

$p_0 = 1$ AND $q_0 = -1/9 \Rightarrow r(r-1) + r - 1/9 = r^2 - 1/9 = 0 \quad r = \pm 1/3$.

$r=-1/3$: $C_m = \dfrac{9(m-4/3)(m-7/3)\,C_{m-1}}{9(m-1/3)^2-1} = \dfrac{(3m-4)(3m-7)\,C_{m-1}}{(3m-1)^2-1}$

$\therefore \quad C_m = \dfrac{(3m-4)(3m-7)\,C_{m-1}}{9m^2-6m+1-1} = \dfrac{(3m-4)(3m-7)\,C_{m-1}}{3m(3m-2)}$

$C_1 \overset{m=1}{=} \dfrac{(3-4)(3-7)}{3(3-2)}\,C_0 = +\dfrac{4}{3}\,C_0$

$C_2 \overset{m=1}{=} \dfrac{(6-4)(6-7)}{6(6-2)}\,C_1 = \dfrac{-2}{24}\,C_1 = -\dfrac{1}{12}\,\dfrac{4}{3}\,C_0 = -\dfrac{C_0}{9}$

$\therefore \quad y_1(x) = c_0\, x^{-1/3}\left[1 + \dfrac{4}{3}x - \dfrac{x^2}{9} + \cdots\right].$

$r=1/3$: $C_m = \dfrac{9(m-2/3)(m-5/3)\,C_{m-1}}{9(m+1/3)^2-1} = \dfrac{(3m-2)(3m-5)\,c_{m-1}}{(3m+1)^2-1}$

$\therefore \quad c_m = \dfrac{(3m-2)(3m-5)}{9m^2+6m+\cancel{1}-\cancel{1}}\,c_{m-1} = \dfrac{(3m-2)(3m-5)}{3m(3m+2)}\,C_{m-1}$

$C_1 \overset{m=1}{=} \dfrac{(3-2)(3-5)}{3(5)}\,C_0 = \dfrac{-2}{15}\,C_0$

$C_2 \overset{m=2}{=} \dfrac{(6-2)(6-5)}{6(8)}\,C_2 = \dfrac{C_2}{12} = -\dfrac{C_0}{90}$

$\therefore \quad y_2(x) = c_0\, x^{1/3}\left[1 - \dfrac{2}{15}x - \dfrac{x^2}{90} + \cdots\right]$

Q2: $u_t = u_{xx} \quad 0 < x < \pi$

BC: $u_x(0,t) = 0 = u(\pi,t)$

IC: $u(x,0) = x(\pi - x)$

LET $u(x,t) = X(x)T(t)$

$X(x)\dot{T}(t) = X''(x)T(t)$

$\div XT$: $\dfrac{\dot{T}(t)}{T(t)} = \dfrac{X''(x)}{X(x)} = -\lambda^2 = \text{CONST}$

TIME EQ: $\dot{T}(t) = -\lambda^2 T(t) \Rightarrow T(t) = De^{-\lambda^2 t}$.

SPACE EQ: $\lambda \neq 0$: $X'' + \lambda^2 X = 0 \quad X'(0) = 0 = X(\pi)$ EIGENVALUE PROBLEM

$X(x) = A\cos\lambda x + B\sin\lambda x \quad X'(x) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$

$0 = X'(0) = B\lambda = 0 \Rightarrow B = 0$

$0 = X(\pi) = A\cos(\lambda\pi) = 0 \quad \lambda_k \pi = \dfrac{(2k+1)\pi}{2} \quad k = 0, 1, 2, \ldots$

$\lambda_k = \left(\dfrac{2k+1}{2}\right) \; k = 0, 1, \ldots \quad X_k = \cos\left(\dfrac{(2k+1)x}{2}\right)$

$\lambda = 0$: $X'' = 0 \Rightarrow X' = B \quad X = A + Bx \quad X'(0) = B = 0 \quad X(\pi) = A = 0$ TRIVIAL SOLN.

THUS EIGENVALUES ARE $\lambda_k = \left(\dfrac{2k+1}{2}\right) \quad k = 0, 1, \ldots$

AND EIGENFUNCTIONS $X_k(x) = \cos\left(\left(\dfrac{2k+1}{2}\right)x\right)$

THE SOLUTION TO THE HEAT EQ IS THUS OF THE FORM

$$u(x,t) = \sum_{k=0}^{\infty} A_k \cos\left(\left(\frac{2k+1}{2}\right)x\right) e^{-\left(\frac{2k+1}{2}\right)^2 t}$$

IC: $$x(\pi - x) = u(x,0) = \sum_{k=0}^{\infty} A_k \cos\left(\left(\frac{2k+1}{2}\right)x\right)$$

TO DETERMINE THE A_k WE PROJECT $x(\pi-x)$ ONTO $X_k(x)$:

$$A_k = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\cos\left(\left(\frac{2k+1}{2}\right)x\right)dx = \frac{8}{\pi}\left\{\frac{4(-1)^k - (2k+1)\pi}{(2k+1)^3}\right\}$$

$$\therefore u(x,t) = \frac{8}{\pi}\sum_{k=0}^{\infty}\left\{\frac{4(-1)^k - (2k+1)\pi}{(2k+1)^3}\right\}\cos\left(\left(\frac{2k+1}{2}\right)x\right) e^{-\left(\frac{2k+1}{2}\right)^2 t}.$$