M 257/316 MIDTERM 2 SECTION 101.

1. $u_t = u_{xx} + (1-x)e^{-t} \quad 0<x<1,\ t>0$

BC: $u(0,t) = 0 \quad u(1,t) = e^{-t}$

IC: $u(x,0) = x + \sin \pi x$.

FIND $W(x,t) = A(t)x + B(t)$ THAT SATISFIES NONZERO BC

$0 = W(0,t) = B(t) \qquad e^{-t} = W(1,t) = A(t).1 \Rightarrow A(t) = e^{-t}$

$\therefore \quad W(x,t) = xe^{-t}$

LET $u(x,t) = W(x,t) + V(x,t)$ AND DETERMINE THE PDE, BC & IC SATISFIED BY $V(x,t)$

$u_t = W_t + V_t = (\cancel{W_{xx}} + V_{xx}) + (1-x)e^{-t} = u_{xx} + (1-x)e^{-t}$

$\therefore -\cancel{x}e^{-t} + V_t = V_{xx} + (1-\cancel{x})e^{-t} \Rightarrow \quad V_t = V_{xx} + e^{-t}$

BC: $\quad 0 = u(0,t) = W(0,t) + V(0,t) = 0 + V(0,t) \Rightarrow \quad V(0,t) = 0$

$\cancel{e^{-t}} = u(1,t) = W(1,t) + V(1,t) = \cancel{e^{-t}} + V(1,t) \Rightarrow \quad V(1,t) = 0$

IC: $\quad \cancel{x} + \sin \pi x = u(x,0) = W(x,0) + V(x,0) = \cancel{x} + V(x,0) \Rightarrow V(x,0) = \sin(\pi x)$

THE EIGENVALUES ASSOCIATED WITH THE BC ON V ARE $\lambda_n = \frac{n\pi}{1} \quad n = 1,2\ldots$

AND THE EIGENFUNCTIONS ARE $X_n = \sin(n\pi x)$

NOW EXPAND $S(x,t) = e^{-t}$ AND $V(x,t)$ IN TERMS OF EIGENFUNTIONS.

$S(x,t) = e^{-t} = \sum_{n=1}^{\infty} \hat{S}_n(t) \sin \lambda_n x \quad$ WHERE $\hat{S}_n(t) = \frac{2}{1}\int_0^1 e^{-t} \sin \lambda_n x \, dx$

$\therefore \hat{S}_n = 2e^{-t}\left[-\cos(n\pi x)/n\pi\right]_0^1 = e^{-t} 2\left[1-(-1)^n\right]/(n\pi) = c_n e^{-t}$

$V(x,t) = \sum_{n=1}^{\infty} \hat{V}_n(t) \sin \lambda_n x$

$0 = V_t - V_{xx} - e^{-t} = \sum_{n=1}^{\infty}\left[\frac{d\hat{V}_n}{dt} + \lambda_n^2 \hat{V}_n - c_n e^{-t}\right] \sin \lambda_n x$; $[\,] = 0$ SINCE $\sin \lambda_n x$ ARE LINEARLY INDP

$\therefore \frac{d\hat{V}_n}{dt} + \lambda_n^2 \hat{V}_n = c_n e^{-t} \Rightarrow \frac{d}{dt}\left[e^{\lambda_n^2 t}\hat{V}_n\right] = c_n e^{(\lambda_n^2 - 1)t}$

$\therefore \quad e^{\lambda_n^2 t}\hat{V}_n(t) = c_n e^{(\lambda_n^2-1)t}/(\lambda_n^2 - 1) + d_n$

$\therefore \quad \hat{V}_n(t) = \frac{c_n e^{-t}}{\lambda_n^2 - 1} + d_n e^{-\lambda_n^2 t}$

$\therefore \quad V(x,t) = \sum_{n=1}^{\infty}\left\{\frac{c_n e^{-t}}{\lambda_n^2 - 1} + d_n e^{-\lambda_n^2 t}\right\} \sin(n\pi x)$

$\sin(\pi x) = \sum_{n=1}^{\infty}\left\{\frac{c_n}{\lambda_n^2 - 1} + d_n\right\} \sin(n\pi x)$

$\therefore \quad 1 = c_1/(\lambda_1^2 - 1) + d_1 \qquad 0 = c_n/(\lambda_n^2 - 1) + d_n$

$d_1 = 1 + c_1/(1-\lambda_1^2) \qquad d_n = c_n/(1-\lambda_n^2)$

$\therefore u(x,t) = xe^{-t} + \left\{\sum_{n=1}^{\infty} \frac{c_n}{\lambda_n^2 - 1}\left(e^{-t} - e^{-\lambda_n^2 t}\right) \sin(\lambda_n x)\right\} + e^{-\pi^2 t} \sin \pi x$

AND $c_n = \frac{2}{\pi}\left[\frac{1-(-1)^n}{n}\right]$

2. $u_{tt} = u_{xx} + 9\pi^2 \sin 3\pi x$

$u(0,t) = 0 = u(1,t)$

$u(x,0) = 0 \quad u_t(x,0) = \sin \pi x$

a) $0 = w_{xx} + 9\pi^2 \sin 3\pi x$

$w_{xx} = -9\pi^2 \sin 3\pi x \quad w_x = 3\pi \cos 3\pi x + A \quad w = \sin 3\pi x + Ax + B$

$0 = w(0) = B \quad 0 = w(1) = 0 + A \quad \Rightarrow \quad w(x) = \sin 3\pi x$

b) $u(x,t) = w(x) + v(x,t)$

PDE: $u_{tt} = (\cancel{w} + v)_{tt} = \{\cancel{w_{xx} + 9\pi^2 \sin 3\pi x}\} + v_{xx} \Rightarrow v_{tt} = v_{xx}$

BC: $0 = u(0,t) = w(0) + v(0,t) = 0 + v(0,t) \quad \Rightarrow v(0,t) = 0$

$0 = u(1,t) = w(1) + v(1,t) = 0 + v(1,t) \quad \Rightarrow v(1,t) = 0$

IC: $0 = u(x,0) = \sin 3\pi x + v(x,0) \quad \Rightarrow v(x,0) = -\sin 3\pi x$

$\sin \pi x = u_t(x,0) = \cancel{w_t(x)} + v_t(x,0) \quad \Rightarrow v_t(x,0) = \sin \pi x.$

c) Let $v(x,t) = X(x)T(t) \Rightarrow \dfrac{\ddot{T}}{T(t)} = \dfrac{X''}{X(x)} = -\lambda^2$

$X]$ $X'' + \lambda^2 X = 0 \quad X(0) = 0 = X(1) \Rightarrow \lambda_n = n\pi \quad n = 1, 2, \ldots \quad X_n = \sin(n\pi x)$

$T]$ $\ddot{T} + \lambda^2 T = 0 \quad T(t) = A\cos(\lambda t) + B\sin(\lambda t)$

$\therefore v(x,t) = \sum_{n=1}^{\infty} [A_n \cos \lambda_n t + B_n \sin \lambda_n t] \sin(\lambda_n x)$

$-\sin 3\pi x = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \Rightarrow A_3 = -1 \quad A_n = 0 \quad n \neq 3.$

$v_t(x,t) = \sum_{n=1}^{\infty} [A_n(-\lambda_n) \sin \lambda_n t + B_n \lambda_n \cos \lambda_n t] \sin \lambda_n x$

$\sin \pi x = v_t(x,0) = \sum_{n=1}^{\infty} B_n (n\pi) \sin(n\pi x) \Rightarrow B_1 = \frac{1}{\pi}, \; B_n = 0 \; n \geq 2.$

$\therefore v(x,t) = -\cos(3\pi t) \sin(3\pi x) + \frac{1}{\pi} \sin(\pi t) \sin \pi x$

$= -\frac{1}{2}[\sin 3\pi(x+t) + \sin 3\pi(x-t)] + \frac{1}{2\pi}[\cos \pi(x-t) - \cos \pi(x+t)]$

c) $v(x,t) = -\frac{1}{2}[\sin 3\pi(x+t) + \sin 3\pi(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \sin \pi s \, ds$

$= -\frac{1}{2}[\sin 3\pi(x+t) + \sin 3\pi(x-t)] + \frac{-1}{2\pi}[\cos \pi(x+t) - \cos \pi(x-t)]$

AS ABOVE

$\therefore u(x,t) = \sin 3\pi x - \frac{1}{2}[\sin 3\pi(x+t) + \sin 3\pi(x-t)] - \frac{1}{2\pi}[\cos \pi(x+t) - \cos \pi(x-t)]$