M257/316 MIDTERM 2 SECTION 101 2018.

Q1 PDE $u_t = u_{xx} + e^{-2t}(\sin 5x - 2x) \qquad 0 < x < \pi/2 \quad t > 0$

BC $u(0,t) = 1 \qquad u_x(\frac{\pi}{2},t) = e^{-2t}$

IC $u(x,0) = 1 + \sin x$

FIRST FIND A FUNCTION $W(x,t) = A(t)x + B(t)$ WITH $W_x = A(t)$ THAT MATCHES THE NONZERO BC:

$$1 = W(0,t) = B(t) \qquad W_x(\tfrac{\pi}{2},t) = A(t) = e^{-2t} \Rightarrow \boxed{W(x,t) = 1 + xe^{-2t}}$$

NOW LET $u(x,t) = W(x,t) + V(x,t)$ THEN PLUG INTO PDE, BC & IC:

PDE: $u_t = W_t + V_t = -2\cancel{x}e^{-2t} + V_t = \overset{\text{W A LINEAR FCN}}{\cancel{W_{xx}}} + V_{xx} + e^{-2t}(\sin 5x - \cancel{2x}) \Rightarrow$ $\boxed{V_t = V_{xx} + e^{-2t}\sin 5x}$

BC: $\cancel{1} = u(0,t) = W(0,t) + V(0,t) = \cancel{1} + V(0,t) \Rightarrow$ $\boxed{V(0,t) = 0}$ (*)

$\cancel{e^{-t}} = u_x(\frac{\pi}{2},t) = W_x(\frac{\pi}{2},t) + V_x(\frac{\pi}{2},t) = \cancel{e^{-t}} + V_x(\frac{\pi}{2},t) \Rightarrow$ $\boxed{V_x(\frac{\pi}{2},t) = 0}$

IC: $\cancel{1} + \sin x = u(x,0) = W(x,0) + V(x,0) = \cancel{1} + x + V(x,0) \Rightarrow$ $\boxed{V(x,0) = \sin x - x}$

THE EIGENVALUES AND EIGENFUNCTIONS FOR THE BVP(*) ARE $\boxed{\mu_n = \frac{(2n-1)\pi}{2(\pi/2)} = (2n-1) \quad n=1,2,\ldots \quad X_n = \sin\mu_n x}$

LET $S(x,t) = e^{-2t}\sin 5x = \sum_{n=1}^{\infty} e^{-2t}\delta_{n3}\sin(2n-1)x = \sum_{n=1}^{\infty} S_n(t)\sin(2n-1)x \Rightarrow \boxed{S_n(t) = e^{-2t}\delta_{n3}}$

NOW LET $V(x,t) = \sum_{n=1}^{\infty} V_n(t)\sin\mu_n x$ AND SUBSTITUTE INTO THE PDE IN (*)

$$0 = V_t - V_{xx} - e^{-2t}\sin 5x = \sum_{n=1}^{\infty}\left\{\frac{dV_n}{dt} + \mu_n^2 V_n - \delta_{n3}e^{-2t}\right\}\sin\mu_n x \Rightarrow \boxed{\dot{V}_n + \mu_n^2 V_n = \delta_{n3}e^{-2t}}$$

APPLYING THE INTEGRATING FACTOR $F = e^{\mu_n^2 t} \Rightarrow \frac{d}{dt}\left[e^{\mu_n^2 t}V_n\right] = \delta_{n3}\,e^{(\mu_n^2-2)t}$

$$\therefore \quad e^{\mu_n^2 t}V_n = \frac{\delta_{n3}e^{(\mu_n^2-2)t}}{(\mu_n^2-2)} + C_n \Rightarrow V_n(t) = \frac{\delta_{n3}e^{-2t}}{(\mu_n^2-2)} + C_n e^{-\mu_n^2 t}.$$

$$\therefore \quad V(x,t) = \sum_{n=1}^{\infty}\left\{\frac{\delta_{n3}e^{-2t}}{\mu_n^2-2} + C_n e^{-\mu_n^2 t}\right\}\sin\mu_n x$$

$$\sin x - x = \sum_{n=1}^{\infty}\left\{\frac{\delta_{n3}}{\mu_n^2-2} + C_n\right\}\sin(2n-1)x \Rightarrow C_n = \delta_{n1} - \frac{\delta_{n3}}{\mu_n^2-2} - \frac{2}{(\frac{\pi}{2})}\int_0^{\pi/2} x\sin(2n-1)x\,dx$$

$$C_n + \frac{\delta_{n3}}{\mu_n^2-2} - \delta_{n1} = \frac{4}{\pi}\left[\frac{x\cos(2n-1)x}{(2n-1)}\Big|_0^{\pi/2} - \frac{1}{(2n-1)}\int_0^{\pi/2} 1\cos(2n-1)x\,dx\right]$$

$$= \frac{4}{\pi}\left[-\frac{1}{(2n-1)^2}\sin(2n-1)x\Big|_0^{\pi/2}\right] = \frac{4}{\pi}\frac{(-1)^n}{(2n-1)^2}$$

$$\therefore \quad C_n = \delta_{n1} - \frac{\delta_{n3}}{\mu_n^2-2} + \frac{4}{\pi}\frac{(-1)^n}{(2n-1)^2}$$

$$\therefore \quad u(x,t) = 1 + xe^{-2t} + \sum_{n=1}^{\infty}\left[\frac{\delta_{n3}}{\mu_n^2-2}e^{-2t} + \left(\delta_{n1} - \frac{\delta_{n3}}{\mu_n^2-2} + \frac{4}{\pi}\frac{(-1)^n}{(2n-1)^2}\right)e^{-\mu_n^2 t}\right]\sin(2n-1)x$$

$$= 1 + xe^{-2t} + \left(\frac{e^{-2t} - e^{-25t}}{23}\right)\sin 5x + e^{-t}\sin x + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}e^{-(2n-1)^2 t}\sin(2n-1)x.$$

Q2: PDE $u_{tt} = u_{xx}$ $\quad 0 < x < \pi$

BC $u(0,t) = \pi t$ $\quad u(\pi,t) = 2\pi t$

IC $u(x,0) = \sin 2x$ $\quad u_t(x,0) = \sin x + \pi + x.$

LET $W(x,t) = A(t)x + B(t)$ BE A FUNCTION CONSTRUCTED TO MATCH THE NON-ZERO BC:

$\pi t = W(0,t) = B(t)$ $\quad u(\pi,t) = A(t)\pi + \pi t = 2\pi t \Rightarrow A(t) = t \Rightarrow \boxed{W(x,t) = \pi t + xt}$

NOW LET $u(x,t) = W(x,t) + V(x,t)$ AND PLUG INTO PDE, BC & IC

PDE: $u_{tt} = \cancel{W_{tt}} + V_{tt} = \cancel{W_{xx}} + V_{xx} \Rightarrow$ $\quad V_{tt} = V_{xx}$

BC: $\cancel{\pi t} = u(0,t) = W(0,t) + V(0,t) = \cancel{\pi t} + V(0,t) \Rightarrow$ $\quad V(0,t) = 0$

$\cancel{2\pi t} = u(\pi,t) = W(\pi,t) + V(\pi,t) = \cancel{2\pi t} + V(\pi,t) \Rightarrow$ $\quad V(\pi,t) = 0$

IC: $\sin 2x = u(x,0) = W(x,0) + V(x,0) = 0 + V(x,0) \Rightarrow$ $\quad V(x,0) = \sin 2x$

$\cancel{\pi} + \cancel{x} + \sin x = u_t(x,0) = W_t(x,0) + V_t(x,0) = \cancel{\pi} + \cancel{x} + V_t(x,0) \Rightarrow$ $\quad V_t(x,0) = \sin x.$

LET $V(x,t) = X(x)T(t)$

$V_{tt} = X(x)\ddot{T}(t) = V_{xx} = X''(x)T(t)$

$\div XT \Rightarrow$ $\quad \dfrac{\ddot{T}(t)}{T(t)} = \dfrac{X''(x)}{X(x)} = \lambda$: A CONSTANT

X] $\lambda \geq 0$ YIELD TRIVIAL SOLN $\Rightarrow \lambda = -\mu^2 < 0 \Rightarrow X'' + \mu^2 X = 0$ $\quad X(0) = 0 = X(\pi)$

$\mu_n = (n\cancel{\pi}/\cancel{\pi}) = n$ $\quad n = 1, 2, \ldots$ $\quad X_n = \sin nx.$

T] $\ddot{T} + \mu^2 T = 0 \Rightarrow T(t) = A\cos\mu t + B\sin\mu t.$

$\therefore V(x,t) = \sum_{n=1}^{\infty} [A_n \cos nt + B_n \sin nt] \sin(nx)$

$V_t(x,t) = \sum_{n=1}^{\infty} [-A_n n \sin nt + B_n n \cos nt] \sin(nx)$

$\sin 2x = \sum_{n=1}^{\infty} A_n \sin nx \Rightarrow A_n = \delta_{n2}$

$\sin x = \sum_{n=1}^{\infty} B_n n \sin nx \Rightarrow B_n n = \delta_{n1} \Rightarrow B_n = \delta_{n1}/n.$

$\therefore u(x,t) = (\pi t + xt) + \sum_{n=1}^{\infty} [\delta_{n2} \cos nt + (\delta_{n1}/n) \sin nt] \sin nx$

$= \pi t + xt + \cos 2t \sin 2x + \sin t \sin x$

$= \pi t + xt + \frac{1}{2}[\sin 2(x+t) + \sin 2(x-t)] + \frac{1}{2}[\cos 2(x-t) - \cos 2(x+t)]$

FROM D'ALEMBERTS SOLUTION

$V(x,t) = \frac{1}{2}[\sin 2(x+t) + \sin 2(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \sin s \, ds$

$= \frac{1}{2}[\sin 2(x+t) + \sin 2(x-t)] + \frac{1}{2}[\cos(x-t) - \cos(x+t)]$ AS ABOVE